

BIFURCATION THEORY OF FUNCTIONAL DIFFERENTIAL EQUATIONS: A SURVEY*

Shangjiang Guo^{1,†} and Jie Li²

Abstract In this paper we survey the topic of bifurcation theory of functional differential equations. We begin with a brief discussion of the position of bifurcation and functional differential equations in dynamical systems. We follow with a survey of the state of the art on the bifurcation theory of functional differential equations, including results on Hopf bifurcation, center manifold theory, normal form theory, Lyapunov-Schmidt reduction, and degree theory.

Keywords Hopf bifurcation, center manifold theory, normal form theory, Lyapunov-Schmidt reduction, degree theory.

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1. Introduction

A functional differential equation (FDE) describes the evolution of a dynamical system for which the rate of change of the state variable depends on not only the current but also the historical and even future states of the system. FDEs arise very naturally in economics, life sciences and engineering, and the study of FDEs has been a major source of inspiration for advancement in nonlinear analysis and infinite dimensional dynamical systems. Therefore, FDEs provides an excellent theoretical platform to develop an interdisciplinary approach understanding complex nonlinear phenomena via appropriate mathematical techniques. There are different types of FDEs arising from important applications: delay differential equations (DDEs) (also referred to as Retarded Functional Differential Equations, RFDEs), Neutral Functional Differential Equations (NFDEs), and Mixed Functional Differential Equations (MFDEs). The classification depends on how the current change rate of the system state depends on the history (the historical status of the state only, or the historical change rate and the historical status), or depends on whether the current change rate of the system state depends on the future expectation of the system. Moreover, we will also see that the delay involved may also depend on the system state, leading to DDEs with state-dependent delay.

The novelty and challenge of fundamental research in the field of FDEs has often been under appreciated. This is specially so in our effort describing the qualitative behaviours of solutions near equilibria or periodic orbits: these qualitative behaviours can be derived from those of finite dimensional (ordinary differential)

[†]the corresponding author. Email address: shangjguo@hnu.edu.cn(S. Guo)

¹College of Mathematics and Econometrics, Hunan University, Changsha, Hunan 410082, People's Republic of China

²School of Physics and Electronics, Hunan University, Changsha, Hunan 410082, People's Republic of China

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systems obtained through a center and center-unstable manifold reduction process, and hence the (local) bifurcation theory which deals with the significant change of these qualitative behaviours is in principle a consequence of the corresponding theory for finite dimensional ordinary differential equations (ODEs). The highly nontrivial and often lengthy calculation of center manifold reduction however not only leads to enormous duplication of calculation efforts, but also prevent us from discovering simple and key mechanisms behind observed bifurcation phenomena due to the infinite dimensionality of FDEs. This in turns makes it difficult to express bifurcation results explicitly in terms of model parameters and to compare and validate different results. Another challenge is the study of the birth and global continuation of bifurcation of periodic solutions and the coexistence of multiple periodic solutions when the parameters are away from the bifurcation/critical values. There has been substantial progress dedicated to this global bifurcation problem, remarkably the presence of a delayed or advanced argument in the nonlinearity can sometimes facilitate the application of topological methods such as equivalent degrees to examine the global continua of branches of periodic solutions, and has inspired interesting development of the spectral analysis of circulant matrices. In the monograph [60], Guo and Wu summarized some practical and general approaches and frameworks for the investigation of bifurcation phenomena of FDEs depending on parameters.

The subsequent sections are devoted to the the art on the bifurcation theory of FDEs, including center manifold theory, normal form theory, Lyapunov-Schmidt reduction, and degree theory. Our concluding section is devoted to an outlook.

2. Center Manifold Reduction

A center manifold at a given non hyperbolic equilibrium is an invariant manifold of the considered differential equation which is tangent at the equilibrium point to the (generalized) eigenspace of the neutrally stable eigenvalues. As the local dynamic behavior *transverse* to the center manifold is relatively simple, the potentially complicated asymptotic behaviours of the full system are captured by the flows restricted to the center manifolds.

