On two equimatchable graph classes

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Received 4 July 2001; received in revised form 26 March 2002; accepted 12 August 2002

Abstract

A graph $G$ is said to be equimatchable if every matching in $G$ extends to (i.e., is a subset of) a maximum matching. In this paper, we use the Gallai–Edmonds decomposition theory for matchings to determine the equimatchable members of two important graph classes. We find that there are precisely 23 3-connected planar graphs (i.e., 3-polytopes) which are equimatchable and that there are only two cubic equimatchable graphs.

Keywords: Matching; Equimatchable; Factor-critical; Randomly matchable

0. Introduction

Let $G$ denote a finite undirected graph with vertex set $V(G)$ and edge set $E(G)$. A set $M \subseteq E(G)$ is a matching in $G$ if the edges in the set $M$ are independent; i.e., no two edges share a common vertex. A matching $M$ is maximal if it is not a proper subset of any other matching and is maximum if, among all matchings in $G$, it is one of largest cardinality. A matching $M$ is called perfect if $V(M) = V(G)$ and near-perfect if $|V(M)| = |V(G)| - 1$. A graph $G$ is said to be factor-critical if $G - v$ has a perfect matching for every vertex $v \in V(G)$. A graph $G$ with a perfect matching is called randomly matchable if every matching in $G$ extends to (i.e., is a subset of) some perfect matching in $G$. Sumner [9] showed that the only such graphs are $K_{n,n}$.

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doi:10.1016/S0012-365X(02)00813-0
and $K_{2n}$ for $n \geq 1$. More generally, a graph $G$ is said to be equimatchable (see [3]) if every matching in $G$ extends to a maximum matching of $G$.

The concept of equimatchability (although not given this name) was first considered in 1974 independently by Meng [6] and Lewin [4] who gave different characterizations of equimatchable graphs. Neither of these characterizations, however, was a “good” characterization of this graph property in the now well-known sense of providing a polynomial algorithm for verifying membership and for verifying non-membership in the class. In [3] it was shown that membership in the class of equimatchable graphs can be polynomially determined. However, the proof of the latter result turned out to be quite technical in nature and gave little insight into the structure of equimatchable graphs in general. Favaron [1] studied those graphs which are simultaneously equimatchable and factor-critical and characterized those with vertex connectivity exactly one or two. She also proved that every equimatchable factor-critical 2-connected graph is Hamiltonian.

In the present paper, equimatchability in several well-known graph families is studied. In particular, those equimatchable cubic graphs and the equimatchable 3-connected planar (i.e., “3-polytopal”) graphs are determined. The Gallai–Edmonds decomposition theory for graphs in terms of their maximum matchings (cf. [5, Chap. 3]) is used extensively to obtain these results. Since it is clear that a graph is equimatchable if and only if each of its components is equimatchable, it will suffice to treat only connected graphs in what follows. (Nevertheless, we shall have occasion to refer to the number of components of a graph $G$ and we shall denote this quantity by $c(G)$.) Finally, we will write $x \sim y$ when vertex $x$ is adjacent to vertex $y$.

1. Equimatchable 3-connected planar graphs

In this section, we study the 3-connected planar graphs, sometimes called the 3-polytopal graphs. We begin with several lemmas.

**Lemma 1.1.** Let $G$ be a graph and suppose $x \in V(G)$. Let $F$ be a matching in $G$ with $x \notin V(F)$, but such that $N_G(x) \subseteq V(F)$. Let $G' = G - (V(F) \cup \{x\})$. Then a matching $M'$ in $G'$ is a maximal matching of $G'$ if and only if $M' \cup F$ is a maximal matching of $G$.

**Proof.** Suppose $M'$ is a maximal matching of $G'$. Then $V(G') - V(M')$ is an independent set of vertices in $G'$. Since $N_G(x) \subseteq V(F)$, $(V(G') - V(M')) \cup \{x\}$ is independent in $G$. But then since $(V(G') - V(M')) \cup \{x\} = V(G) - V(F \cup M')$, $F \cup M'$ is a maximal matching in $G$.

Conversely, suppose $M' \cup F$ is a maximal matching in $G$. Then $(V(G) - V(F \cup M')) = (V(G') - V(M')) \cup \{x\}$ is independent in $G$. So $V(G') - V(M')$ is an independent set in $G'$ and hence $M'$ is a maximal matching in $G'$. \(\square\)

At this point, let us recall some basic results from the theory of so-called Euler contributions. (See [8, pp. 348–349].) The reader is surely familiar with the simple
result that any connected plane graph has a vertex of degree at most five. Lebesgue showed that one can “fine tune” this result considerably by classifying the possible face sizes present at certain vertices of small degree. Suppose \( v \) is a vertex of degree \( n \) of a plane graph. If the faces at vertex \( v \) are arbitrarily numbered \( F_1, F_2, \ldots, F_n \), we say that vertex \( v \) is of type \( (x_1, x_2, \ldots, x_n) \) if face \( F_i \) has \( x_i \) vertices in its boundary.

