Parameter Estimation Of The Diagonal Of The Modified Riesz Distribution

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Abstract—In this paper, we introduce the diagonal of the modified Riesz distribution defined in \( \mathbb{R}^r, r \geq 2 \). We propose the estimators of different parameters of this new distribution by two method, firstly by the moment method and secondly by the maximum likelihood method. The performance of the both estimators are studied by calculating the Mean Square Error (MSE).

Keywords—Diagonal of the modified Riesz distribution, Moment Method, Maximum Likelihood Method, Mean Square Error.

I. INTRODUCTION

Under the approach based on the theory of Jordan algebras, a family of distributions on symmetric cones, termed the Riesz distributions, was first introduced by Hassairi and Lajmi [8] under the name of Riesz natural exponential family (Riesz NEF); it was based on a special case of the so-called Riesz measure from Faraut and Koranyi [5], going back to Riesz [15]. This Riesz distribution generalizes the matrix multivariate Gamma and Wishart distributions, containing them as particular cases [11].

In this paper, we introduce the form of the diagonal of modified Riesz distribution. In a particular case, these distributions are reduced to multivariate Gamma distribution. In the literature, the multivariate gamma distributions on \( \mathbb{R}^r \) have several non-equivalent definitions. Many authors require only that the marginal distributions are ordinary gamma distributions [3]. However, the family of distributions satisfying this condition is very large and in order to reduce the size of the family of multivariate gamma distributions, Moran and Vere-Jones [13], Griffiths [6] and Bapat [14] define the multivariate gamma distributions by the form of their characteristic functions their Laplace transforms defined recently by Bernardoff [3] using the notion of the affine polynomial. Louati [11] describe the class of the generalized multivariate gamma distributions and study the statistical model obtained by the mixture of this distribution with the Riesz one on the space of symmetric matrices.

The gamma distribution has been used in different domains applications such as engineering, business, image segmentation and signal processing.

The present paper is structured as follows. Section II presents some definitions of the Riesz distribution. Section III recalls some important results on diagonal of the modified Riesz distribution with different case. Section IV studies estimators of the unknown parameters of a bivariate diagonal of the modified Riesz distribution. These estimators are based on the classical maximum likelihood method and method of moments. Simulation results illustrating the performance of both estimators are presented in Section V. Conclusion is finally reported in section VI.

II. RIEZ DISTRIBUTION

In this section, we recall the basic results of Riesz families [1] [5] [8] [9] [11].

Let \( E \) be the Euclidean space of \( (r,r) \) real symmetric matrices equipped with the scalar product \( \langle x,y \rangle = tr(\Sigma y) \) and \( \Omega \) denotes the cone of \( (r,r) \)-symmetric positive definite matrices. For \( x = (x_{ij})_{i \leq j \leq r} \) in \( E \) and \( 1 \leq k \leq r \), we define the sub-matrices \( P_k(x) = (x_{i \leq j \leq k}) \) and \( \Delta_k(x) \) denote the determinant of the \((k,k)\) matrix \( P_k(x) \). Then the generalized power of \( x \) in the cone \( \Omega \) of positive definite elements of \( E \) is defined, for \( \mathbf{s} = (s_1, s_2, ..., s_r) \in \mathbb{R}^r \), by

\[
\Delta_s(x) = \Delta_{s_1}(x)^{s_1} \Delta_{s_2}(x)^{s_2} \cdots \Delta_{s_r}(x)^{s_r}.
\]