Center manifold theory plays an important role in the study of the stability of dynamical systems when the equilibrium point is not hyperbolic. The combination of this theory with the normal form approach was used extensively to study parameterized dynamical systems exhibiting bifurcations. The center manifold theorem provides, in this case, a means of systematically reducing the dimension of the state spaces which need to be considered when analyzing bifurcations of a given type. In fact, after determining the center manifold, the analysis of these parameterized dynamical systems is based only on the restriction of the original system on the center manifold whose stability properties are the same as the ones of the full order system.

The classical center manifolds theory of equilibria, since first is introduced by Pliss [101] and Kelley [81] in the 1960's and later is developed by Carr [22], Hirsch et al. [66], Sijbrand [107], Guckenheimer and Holmes [44], Vanderbauwhede [114] and others. For recent developments in the approximation of center manifolds see Jolly and Rosa [78]. Use of this for the study of bifurcation problems owes a lot to the paper of Lanford [88]. Center manifold theory for equilibrium solutions to MFDEs that was developed in [73].

As a DDE generates a semiflow in an infinite dimensional Banach space, one naturally first reduces the semiflow to a flow in the finite dimensional center manifold, and then calculates the normal form of the reduced flow; See for example, [47,50,51,56,58,59]. However, in most literatures [16,18–20,37,38], it is assumed that the equilibrium point is always fixed at the origin. As stated before, this is not true in a general physical system or an engineering problem. In the case where the equilibrium point isn't always fixed at the origin as the perturbation parameter α varies, one may first ignore the perturbation parameter α and compute the center manifold as well as normal form, and then add unfolding to the resulting normal form. In other words, the normal form of the original system with parameters is equal to the normal form of the *reduced* system plus the unfolding. This way it greatly reduces the computation effort, with the cost that it does not provide the nonlinear transformation between the original system and the normal form. See, for example, [16, 18, 19], in the context of functional differential equations with symmetries.

Guo and Man [57] provided a general framework to obtain the reduced equation on the center manifold in the case where the equilibrium point isn't always fixed at the origin as the perturbation parameter varies. The approach of Guo and Man [57] starts with the consideration of the structures of the center spaces associated with finite subsets of eigenvalues of the infinitesimal generator for the linearized equations of FDEs at the singularity, and then follows by enlarging the phase space in such a way that FDEs can be rewritten as an abstract ODE in a Banach space. Then the center manifold theorem for this abstract ODE can be employed to obtain the reduced equation on the center manifold, which may inherit the symmetry of the original system. This approach and general results can be illustrated by some applications to fold and Bogdanov-Takens bifurcations.

3. Normal form theory

Normal forms theory provides one of the most powerful tools in the study of nonlinear dynamical systems, in particular, in the stability and bifurcation analysis. In the context of finite-dimensional ordinary differential equations (ODEs), this theory can be traced back as far as Euler. However, Poincaré [102] and Birkhoff [12] were the first to bring forth the theory in a more definite form. Now, many systematic procedures for constructing normal forms have been developed. A method of Lie brackets is given in Chow and Hale [27], Takens [111] and Ushiki [113], a method using an inner product in the space of homogeneous polynomials is given in Elphick et al. [35] and Ashkenazi and Chow [6], a method for direct computations is given in Bruno [15] and Chen and Della Dora [23], a method using the Carleman linearization is given in Tsiligiannis and Lyberatos [112] and Chen and Della Dora [24]. The nilpotent case is treated in Cushman and Sanders [30] using the representation theory of $sl_2(\mathbb{R})$. Recently, the normal form for a generalized Hopf bifurcation is expressed as a 4-dimensional real system by Cushman and Sanders [31] and as a 2-dimensional complex system by Elphick et al. [35] and Iooss and Adelmeyer [74].