Proof. Let \( G \) be a connected plane graph. We may also assume that \( G \) is 3-connected, a contradiction. Hence we may assume that \( x \) is a control vertex. Then \( G \) contains a matching \( F \) such that \( x \) is a control vertex of \( G \). Then \( G \) contains a matching \( F \) such that \( v \) is a control vertex of \( G \). Then \( G \) contains a matching \( F \) such that \( \Phi(v) \) denotes the degree of vertex \( v \). The following is then a simple result due essentially to Lebesgue [2].

Lemma 1.2. If \( G \) is a connected plane graph, then \( \sum v \Phi(v) = 2 \).

An immediate consequence of Lemma 1.2 is that in any plane graph there must exist vertices \( v \) having \( \Phi(v) > 0 \). Following the terminology of Plummer [8], we will call any such vertex a control vertex.

A second simple result is the following lemma.

Lemma 1.3. Let \( G \) be a connected plane graph in which each face has size at least 3. Then for all \( v \in V(G) \), \( \Phi(v) \leq 1 - \deg(v)/6 \).

It then follows immediately that any 3-connected plane graph has control vertices of degree 3, 4 or 5.

Lemma 1.4. Let \( G \) be a 3-connected plane graph of order at least 7 and let \( x \) be a control vertex of \( G \). Then \( G \) contains a matching \( F \) such that \( |F| \leq 3 \), \( x \not\in V(F) \), \( N_G(x) \subseteq V(F) \) and \( N_G(x) \cap V(e) \neq \emptyset \) for each \( e \in F \).

Proof. Since \( x \) is a control vertex, \( 3 \leq \deg(x) \leq 5 \). If \( \deg_G(x) = 4 \), then by the argument in the proof of Theorem 3.3 of Plummer [8], the desired matching \( F \) exists.

Suppose next that \( \deg_G(x) = 5 \). Then by the arguments in [8], vertex \( x \) is of type \( (3, 3, 3, 3, a) \) for some \( a = 3, 4, 5 \). Let \( N_G(x) = \{x_1, x_2, x_3, x_4, x_5\} \). (We may assume that \( x_1, x_2, x_3, x_4 \) and \( x_5 \) appear in clockwise consecutive order in the face of \( G-x \). We may also assume that \( x_1, x_2, x_3, x_4, x_5 \in E(G) \).

If \( N_G(x_1) \not\subseteq \{x, x_2, x_3, x_4, x_5\} \), choose \( y \in N_G(x_1) - \{x, x_2, x_3, x_4, x_5\} \) and then \( \{x_1, y, x_2, x_3, x_4, x_5\} \) suffices as our matching \( F \). Hence we may assume that \( N_G(x_1) \subseteq \{x, x_2, x_3, x_4, x_5\} \). Similarly, we may assume that \( N_G(x_5) \subseteq \{x, x_1, x_2, x_3, x_4\} \). If \( x_1 x_5 \not\in E(G) \), then \( \{x_3, x_4\} \cap N_G(x_1) \not\subseteq \emptyset \) and \( \{x_2, x_3\} \cap N_G(x_5) \not\subseteq \emptyset \), since \( \deg_G(x_1) \geq 3 \) and \( \deg_G(x_5) \geq 3 \). However, if \( x_4 \in N_G(x_1) \), then \( x_2 x_3, x_3 x_5 \not\in E(G) \) by planarity. Hence \( N_G(x_1) = \{x, x_2, x_3\} \). Similarly, \( N_G(x_5) = \{x, x_3, x_4\} \). But then \( \{x, x_3\} \) separates \( x_2 \) and \( x_4 \) and hence \( G \) is not 3-connected, a contradiction. Hence we may assume that \( x_1 x_5 \in E(G) \).
If \( N_G(x_i) \subseteq \{x_1,x_2,x_3,x_4,x_5\} \) for all \( i = 1,\ldots,5 \), then \( |V(G)| = 6 \), a contradiction. So \( N_G(x_i) \not\subseteq \{x_1,x_2,x_3,x_4,x_5\} \) for some \( i, 1 \leq i \leq 5 \), and by the same argument as before, \( G \) has a matching of the type required.

Finally, suppose \( \text{deg}_G(x) = 3 \); say \( N_G(x) = \{x_1,x_2,x_3\} \) without loss of generality. Choose \( y_1 \in N_G(x_1) - \{x\} \) so that \( x_1,y_1 \), \( x_2,y_2 \) and \( x_3,y_3 \) belong to three different facial cycles at \( x \). If \( \{x_1,y_1,x_2,y_2,x_3,y_3\} \) is independent, it is a matching of the type required. If not, without loss of generality we may assume that \( y_1 = x_2 \). If \( \{x_1,x_2,x_3,y_3\} \) is independent, it is a matching of the type required. If not, we may suppose that \( y_3 = x_1 \). By the same argument, we may also assume that \( y_2 = x_3 \). If \( N_G(x_i) \not\subseteq \{x_1,x_2,x_3\} \) for some \( i, 1 \leq i \leq 3 \), we may assume that \( i = 1 \). Choose \( y \in N_G(x_1) - \{x_1,x_2,x_3\} \). Then \( \{yx_1,x_2,x_3\} \) is a matching of the type required. Hence we may assume that \( N_G(x_i) \subseteq \{x,x_1,x_2,x_3\} \) for each \( i, 1 \leq i \leq 3 \). But then since \( G \) is connected, \( |V(G)| = 4 \), a contradiction. \( \square \)

**Lemma 1.5.** Let \( G \) be a factor-critical equimatchable 3-connected planar graph and choose \( x \in V(G) \). Let \( F \) be a matching with \( x \not\in V(F) \) and \( N_G(x) \subseteq V(F) \). Then each component of \( G - (V(F) \cup \{x\}) \), if any, is either \( K_2 \), \( K_4 \) or \( C_4 \).