For \( \mathbf{s} = (s_1, s_2, ..., s_r) \) satisfying the conditions \( s_i > \frac{i-1}{2} \), the absolutely continuous Riesz measure is defined by

\[
L_{\mathbf{s}}(dx) = \frac{\Delta_{-\mathbf{s}}(x)}{\Gamma_{\mathbf{s}}(s)} \Delta_{\mathbf{s}}(x)(dx),
\]
where \( n = \frac{r(r+1)}{2} \) is the dimension of \( E \) and 
\[
\Gamma_\Omega(p) = (2\pi)^{\frac{r}{2}} \prod_{i=1}^{r} \Gamma(s_j - \frac{1}{2}(i-1)).
\]
A result due to [7] says that for all \( \theta \in -\Omega \), this Laplace transform is defined by 
\[
L_\Omega(\theta) = \int_\mathbb{E} e^{\theta^T r} R_i(dx) = \Delta_i((-\theta^{-1})),
\]
and for \( s \) satisfying the conditions \( s_j > \frac{i-1}{2} \), the Riesz distribution is given by 
\[
R(s, \sigma)(dx) = \frac{e^{-\sigma s x} \Delta_i\left(\frac{s_j}{2}\right)}{\Gamma_\Omega(s)\Delta_i\left(\sigma^{-1}\right)} l_\Omega(x)(dx).
\]
When \( s_1 = s_2 = \ldots = s_r = p > \frac{r-1}{2} \), \( R(s, \sigma) \) reduces to the Wishart distribution 
\[
W(p, \sigma) = \frac{1}{\Gamma_\Omega(p)\det(\sigma^{-r})} e^{-\sigma s x} \det(x)^{p-\frac{n}{2}} l_\Omega(x).
\]

III. A PROPOSED MODEL OF THE DIAGONAL OF THE RIESZ DISTRIBUTION

In this section, we introduce a model of the diagonal of the modified Riesz distribution. The model is based to Laplace transform of Wishart distribution. In our work, we study the case of \( r = 2 \).

Let \( X \) be a random Riesz \((2,2)\)-matrix with parameters 
\[
s = (s_1, s_2) \in [0, +\infty] \times [0, +\infty] \quad \text{and} \quad \theta \in I_2 - \Omega,
\]
where \( I_2 \) is the identity matrix of order 2. Then, according to equation (1), the Laplace transform of \( X \) is defined by

\[
L(\theta) = \int_\mathbb{E} e^{\theta^T r} R_i(dx) = \Delta_i((-I_2 - \theta)^{-1}).
\]

To obtain our model, we use an affine transformation to the random matrix \( X \) given by \( \tilde{X} = \Sigma X \) where 
\[
\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix},
\]
\( \Sigma_{11} > 0, \Sigma_{22} > 0 \) and \( \Sigma_{12} \Sigma_{21} - \Sigma_{11} \Sigma_{22} > 0 \).

The proposed model of the diagonal of the modified Riesz is given by 
\( Y = (Y_1, Y_2) = \text{diag}(\tilde{X}) \) (the diagonal elements of the matrix \( \tilde{X} \)).

To define the Laplace transform \( L(\theta) \) of \( Y \), we introduce the following proposition.

**Proposition III.1.** Let \( s = (s_1, s_2) \in [0, +\infty] \times [0, +\infty] \) and \( \theta \in I_2 - \Omega \), then

\[
L(\theta) = [(1 - \theta_{1,1} \Sigma_{11})(1 - \theta_{2,2} \Sigma_{22}) - \theta_{1,2} \Sigma_{12}^2 - \theta_{2,1} \Sigma_{21}^2]^{s_2} \times [1 - \theta_{1,2} \Sigma_{12}]^{(s_2)^{-1}}.
\]

**Proof**
Firstly, let \( Y = \text{diag}(\tilde{X}) \) be a random vector of \( IR^2 \) whose elements are the diagonal elements of the matrix \( \tilde{X} \). Then, for all \( \theta \in I_2 - \Omega \), we have

\[
E(e^{\theta^T(Y_1, Y_2)}) = E(e^{\theta^T(r \tilde{X})}) = E(e^{x^T(\tilde{X}^T \Sigma \tilde{X})}) = L(\theta \Sigma).
\]

Secondly, if \( \theta_{ij} = 0 \), for \( i \neq j \), then

\[
L(\theta) = E(e^{\theta^T r}) = L_\Sigma(\text{Diag}(\theta \Sigma) = [1 - \theta_{1,2} \Sigma_{12}]^{s_2} \times [1 - \theta_{1,2} \Sigma_{12}]^{s_2^{-1}}.
\]

In the remainder of this section, we will define the distribution of \( Y \) in both case \( s_i = s_2 \) and \( s_i > s_2 \). The following propositions (III.2) and (III.3) give the definition of the probability density function (pdf) \( f_{y_i}(y_i) \) of \( Y \).