The basic idea of normal form consists of employing successive, near-identity, nonlinear transformations, which leads to a differential equation in a simpler form, qualitatively equivalent to the original system in the vicinity of a fixed equilibrium point, thus hopefully greatly simplifying the dynamics analysis. As we develop the method, three important characteristics should become apparent. (i) The method is

local in the sense that the coordinate are generated in a neighborhood of a known solution. For our purposes, the known solution will be an equilibrium. (ii) In general, the coordinate transformations will be nonlinear of the dependent variables. However, the important point is that coordinate transformations are found by solving a sequence of problems. (iii) The structure of the norm form is determined entirely by the linear part of the vector field. A key notion in normal form reduction is that of resonance. In particular, the Jacobian matrix of the system, evaluated at the equilibrium point determines which monomials in the formal expansion of the system are resonant and cannot be removed by any smooth coordinate transformation.

The principal difficulty in developing a normal form theory for FDEs is the fact that the phase space is not finite dimensional. The first work in this direction, in such a way to overcome this difficulty, is due to Faria and Magalhães [37, 38], who considered an RFDE as an abstract ODE in an adequate infinite-dimensional phase space which was first presented in the work of Chow and Mallet-Paret [28]. This infinite dimensional ODE was then handled in a similar way as in the finite dimensional case. Through a recursive process of nonlinear transformations, Faria and Magalhães [37, 38] succeeded to reduce to a simpler infinite dimensional ODE so defined as a normal form of the original RFDE. Faria and Magalhães [37, 38] illustrated that their method provides an efficient algorithm for approximating normal forms for a RFDE directly without computing beforehand a local center manifold near the singularity. This is important as this approach does not lead to the loss of the explicit relationships between the coefficients in the normal form obtained and the coefficients in the original RFDE. Guo, Chen, and Wu [49] developed an effective approach to compute normal forms on sub-center manifolds for equivariant FDEs near equilibria and use the normal forms to study the qualitative behavior of solutions on those manifolds.

4. Lyapunov-Schmidt Reduction

Generally, particular types of solutions of a differential equation, such as a fixed point, relative equilibrium, or a periodic orbit can be found by determining the zeros of an appropriate map F and applying the Lyapunov-Schmidt procedure. The Lyapunov-Schmidt reduction results in the so-called bifurcation equations, a finite set of equations, equivalent to the original problem. This finite set of equations may inherit the symmetry properties of the original system if the reduction is done properly. For example, if we are looking for periodic solution, the map F has a natural symmetry group \mathbb{S}^1 representing phase shifts along the periodic solution.

It would be interesting to know for what values of parameter, say α , solutions of the bifurcation equation disappear or are created. These particular values of α are called bifurcation values. Now there exists an extensive mathematical machinery called singularity theory (see Golubitsky and Guillemin [41]) which deals with such questions. Singularity theory is concerned with the local properties of smooth functions near a zero of the function. It provides a classification of the various cases based on codimension. The reason this is possible is that the codimension k submanifolds in the space of all smooth functions having zeros can be described algebraically by imposing conditions on derivatives of the functions. This gives us a way of classifying the various possible bifurcations and of computing the proper unfoldings.