**Proof.** Let \( G' = G - (V(F) \cup \{x\}) \) and suppose \( V(G') \neq \emptyset \). Let \( M' \) be a maximal matching of \( G' \). Then by Lemma 1.1, \( M' \cup F \) is a maximal matching of \( G \) which does not cover \( x \). Since \( G \) is factor-critical, all other vertices of \( G \) are covered by this matching and hence \( M' \) is a perfect matching of \( G' \). This implies that \( G' \) is randomly matchable. But then by planarity and Theorem 1 of Sumner [9], every component of \( G' \) is either \( K_2 \), \( K_4 \) or \( C_4 \). \( \square \)

**Lemma 1.6.** Let \( G \) be a 3-connected, planar, factor-critical and equimatchable graph. Let \( u \) be any vertex of \( G \) and let \( F \) be a matching such that \( u \not\in V(F) \), \( N_G(u) \subseteq V(F) \), \( N_G(u) \cap V(e) \neq \emptyset \) for each \( e \in F \) and \( |F| = 2 \) or 3. Let \( G' = G - V(F) - \{u\} \). Then, if \( G' \neq \emptyset \), \( G' \) is connected.

**Proof.** Assume that \( G' \neq \emptyset \) and that \( G' \) is not connected, so that \( G' \) consists of at least two components. Let \( X_1 \) and \( X_2 \) be two different components of \( G' \). Recall that by Lemma 1.5, each of \( X_1 \) and \( X_2 \) contains an even number of vertices (namely either two or four vertices each).

First we assert the following:

**Claim.** Let \( xy \) be one of the edges of \( F \). If \( (N_G(x) \cup N_G(y)) \cap (V(X_1) \cup V(X_2)) \neq \emptyset \), then (i) if \( N_G(x) \cap V(X_1) \neq \emptyset \), then \( N_G(y) \cap V(X_2) = \emptyset \) and (ii) if \( N_G(x) \cap V(X_2) \neq \emptyset \), then \( N_G(y) \cap V(X_1) = \emptyset \).

By symmetry it will suffice to prove (i). So suppose that \( N_G(x) \cap V(X_1) \neq \emptyset \) and \( N_G(y) \cap V(X_2) \neq \emptyset \). Choose \( x' \in N_G(x) \cap V(X_1) \) and \( y' \in N_G(y) \cap V(X_2) \). Then \( \{x',y'\} \cup (F - xy) \) cannot extend to a maximum matching in \( G \) because \( X_1 - \{x'\} \) and \( X_2 - \{y'\} \) are both odd components of \( G - \{u,x',y'\} - V(F) \). So \( G \) is not equimatchable, a contradiction, and (i) is proved and hence by symmetry so is the Claim.
We now consider two cases:

**Case 1:** Suppose \(|E(F)| = 2\). Say \(F = \{x_1 y_1, x_2 y_2\}\).

Then since \(G\) is 3-connected, \(|N_G(X_1) \cap V(F)| \geq 3\). Thus we may assume without loss of generality that \(\{x_1, y_1, x_2, y_2\} \subseteq N_G(X_1) \cap V(F)\). But by the Claim, \(\{x_1, y_1, y_2\} \cap N_G(X_2) = \emptyset\). Hence \(x_2\) is a cut vertex of \(G\) contradicting the fact that \(G\) is 3-connected.

**Case 2:** So suppose \(|E(F)| = 3\). Say \(F = \{x_1 y_1, x_2 y_2, x_3 y_3\}\).

We consider two subcases.

**Case 2.1:** Suppose \(\{x_1, x_2, x_3\} \subseteq N_G(X_1)\). Then by the Claim, \(\{y_1, y_2, y_3\} \cap N_G(X_2) = \emptyset\). But since \(G\) is 3-connected, \(|N_G(X_2) \cap V(F)| > 3\), and hence \(N_G(X_2) \cap V(F) = \{x_1, x_2, x_3\}\). But then if one contracts each of \(x_1 y_1, x_2 y_2, x_3 y_3, x_1\) and \(x_2\), the five resulting vertices, together with vertex \(u\), form the six vertices of a \(K_{3,3}\)-minor of \(G\), a contradiction to the fact that \(G\) is planar.

**Case 2.2:** Suppose \(\{x_1, y_1, x_2\} \subseteq N_G(X_1) \cap V(F)\). Then by the Claim, \(\{x_1, y_1, y_2\} \cap N_G(X_2) = \emptyset\) and so again since \(G\) is 3-connected, \(N_G(X_2) \cap V(F) = \{x_2, x_3, y_3\}\). But then \(y_2 \notin N_G(X_1) \cap V(F)\) and \(y_2 \notin N_G(X_2) \cap V(F)\).