**A. Case 1: \( s_i = s_2 \).**

**Proposition III.2.** Let \( s_i = s_2, \theta \in -\Omega \) and \( \Sigma \) is a positive definite matrix. The distribution of the random variable \( Y \) is a bivariate Gamma distribution \( \text{BGD} \) with pdf is defined by 

\[
f_{y_i}(y_1, y_2) = \exp\left(-\frac{\Sigma_{11} y_1 + \Sigma_{22} y_2}{\Sigma_{12} \Sigma_{22} - \Sigma_{11} \Sigma_{21}} \right) \frac{y_1^{s_1-1} y_2^{s_2-1}}{(\Sigma_{11} - \Sigma_{12} \Sigma_{22})^2} \Gamma(s_i)(s_j)
\]

where \( 1_{[0, +\infty]}(y_1, y_2) \) is the indicator function defined on \( [0, +\infty]^2 \), \( \delta = \frac{\Sigma_{12}^2}{(\Sigma_{11} \Sigma_{22} - \Sigma_{12} \Sigma_{21})^2} \), and \( f_{y_i}(z) \) is given by 

\[
f_{y_i}(z) = \frac{\delta}{k!} \Gamma(s_i + k) z^{k}.
\]

**Proof**
If \( s_i = s_2, \Sigma_{11} > 0, \Sigma_{22} > 0 \) and \( \Sigma_{12} \Sigma_{21} - \Sigma_{11} \Sigma_{22} > 0 \), then the Laplace transform of \( Y \)

\[
L(\theta) = [(1 - \theta_{1,1} \Sigma_{11})(1 - \theta_{2,2} \Sigma_{22}) - \theta_{1,2} \Sigma_{12}^2 - \theta_{2,1} \Sigma_{21}^2]^{s_2} \times [1 - \theta_{1,2} \Sigma_{12}]^{s_2^{-1}}.
\]

According to [4], if \( \theta \in -\Omega \), then this Laplace transform is related to a bivariate Gamma distribution with pdf defined in equation (5).

**Proposition III.3.** The marginal distribution \( Y_{i,i} \), \( i = 1, 2 \), is distributed according to a univariate Gamma distribution with pdf defined by 

\[
f_{y_i}(y_i) = \frac{\gamma y_i^{s_1-1} \exp\left(-\frac{y_i}{\Sigma_{ii}}\right)}{\Sigma_{ii} \Gamma(s_i)} 1_{[0, +\infty]}(y_i).
\]

where \( s_i > 0 \) represent the shape parameter and \( \Sigma_{ii} > 0 \) is the scale parameter.

To define the parameters of the random variable \( Y \), we
introduce the following proposition.

**Proposition III.4.** The moments and the covariance of the random variable $Y$, with pdf defined in equation (5), is given by

$$ E[Y_1] = s_1, \quad E[Y_2] = s_2 $$

$$ \text{var}(Y_1) = s_1^2, \quad \text{var}(Y_2) = s_2^2 $$

$$ \text{cov}(Y_1, Y_2) = s_1 s_2. $$

This different moments cited in Proposition III.4 are obtained by differentiating of the expression of the Laplace transform defined in equation (2), where $x_i = \vartheta_i$, with respect to $i, j = 1, 2$.