Lyapunov-Schmidt reduction is a very effective method to investigate the phenomenon of Hopf bifurcation, which concerns the birth of a periodic solution from an equilibrium solution through a local oscillatory instability. Stech [109] used the Lyapunov-Schmidt reduction method and generalized a proof given by De Oliveira and Hale [32] in the case of ODEs to infinite delay differential equations. Stech [109] also gave a computational scheme of bifurcation elements via an asymptotic expansion of the bifurcation function. Staffans [108] established the theorem in a case analogous to Stech's for neutral functional differential equations, using the Lyapunov-Schmidt reduction method. Guo and Wu [61] presented a treatment of resonant Hopf bifurcation for DDEs on the basis of Lyapunov-Schmidt reduction. In the process explicit expressions in terms of the coefficients of the original systems are obtained to determine whether either no branch, or one, or two branches of periodic solutions exist as the bifurcation parameter varies. With these expressions at our disposal, the study of resonant Hopf bifurcation in concrete DDE system can now be performed without having to resort to lengthy computations associated to the center manifold reduction. This will be shown by a case study on the van der Pol oscillator with delayed feedback in [61]. By means of a Lyapunov-Schmidt reduction, Guo, Lamb, and Rink [55] have addressed an elementary question whether the monochromatic and bichromatic wave trains of the linear FPU lattice continue to exist in the nonlinear lattice. This is a way of reducing an advance-delay differential equations to a finite-dimensional bifurcation equation. This method was extended by Zhang and Guo [121] to study the existence and branching patterns of wave trains in a two-dimensional lattice with linear and nonlinear coupling between nearest particles and a nonlinear substrate potential. The works [55, 121] also show how the particle-shift \mathbb{Z} -equivariance, the time reversal symmetry, and the Hamiltonian structure manifest themselves in the reduced bifurcation equation.

5. Degree theory

Many applications, including some bifurcation problems of functional differential equations, lead to the problem of finding all zeros of a mapping $f: U \subseteq X \rightarrow X$, where X is some (real) Banach space. The basic idea of degree theory is as follow. Given a (sufficiently smooth) domain U with enclosing Jordan curve ∂U , we have defined a degree $\deg(f, U, z_0) = n(f(\partial U), z_0) = n(f(\partial U) - z_0, 0) \in \mathbb{Z}$, which counts the number of solutions of $f(z) = z_0$ inside U . The invariance of this degree with respect to certain deformations of f allowed us to explicitly compute $\deg(f, U, z_0)$ even in nontrivial cases. Degree theory has been developed for various classes of mappings. For relevant results on topological degree, see, for example, [7, 8, 76, 77, 82–86, 86]. Moreover, similar ideas also appears in the definitions of Fuller index. See, for example, Chow and Mallet-Paret [29].

In 1912, Brouwer [13] introduced the so-called Brouwer degree in \mathbb{R}^n . See Brouwer [14], Sieberg [106] for historical developments. In 1934, Leray and Schauder [93] generalized Brouwer degree theory to an infinite Banach space and established the so-called the Leray-Schauder degree. It turns out that the Leray-Schauder degree is very powerful tool in proving various existence results for nonlinear differential equations. As in the previous section, we study nonlinear parameter-dependent problem $F(u, \alpha) = 0$, where $F: E \times \mathbb{R} \rightarrow X$ is a C^1 -map such that $F(0, \alpha) = 0$ for all $\alpha \in \mathbb{R}$, $E \subseteq X$ is an open neighborhood of 0 (possibly $E = X$). Note that $F(\cdot, \alpha)$ has the trivial zero point for all values of α . We shall now consider the question of bifur-

cation from this trivial branch of solutions and demonstrate the existence of global branches of nontrivial solutions bifurcating from the trivial branch. If $X = \mathbb{R}^n$, then we use the Brouwer degree; if X is an infinite-dimensional (real) Banach space, then we assume that $F(x, \alpha) = x + f(x, \alpha)$ and that $f: E \times \mathbb{R} \rightarrow X$ is completely continuous. Thus, for $F(\cdot, \alpha)$ the Leray-Schauder degree is applicable. The application of degree theory to bifurcation theory goes back to Krasnoselskii [191]. Global bifurcation theorem of the following type were first proved by Rabinowitz [251]. Several generalizations have been given by Ize et al. [76, 77], Krawcewicz et al. [83–86, 86], and Nussbaum et al. [97–100].