Since \(\deg_{G}(y_2) \geq 3\), \((V(F) - \{x_2, y_2\}) \cap N_G(y_2) \neq \emptyset\). Without loss of generality, we may assume that \(y_1, y_2 \in E(G)\). Let \(x' \in X_1\) be adjacent to \(x_1\) and let \(y' \in X_2\) be adjacent to \(x_2\). Then the matching \(\{x_1 x', y_1 y_2, x_2 y', x_3 y_3\}\) cannot extend to a maximum matching, for both \(X_1 - x'\) and \(X_2 - y'\) contain an odd number of vertices. \(\square\)

**Corollary 1.7.** If \(G\) is a 3-connected, planar, factor-critical, equimatchable graph, then \(|V(G)| \leq 11\).

**Proof.** Suppose \(|V(G)| > 11\). Graph \(G\) has a control vertex \(u\) and, by Lemma 1.4, a matching \(F\) containing \(N_G(u)\), but not \(u\), where \(|E(F)| = 2\) or \(3\), \(N_G(u) \cap V(e) \neq \emptyset\) for each \(e \in F\) and each component of \(G' = G - V(F) - \{u\}\) has either 2 or 4 vertices by Lemma 1.5. But by Lemma 1.6, if the graph \(G'\) is not empty, it is connected and hence by Lemma 1.5 it is either \(K_2\), \(K_4\) or \(C_4\). The Corollary follows. \(\square\)

**Lemma 1.8.** If \(G\) is 3-connected, planar, and equimatchable, then

(i) if \(G\) contains a perfect matching, \(G \cong K_4\), while

(ii) if \(G\) does not contain a perfect matching, \(G\) is factor-critical and hence \(|V(G)| = 5, 7, 9\) or 11.

**Proof.** Let \(G\) be 3-connected planar and equimatchable. If \(G\) contains a perfect matching, then \(G\) is randomly matchable and hence by Theorem 1 of Sumner [9] and planarity, \(G\) must be \(K_4\).

Now suppose \(G\) does not contain a perfect matching. Following the notation of Lovász and Plummer [5, Chap. 3], let \(\{D, A, C\}\) denote the Gallai–Edmonds decomposition of graph \(G\). That is to say, let \(D\) be the set of all vertices of \(G\) which are left unmatched by at least one maximum matching of \(G\), let \(A\) be the neighbors of vertices in \(D\) which are not themselves in \(D\), and let \(C = V(G) - D - A\). Then by the Gallai–Edmonds theorem, each component of \(D\) is factor-critical. Moreover, by Lemma 1 of Lesk et al. [3], \(C = \emptyset\) and \(A\) is independent.
First, suppose that $A \neq \emptyset$. Consider any (necessarily odd) component of $D$, say $D_i$. If $|V(D_i)| \geq 3$, then since $G$ is 3-connected, there must be at least two independent edges joining $D_i$ to $A$. But by Theorem 3.2.8 of Lovász and Plummer [5], part (4), this is impossible. So $D_i$ is a singleton for all $i$ and hence $G$ is bipartite.

Choose a vertex $v \in A$. Now since $G$ is 3-connected and planar, there must exist a “wheel” at $v$; that is, the vertices other than $v$ itself which lie on the union of the boundaries of the faces at $v$ form a cycle which we shall denote by $C_v$. But then $C_v$ is even, since $G$ is bipartite, and so let $M_v$ be a matching consisting of every second edge in $C_v$, taken, say, clockwise. Then $M_v$ extends to a maximal matching $M$ which, in turn, must be maximum, since $G$ is equimatchable. But $M$ cannot cover $v$, contradicting the fact that every vertex in $A$ must be matched by every maximum matching in $G$.

So $A = \emptyset$ and hence $G = D$ and $G$ is factor critical (and hence odd).

**Corollary 1.9.** If $G$ is 3-connected planar and equimatchable, then either $G \cong K_4$ or $|V(G)| = 5, 7, 9$ or 11.

**Proof.** Immediate via Corollary 1.7 and Lemma 1.8.

The next result will reduce considerably the number of cases which must be checked in our final theorem.

**Lemma 1.10.** If $G$ is a 3-connected equimatchable plane graph and contains a subgraph isomorphic to $K_4$, then $|V(G)| \leq 7$.

**Proof.** Suppose $|V(G)| > 7$. Hence $G \not\cong K_4$ and hence by Corollary 1.9, $|V(G)| = 9$ or 11. Suppose $G$ contains a subgraph $H$ isomorphic to $K_4$.

**Case 1:** Suppose $H$ has a vertex of degree 3. Label the vertices of $H$ by $a, b, c, d$ such that $\deg(d) = 3$. Since $|V(G)| \geq 9$ and $G$ is 3-connected, there exists an edge joining a vertex of triangle $abc$ to a fifth vertex $e$ in the region bounded by $abc$. Without loss of generality, suppose $e \sim a$. Let $F = \{bc, ae\}$. Then if $G' = G - V(F) - \{d\}$, by Lemmas 1.5 and 1.6 we have that $G' \cong K_2, K_4$ or $C_4$. Moreover, since $|V(G)| \geq 9$, $G' \not\cong K_2$ and hence $|V(G)| = 9$.