Similarly, the problem of finding a nontrivial periodic solution to a differential equation can be solved by using the so-called \mathbb{S}^1 -equivariant degree. Equivariant degree theory was developed by Ize et al. [76, 77] and independently by Geba et al. [40], which in the case of G -symmetric (with respect to a general compact Lie group G) equation provides a topological tool to classify the solutions according to their symmetry properties (in the same way as the Brouwer degree is applied to equations without symmetries). This method provides an effective and complete method for a full analysis of the symmetric Hopf bifurcation problems. It allows to directly translate the equivariant spectral properties of the characteristic operator (associated with the system) into a topological invariant containing the information related to the occurrences of the Hopf bifurcation, the symmetric structure of the bifurcating branches of non-constant periodic solutions, and their multiplicity. It is important to point out, that in the case of a complete classification of a symmetric Hopf bifurcation, it is necessary to use the full primary equivariant degree [8] of the associated maps. The computations of such a full primary degree can be done based on the standard properties and the so-called multiplication tables. Guo et al. [48, 52, 53] employed \mathbb{S}^1 -equivariant degree to investigate the spatio-temporal patterns of nonlinear oscillations in ring networks with delay. Recently, Hu and Wu [70, 71] provided a general tool and framework for studying the Hopf bifurcation problem, and in particular, the global continuation of local bifurcation of periodic solutions of state-dependent DDEs from an equivariant-degree point of view.

6. Bifurcation of FDEs

For equilibria of flows, a (generic) codimension one bifurcation means that the crossing of the stability region (the imaginary axis) is taking place with either one eigenvalue of the linear part going through 0, or one pair of complex conjugate eigenvalues crossing the imaginary axis. As pointed out repeatedly by Arnold [5], examples of Hopf bifurcation can be found in the work of Poincaré [103]. The first specific study and formulation of a theorem was due to Andronov [3]. However, the work of Poincaré and Andronov was concerned with two-dimensional vector fields. The existence of such a bifurcation was found in the context of a general n -dimensional ODEs by Hopf [68] in 1942. This was before the discovery of the center manifold theorem. For these reasons we usually refer to this kind of bifurcation as Poincaré-Andronov-Hopf bifurcation.

In the seventies of last century, Hsu and Kazarinoff [69], Poore [104], Marsden and McCracken [95] and others discussed in their works the computation of important features of the Hopf bifurcation, especially the direction of bifurcation and dynamical aspects (stability, attractiveness, etc), both from theoretical and numerical standpoints. A very important new achievement was the proof by Alexander

and Yorke [2] of what is known as the global Hopf bifurcation theorem, which roughly speaking describes the global continuation of the local branch. The theory was also extended to allow further degeneracies (more than two eigenvalues crossing the imaginary axis, or multiplicity higher than one, etc) leading notably to the development of the generalized Hopf bifurcation theory (Bernfeld, Negrini and Salvadori [9, 10], Negrini and Salvadori [96]).

The first results on Hopf bifurcation for retarded FDEs date back to work by Chafee in 1971. However, according to Hale [62], the first proof of the Hopf bifurcation theorem for RFDEs under analytically computable conditions was presented by Chow and Mallet-Paret [28] in 1977. Since then, a considerable number of studies have been done by many authors, treating many aspects related to bifurcation of periodic solutions. For existence, uniqueness and regularity of the bifurcating branch, several approaches have been undertaken: the averaging method was notably developed by Gumowski [45] and Chow and Mallet-Paret [28]. Another approach, based on integral manifolds, was developed by Hale [63] and was further extended to the case of infinite delay by Stech [109]. Arino [4] treated the same problem by formulating an *adapted* implicit function theorem. Adimy [1] proved a Hopf bifurcation theorem using the integrated semigroup theory. Diekmann et al. [33] tackled the problem due to the lack of regularity of the solution operator associated with a delay equation. Using the sun-star theory of dual semigroups, they reduced the problem of bifurcation, on a center manifold, to a planar ODE. In [37, 38], Faria and Magalhães studied the Hopf bifurcation problem by developing a normal form theory for retarded FDEs. Sieber [105] proved that periodic boundary-value problems for DDEs are locally equivalent to finite-dimensional algebraic systems of equations and then uses this equivalence theorem to provide a complete proof for the local Hopf Bifurcation Theorem for state-dependent DDEs, including the regularity of the emerging periodic orbits. Hu and Wu [70, 71] studied the Hopf bifurcation problem of state-dependent DDEs from an equivariant degree point of view. By means of S^1 -equivariant degree coupled with a higher dimensional Bendixson criterion for ODEs due to Li and Muldowney [94], Wei and Li [116] established the global extension of the local Hopf branch in a delayed Nicholson blowflies equation.

Local Hopf bifurcation theorems for evolution equation in a Banach space with delays have been proved in [119] (see Theorem 4.6 on page 211) and Faria [36]. Hopf bifurcating periodic solutions can be found by means of a center manifold (see, for example, [17, 25, 72, 110]). In addition, dynamical behavior near spatially homogeneous equilibriums of diffusive systems has been investigated by some researchers. However, the discussion of dynamical behavior near a spatially nonhomogeneous steady-state solution is very difficult since the characteristic equation is no longer algebraic equations. The pioneer work about the dynamics near a spatially nonhomogeneous steady-state solution is Busenberg and Huang [17]. Motivated by the idea used in [17], some researchers investigated the existence, uniqueness, and stability of spatially nonhomogeneous steady-state solutions for some 1-dimensional and 2-dimensional population models (see, for example, [17, 25, 72, 110]). In [17, 25, 72, 110], however, the authors did not investigate the multiplicity of spatially nonhomogeneous steady-state solutions. Different from Busenberg and Huang [17], Guo [46] employed a Lyapunov-Schmidt reduction to investigate the existence, stability, and multiplicity of spatially nonhomogeneous steady-state solution and periodic solutions for a reaction-diffusion model with nonlocal delay effect and Dirichlet boundary condition, which includes such systems investigated in [17, 25, 72, 110] and also

some general systems with distributed delay as special cases. Moreover, Guo [46] derived the formula to determine the bifurcation direction and the stability of Hopf bifurcating periodic solutions.

If a system of FDEs is symmetric, i.e., it has a nontrivial group of symmetries, one expects that the system has symmetric orbits, symmetric fixed points and periodic orbits, symmetric attractors or repellers. Also, symmetric steady states can generate symmetric patterns in the state space of the system. In 1998, Wu [118] employed the same techniques as that in [62] to establish a Hopf bifurcation for RFDEs with symmetry under the condition that the imaginary eigenspace is isomorphic to the direct sum of two copies of the same absolutely irreducible representation. Guo and Lamb [54] developed the theory of equivariant Lyapunov-Schmidt procedure in NFDEs with symmetry to set up a more general equivariant Hopf bifurcation theory and obtained some important explicit formulas giving the relevant coefficients for the determinations of the monotone of the periods and Hopf bifurcation direction of the bifurcating symmetric periodic solutions directly in terms of the coefficients of the original equations.

Typically, branches of solutions bifurcate from the original equilibrium and are approximated to leading order by the corresponding eigenfunctions at singularities, these branches are often referred to as modes. Generically we expect, in a one-parameter system, to have only one critical mode. Multiple critical modes are expected in systems with more than one parameter. A secondary bifurcation is thought of as resulting from an interaction of several critical modes, called mode interaction. Since there are two types of critical modes (steady-state and Hopf) there may exist four types of mode interactions in two-parameter systems: (a) Bogdanov-Takens bifurcations, (b) fold/fold bifurcation, (c) Hopf/fold, (d) Hopf/Hopf. For example, the interaction of a fold bifurcation with a Hopf bifurcation can lead to much richer dynamics than just the expected equilibria and periodic solutions, including the possibility of an invariant 2-torus on which the flow may be periodic or quasi-periodic, see Gavrilov [39], Langford [89], Guckenheimer [43], Iooss & Langford [75]. As this torus grows fatter, generic perturbations can also lead to chaotic dynamics, see Holmes [67], Langford [90–92]. Recently, Guo and his coworkers obtained stable or unstable equilibria, periodic solutions, quasi-periodic solutions, and sphere-like surfaces of solutions in a two-coupled-neuron network [50], a ring network [47, 58], and a hierarchically organized network [56]. A general mathematical framework for a Bogdanov-Takens and a triple-zero (with geometric multiplicity one) bifurcations of a general class of DDEs has been developed by Campbell and Yuan [21] via a center manifold projection and derivation of the normal forms.