**Case 1.1:** Suppose $G' \cong K_4$. Then $G'$ lies in the region bounded by $abc$ (along with vertex $e$). Label the vertices of $G'$ with $w, x, y, z$ in such a way that $xyz$ separates vertex $w$ from $H$. Now by 3-connectivity and Menger’s Theorem, there are three internally disjoint paths joining $d$ and $w$. Moreover, since $|V(G)| = 9$, each of these three paths must consist of a single edge. Thus there is a matching of $\{x, y, z\}$ into $\{a, b, c, e\}$. So by symmetry, relabel $V(G')$, if necessary, so that $x \sim b$.

But then by the planarity of $G$, $M = \{ac, bx, yz\}$ is a matching which does not extend to a near-perfect matching, contradicting Lemma 1.8 (ii).

**Case 1.2:** So we may assume that $G' \cong C_4$. Then, as before, $G'$ lies in the interior of region $abc$ along with vertex $e$. Again, by 3-connectivity and Menger’s Theorem, there exists a matching $M$ of size 3 from $V(G')$ into $\{a, b, c, e\}$.
Case 1.2.1: Suppose \( M \) matches \( \{a, b, c\} \) into \( V(G') \); say \( V(G') \) is labeled such that \( M = \{ax, by, ez\} \) and \( w \) is the fourth vertex of \( V(G') \). But then \( M' = \{ae, by, cz\} \) does not extend.

Case 1.2.2: Suppose \( M \) matches \( \{a, b, e\} \) into \( V(G') \); say via edges \( ax, by \) and \( ez \). Then \( M' = \{ac, by, ez\} \) does not extend.

Case 1.2.3: Suppose \( M \) matches \( \{b, c, e\} \) into \( V(G') \); say via \( bx, cy \) and \( ez \). Then \( M' = \{ae, bx, cy\} \) does not extend.

Case 2: So we may assume that \( G \) contains a subgraph \( H \cong K_4 \), but that every such \( K_4 \) subgraph has the property that all its vertices have degree at least 4. Choose any such \( H = K_4 \) and label \( V(H) = \{a, b, c, d\} \). If any of the four regions determined by the embedding of the subgraph \( H \) contains exactly one interior vertex, then by 3-connectivity there must be a \( K_4 \) with a vertex of degree 3 and we are back in Case 1.

Case 2.1: Suppose that one of the four regions determined by \( H \), say \( bcd \), contains exactly two vertices of \( G \) in its interior. Call these two vertices \( e \) and \( f \).

Case 2.1.1: Suppose \( \deg(e) = 3 \).

Case 2.1.1.1: Suppose that \( e \sim b, c \) and \( d \). Then without loss of generality, suppose \( f \) is interior to region \( bde \). Then \( G \) must contain a \( K_4 \) containing vertex \( f \). But by 3-connectivity, \( e \sim f \) and hence \( \deg(e) \geq 4 \), a contradiction.

Case 2.1.1.2: Suppose \( e \sim b \). Thus \( e \sim c, d \) and \( f \). Now \( f \) is not interior to region \( cde \) or else we are back in Case 1. So \( f \) is interior to the region bounded by \( bcd \).

Now if \( F = \{bd, ce\} \), then by Lemmas 1.5 and 1.6, \( G' = G - V(F) - \{f\} = K_2, \ K_4 \) or \( C_4 \). Since \( |V(G)| \geq 9 \), \( G' \not\cong K_2, \ K_4, \) or \( C_4 \). But then \( |V(G)| = 9 \) and \( a \in V(G') \). If \( G' \cong K_4 \), then one of its vertices must have degree 3 and again we are back in Case 1. So we may assume that \( G' \cong C_4 \).

Suppose \( G' \) is interior to region \( acd \). Then label \( V(G') \) by \( axyz \) in clockwise order. Now \( y \sim a \), so again by 3-connectivity, \( y \sim c \) or \( y \sim d \). If \( y \sim c \), then \( M = \{yc, ab, ce\} \) does not extend; while if \( y \sim d \), then \( M = \{yd, ab, ce\} \) does not extend.

Next, suppose \( G' \) is interior to region \( abd \). Then again label \( V(G') \) as \( axyz \) in clockwise order. Now \( y \sim a \), so \( y \sim b \) or \( y \sim d \). But if \( y \sim b \), \( M = \{by, ad, cd\} \) does not extend; while if \( y \sim d \), \( M = \{ab, dy, ce\} \) does not extend.

So we may suppose that \( G' \) is interior to region \( abc \). Once again let \( V(G') \) be labeled \( axyz \) in clockwise order. As before, \( y \sim a \), so by 3-connectivity, \( y \sim b \) or \( y \sim c \). But if \( y \sim b \), then \( M = \{ad, by, ce\} \) does not extend and if \( y \sim c \), then \( M = \{de, ab, cy\} \) does not extend.