The Kaplan-Yorke's method [79, 80] has a clear advantage that it may change the problem of finding periodic solutions of DDEs into the problem of finding periodic solutions of associated Hamiltonian systems. This technique has been widely applied in the literature, we refer to [11, 64, 65, 80, 117, 120]. In particular, this method has been developed by Han [64] to study Hopf and saddle-node bifurcations of periodic solutions with certain periods.

The first results on bifurcation from periodic solutions in retarded FDEs dated back to a work by Walther [115], who considered the bifurcation from slowly oscillating periodic solutions of the following scalar retarded FDE

$$\frac{d}{dt}x(t) = -\alpha f(x(t-1)) \quad (6.1)$$

under some symmetry conditions on f . Dormayer [34] considered (6.1) with a class

of nonmonotone functions f , periodic solutions y with the more general symmetry: $y(\cdot + \tau) = -y$ for some $\tau > 0$, bifurcate from the primary branch at some critical parameter. Their initial values lie on a smooth curve, and $\tau \neq 2$ except at the bifurcation point. However, the relevant results in the aspect are rare. If an equivariant FDE has a nontrivial periodic solution $p(t)$ at some parameter value, with (minimal) period ω with symmetry Σ . Then it is natural to ask these questions: What happens to the periodic solution $p(t)$ when the parameter changes? Is there any kinds of saddle-node bifurcation, period-doubling bifurcation, and Hopf bifurcation? If yes, how to judge it? These seem to be open questions.

7. Discussion

In this paper we have presented a compact survey of the literature on bifurcation theory of FDEs. FDEs with symmetry have certainly received a lot of interest in recent years, and a lot of interesting results have been obtained. However, given the importance and relevance of FDEs, there is still a range of problems to be tackled. A theme throughout this survey has been the relation of dynamical systems to delay dynamical systems. The main task for the future seems to be bringing the bifurcation theory of FDEs to a similar maturity as that of ODEs.

On the other hand, the study of dynamical systems with symmetries has become well established as a major branch of nonlinear systems theory. The current interest in the field dates mainly back to the appearance of the *equivariant branching lemma* of Vanderbauwhede and Cicogna [26] and the *equivariant Hopf bifurcation theorem* of Golubitsky and Stewart [42], both of which are reviewed in the book by Golubitsky, Stewart and Schaeffer. Since then important new theories have been developed for more complex dynamical phenomena, including the existence, stability and bifurcations of systems of heteroclinic connections, and the symmetry groups and bifurcations of chaotic attractors. To a large extent the phenomenal growth in the subject has been due to its effectiveness in explaining the bifurcations and dynamical phenomena that are seen in a wide range of physical systems including coupled oscillators, reaction diffusion systems, convecting fluids and mechanical systems. A local symmetric bifurcation theory for FDEs can be derived from that of ODEs, but since some special properties of spatiotemporal symmetry of FDEs may be reflected generically in the reduced finite dimensional systems, one can and should make general observation about the particular bifurcation patterns of symmetric FDEs. Moreover, because the present theory for equivariant dynamical systems is powerful and successful, it seems most desirable to adopt an approach that smoothly connects to the theory for FDEs with symmetry. In order to achieve this, the introduction of a more systematic use of group (representation) theory for symmetry groups would be useful.

Other future directions of research might include the study of more general space-time symmetries of FDEs, and symmetry properties of partial FDEs that involve transformations of both the dependent and independent variables.

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