Case 2.1.1.3: So we may suppose that \( e \sim b \) and hence by symmetry, also that \( e \sim c \). Suppose the third neighbor of \( e \) is \( f \). If \( f \) is interior to region \( bce \), then again we have a \( K_4 \) with a vertex of degree 3 and we are back to Case 1. So we may suppose that \( f \) is interior to region \( bcd \). Now if \( f \sim b \) and \( f \sim c \), again we get a \( K_4 \) with a vertex of degree 3 and we are in Case 1. So either \( f \sim b \) or \( f \sim c \); without loss of generality, assume that \( f \sim b \). Thus \( f \sim c \) and \( d \) by 3-connectivity. Letting \( F = \{be, cd\} \), by Lemmas 1.5 and 1.6 we have \( G' = G - V(F) - \{f\} = K_2, \ K_4 \) or \( C_4 \). Once again since \( |V(G)| \geq 9 \), \( G' \not\cong K_2, \ K_4, \) or \( |V(G)| = 9 \) and \( a \in V(G') \). Now if \( G' \cong K_4 \), then by planarity, \( G' \) contains a vertex of degree 3 and we are back to Case 1. So \( G' \cong C_4 \).
If $G'$ is interior to region $acd$, then if $M = \{ad, ce\}$, $M$ does not extend. Suppose $G'$ is interior to region $abd$. Let $G'$ be denoted by $axyz$ in clockwise order. Now $y \sim a$ and so again by 3-connectivity, $y \sim b$ or $d$. But if $y \sim b$ then $M = \{ad, by, ce\}$ does not extend, while if $y \sim d$, $M = \{ab, cf, dy\}$ does not extend.

Hence $G'$ is interior to region $abc$. Again, let $G'$ be denoted by $axyz$ in clockwise order. By 3-connectivity, $y \sim b$ or $y \sim c$. But if $y \sim b$, $M = \{ad, by, ce\}$ does not extend, while if $y \sim c$, $M = \{ab, cy, ef\}$ does not extend.

Case 2.1.2: So we may suppose that $\deg(e) = 4$, and, by symmetry, also that $\deg(f) = 4$. But this is impossible by planarity.

So we may assume that none of the four regions of $abcd$ contains exactly one or exactly two vertices in its interior. So since $|V(G)| \geq 9$, some region contains at least three internal vertices.

Case 2.2: Suppose one of these four regions, without loss of generality, say $bcd$, contains exactly three vertices. Denote these vertices by $x$, $y$ and $z$. Then by 3-connectivity and Menger’s Theorem, we may suppose, without loss of generality, that $b \sim x$, $c \sim y$ and $d \sim z$. Then if $F = \{ab, cy, dz\}$, $G' = G - F - \{x\} \cong K_2$, $K_4$ or $C_4$. If $G' \cong K_2$, then one of the regions of $abcd$ contains exactly two vertices, a case already treated above. So we may assume that $G' \cong K_4$ or $C_4$. But as before, if $G \cong K_4$, then $G'$ contains a vertex of degree 3 by planarity, and we are once again in Case 1. So we may assume that $G' \cong C_4$ and hence $|V(G)| = 11$.

Case 2.2.1: Suppose $G'$ is interior to $abd$. By 3-connectivity and Menger’s Theorem, there is a matching of the form $ax', by', dz'$ where $\{x', y', z'\} \subseteq V(G')$. Let $w'$ denote the fourth vertex of $V(G')$. If $x'$ and $y'$ separate $w'$ and $z'$ on $G' = C_4$, then $M = \{ax', by', cy, dz\}$ does not extend. So we may assume that $x'$ and $y'$ are consecutive on $G'$ in clockwise order. But then if $V(G') = \{x', y', z', w'\}$ in clockwise order, $M = \{ax', bx, cy, dz'\}$ does not extend, while if $V(G') = \{x', y', w', z'\}$ in clockwise order, $M = \{ac, by', dz'\}$ does not extend.

So we may assume that $G'$ is not interior to $abd$ and also, by symmetry, that $G'$ is not interior to $acd$.

Case 2.2.2: Thus we may assume that $G'$ is interior to region $abc$. Still again by 3-connectivity and planarity, we may suppose there is a matching of $\{a, b, c\}$ into $V(G')$. Label $V(G')$ in such a way that $a \sim x'$ and $b \sim y'$.

Case 2.2.2.1: First suppose that $y' \sim x'$ on cycle $G'$. Then let the remaining two vertices of $G'$ be $z'$ and $w'$ where $z' \sim x'$. Now $M = \{x'z', by', cd\}$ must extend to a near-perfect matching, so this implies by planarity that $a \sim w'$. But then $\{w', x'\}$ is a 2-cut in $G$, a contradiction.

Case 2.2.2.2: So finally suppose that $x' \sim y'$ on $G'$. But then $M = \{ax', by', cy, dz\}$ does not extend, a contradiction.

Thus we may assume that each of the four regions determined by the $K_4$ subgraph on vertices $a$, $b$, $c$ and $d$ contains either no interior vertices or at least four interior vertices. But since $|V(G)| = 9$ or 11, this means that exactly one region contains five vertices (in the case when $|V(G)| = 9$) or exactly one region contains seven vertices (in the case when $|V(G)| = 11$). But in either case, the $K_4$ subgraph on the vertices $a$, $b$, $c$ and $d$ must have a vertex of degree 3 and by Case 1, the proof is complete. □
We are now prepared for our final result.

**Theorem 1.11.** If $G$ is 3-connected, planar and equimatchable, then $|V(G)| \leq 9$ and must be one of the following 23 graphs all shown in Fig. 1:

(a) If $|V(G)| = 4$, then $G = K_4$;
(b) if $|V(G)| = 5$, then $G$ is one of the two graphs shown;
(c) if $|V(G)| = 7$, then $G$ is one of $G_1, \ldots, G_{19}$; and
(d) if $|V(G)| = 9$, then $G = G_{30}$.

**Proof.** As a complete proof is both long and tedious, we will give only an outline.

One can quickly check that both (a) and (b) are true.

We turn next to (c). Suppose first that $|V(G)| = 7$ and $G$ contains no vertex of degree 6. Suppose now that $G$ contains a vertex $x$ of degree 5. Since $|V(G)| = 7$, $x$ is of type $(3,3,3,3)$ or $(3,3,3,4)$. Suppose first that $x$ is of type $(3,3,3,3)$. Then using Menger’s Theorem, planarity and equimatchability, one arrives at the conclusion that $G$ must be one of the (non-isomorphic) graphs $G_1, \ldots, G_6$ see in Fig. 1. Then assuming that $x$ is of type $(3,3,3,3,4)$, one similarly arrives at the fact that $G$ must be $G_7$ or $G_8$.

Now suppose that $|V(G)| = 9$ and suppose that $x$ has degree 4. Then since $|V(G)| = 7$, at least two faces at $x$ must be triangular. In fact, vertex $x$ must be one of the types $(3,3,3,3,3,3,3,4)$, $(3,3,3,4,3,3,3,4)$, $(3,4,4,3,3,3,4)$ or $(3,3,3,5)$. Then using 3-connectivity and equimatchability, as well as Lemmas 1.5 and 1.8, we find that $G$ must be one of $G_9, \ldots, G_{15}$.

Finally, assuming that $G$ contains a vertex of degree 6, it is easy to see that $G$ must be one of $G_{16}, \ldots, G_{19}$.

Now suppose $|V(G)| = 11$. It is known that every 3-connected planar graph must contain a vertex of one of the types shown in Table 1 of Ore and Plummer [7]. (NB: The reader should recall that the solutions presented in Table 1 are just the “maximal” ones. That is, for example, if vertex type $(a,b,c)$ is a solution found in Table 1 and if $d' \leq a, \ b' \leq b$ and $c' \leq c$, then type $(d',b',c')$ is also a solution.) Furthermore, by Lemma 1.10, $G$ does not contain a vertex of type $(3,3,3)$. Then using Lemmas 1.5 and 1.6, we proceed through the list of types in Table 1 (in the order listed there) to find that the only type to yield an equimatchable graph is $(4,4,4)$. We give some additional details for this case.

Suppose $x$ is of type $(4,4,4)$ and suppose that the hexagon surrounding $x$ is labeled $v_1, \ldots, v_6$ (clockwise) where $x$ is adjacent to vertices $v_1, v_3$ and $v_5$. Note that $\{v_1, v_3, v_5\}$ is an independent set since $G$ is 3-connected and planar. Note also that if $v_2$ is adjacent to $v_6$, then vertex $v_1$ is of type $(3,4,4)$ and this type has already been considered and found to yield no 3-connected planar equimatchable graph on nine or eleven vertices. So we may assume that $v_2$ is not adjacent to $v_6$ and hence by symmetry, $\{v_2, v_4, v_6\}$ is an independent set also.

Letting $M = \{v_1 v_6, v_2 v_3, v_4 v_5\}$, we see by Lemmas 1.5 and 1.6 that $G - V(M) - \{x\}$ must be either $K_2$ or a $C_4$. Assume first that $|V(G)| = 9$ and hence that $G - V(M) - \{x\} = K_2$. Label the $K_2$ as $v_7 v_8$. Suppose $v_1$ is adjacent to $v_7$. Then $M' = \{v_1 v_7, v_3 v_4, v_5 v_6\}$
implies that $v_2$ is adjacent to $v_8$ and then by symmetry, $v_6$ is adjacent to $v_8$ also. Now $\deg(v_7) \geq 3$, so we may assume, without loss of generality, that $v_7$ is adjacent to vertex $v_2$. Then if $\deg(v_7) = 3$, vertex $v_7$ must be of type $(3, 3, 4)$ which we have already treated.
in our sequence of types in Table 1. So we may assume that vertex \( v_7 \) is adjacent to \( v_6 \) also. Now since \( \deg(v_4) \geq 3 \), vertices \( v_4 \) and \( v_8 \) must also be adjacent and hence \( \deg(v_4) = 3 \). Suppose \( v_3 \) is adjacent to \( v_8 \). Then \( v_4 \) is of type \((3,3,4)\) or \((3,4,4)\) and both of these types have been previously considered in our Table 1 list. So we may assume that \( v_3 \) is not adjacent to \( v_8 \) and by symmetry, \( v_5 \) is not adjacent to \( v_8 \) also.

But then \( G = G_{20} \) in Fig. 1.

One then proceeds through the remaining cases where \( x \) is of degree 4. Finally, a simple computation shows that there is no 3-connected, planar graph on 9 or 11 vertices having minimum degree 5.

\[ \square \]

2. Equimatchable cubic graphs

Using the Gallai–Edmonds decomposition theorem, one can easily characterize the equimatchable cubic graphs.

**Theorem 2.1.** If \( G \) is a connected cubic equimatchable graph, then \( G \) is either isomorphic to \( K_4 \) or to \( K_{3,3} \).

**Proof.** If \( G \) contains a perfect matching, it is randomly matchable and hence by Theorem 1 of Sumner [9], since \( G \) is cubic, it is isomorphic to either \( K_4 \) or to \( K_{3,3} \).

So suppose \( G \) does not contain a perfect matching and let \( \{D, A, C\} \) again denote the Gallai–Edmonds decomposition of \( V(G) \) as already described in Section 1. Then by Lemma 1 of Lesk et al. [3], \( C = \emptyset \) and \( A \) is independent. Moreover, since \( G \) contains no perfect matching, \( c(D) \geq |A| + 1 \), and hence by parity, \( c(D) \geq |A| + 2 \). Moreover, each component of \( D \) is factor-critical and hence odd.

If \( A = \emptyset \), then \( G \) is disconnected, a contradiction. So \( A \neq \emptyset \) and it follows that \( D \) has at least three components. Since there are 3\(|A|\) edges between \( A \) and the \(( \geq |A| + 2 \) odd components of \( D \), there must be some component \( D_1 \) of \( D \) which receives no more than two edges from \( A \). But then by parity, \( D_1 \) cannot receive an even number of edges from \( A \), and so \( D_1 \) must receive exactly one edge from \( A \). (Moreover, since \( G \) is cubic, it also follows that \([V(D_1)] \geq 3\).) Denote by \( v \) the unique neighbor vertex of \( D_1 \) in \( A \) and suppose \( d_1 \) is the vertex of \( D_1 \) adjacent to \( v \). Now \( v \) is adjacent to no more than two other components of \( D \).

Suppose first that \( v \) is adjacent to exactly one component \( D_2 \neq D_1 \). So there exist two distinct vertices \( d_2 \) and \( d_2' \) in \( V(D_2) \) such that \( v \) is adjacent to each. Now since \( G \) is cubic and \( D_2 \) is odd, by parity there must be at least one edge from \( D_2 \) to \( A - \{v\} \). Denote such an edge by \( v'd_2 \), where \( v' \in A - \{v\} \) and \( d_2 \in V(D_2) \). Let \( M_2 \) be a perfect matching of \( D_2 - d_2 \) and let \( M_1 \) be a perfect matching of \( D_1 - d_1' \), where \( d_1' \) is any vertex of \( D_1 \) different from \( d_1 \). Then \( M_1 \cup M_2 \cup \{v'd_2\} \) is a matching in \( G \) which therefore must extend to a maximum matching \( M \) of \( G \). But \( v \) is not covered by \( M \), a contradiction of the fact that every maximum matching of \( G \) covers every vertex of \( A \).

So we may suppose that \( v \) is adjacent to exactly two other components of \( D \) which we shall denote by \( D_2 \) and \( D_3 \). For \( i = 1, 2, 3 \), let \( d_i \) denote the neighbor of \( v \) in component \( D_i \).
Case 1: Suppose that $|V(D_2)| \geq 3$ and $|V(D_3)| \geq 3$. For $i = 1, 2, 3$, choose a second vertex $d'_i \in V(D_i)$ such that $d'_i \neq d_i$. Since each component $D_i$ is factor-critical, choose a perfect matching $M'_i$ of $D_i - d'_i$, for $i = 1, 2, 3$. Then $M' = M'_1 \cup M'_2 \cup M'_3$ is a matching in $G$ which covers $N(v)$, but not vertex $v$. Thus $M'$ extends to a maximum matching $M$ of $G$ which does not cover vertex $v$, contradicting the assumption that $v \in A$.

Case 2: So suppose at least one of $D_2$ and $D_3$ consists of a single vertex; say without loss of generality that $V(D_2) = \{d_2\}$.

Suppose first that $|V(D_3)| \geq 3$. Choose a vertex $a \in N(d_2)$ such that $a \neq v$. For $i = 1, 3$, choose $d'_i \neq d_i$ in $D_i$ and choose a perfect matching $M'_i$ of $D_i - d'_i$. Then $M' = M'_1 \cup M'_3 \cup \{ad_2\}$ is a matching in $G$ covering $N(v)$, but not $v$. So $M'$ extends to a maximum matching $M$ of $G$ which does not cover vertex $v$, again contradicting the assumption that $v \in A$.

So finally assume that $|V(D_2)| = |V(D_3)| = 1$. Then since $G$ is cubic, there exist distinct vertices $a_2$ and $a_3$ in $G$ such that $a_2 \neq v \neq a_3$, $d_2$ is adjacent to $a_2$ and $d_3$ is adjacent to $a_3$. Choose $d'_1 \neq d_1$, $d'_1 \in V(D_1)$ and let $M'_1$ be a perfect matching of $D_1 - d'_1$. Then $M' \cup \{d_2a_2, d_3a_3\}$ is a matching in $G$ which covers $N(v)$, but not $v$, and so $M'$ extends to a maximum matching of $G$ which does not cover vertex $v$, again a contradiction.

Thus there is no connected, cubic, equimatchable graph without a perfect matching and the theorem is proved. \(\Box\)

Acknowledgements

The authors wish to thank one of the referees for spotting a flaw in our original proof of Lemma 1.10.

References