**B. case 2:** $s_2 > s_1$.

The Laplace transform of $Y$ according to the distribution of the diagonal of the modified Riesz, given in Proposition III.1 is a product of two Laplace transform of two independent random variables according respectively the univariate gamma distribution with shape parameter $s_2 - s_1$ and scale parameter $\Sigma_{22}$ and the bivariate gamma distribution with parameters $s_1, \Sigma_{11}, \Sigma_{12}$ and $\Sigma_{22}$, where $\Sigma_{12} = \Sigma_{12} - \Sigma_{12}^2$. We can define the bivariate random variable $Y$ by

$$ Y = (Z_1, Z_2 + T), $$

with:

- $T$ is a univariate gamma distribution with shape parameter $s_2 - s_1$ and scale parameter $\Sigma_{22}$ with pdf $f_T$ defined in (6),
- $Z = (Z_1, Z_2)$ is a BGD whose pdf $f_{Z|Z}$ defined in equation (5),
- $T$ and $Z$ are independent.

**Proposition III.5.** Let $0 < s_1 < s_2$ and $\Sigma$ is a positive definite matrix. In this case, the distribution of the random variable $Y$ is defined by

$$ f_{2,Y}(y_1, y_2) = \frac{1}{\Sigma_{22}^{-\frac{1}{2}}} (\Sigma_{11}^{-1} - \Sigma_{12}^{-1})^{-\frac{1}{2}} \exp \left( -\frac{1}{2} \left( \begin{array}{c} y_1 \\ y_2 \\ \end{array} \right)^T \left( \Sigma_{11}^{-1} - \Sigma_{12}^{-1} \right)^{-1} \left( \begin{array}{c} y_1 \\ y_2 \\ \end{array} \right) \right) $$

$$ \times \frac{\Gamma(s_1 - \frac{1}{2}) \Gamma(s_2 - \frac{1}{2})}{\Gamma(s_1) \Gamma(s_2)} \Phi_3 \left( (y_2 - s_1; s_2; h, \vartheta_2, \delta \vartheta_2) \right) \right| \left. \left( (y_1, y_2) \right) \right), $$

where $h = \delta \Sigma_{22}^{-1} \Sigma_{11}^{-1}$ and $\Phi_3$ is the so-called Horn function.

**Proof**

Firstly, by using the independence assumption between $Z$ and $T$, the density of $Y$ can be expressed as

$$ f_{2,Y}(y_1, y_2) = \int_{0}^{\infty} f_{1,Z}(y_1, u) f_{T}(y_2 - u) \text{d}u. \quad (8) $$

Secondly, by making the following change of variable $w = \frac{u}{y_2}$ in formula (8) and by using the following series expansion

$$ e^w = \sum_{j=0}^{\infty} \frac{w^j}{j!}, $$

we have the desire result.

To define the parameters of the random variable $Y$, we introduce the following proposition.

**Proposition III.6.** The moments and the covariance of the random variable $Y$, with pdf defined in equation (5), is given by

$$ E[Y_1] = s_1, \quad E[Y_2] = s_2 $$

$$ \text{var}(Y_1) = s_1^2, \quad \text{var}(Y_2) = s_2^2 $$

$$ \text{cov}(Y_1, Y_2) = s_1 s_2. $$

This different moments cited in Proposition III.6 are obtained by differentiating of the expression of the Laplace transform defined in (III.1), where $0 < s_1 < s_2$, with respect to $\vartheta_i, i, j = 1, 2$.

**IV. Parameters estimation**

In this section, we estimate the parameters of the diagonal of the modified Riesz distribution in this to cases ($s_1 = s_2$ and $s_2 > s_1$). We assume that $(s_1, s_2)$ is a known parameters.

Therefore, the diagonal of the modified Riesz distribution is characterized by a unknown parameters $\Pi = (\Sigma_{11}, \Sigma_{12}, \Sigma_{22})$, where $\Sigma_{12} = \Sigma_{11} - \Sigma_{12}^2$.

Let $Y = (Y^1, ..., Y^n)$, where $Y^i = (Y^i_1, Y^i_2)$, $n$ vectors is distributed according to a diagonal of the modified Riesz distribution with the unknown parameter vector $\Pi$.

**A. Case 1:** $s_2 = s_1 = s$.

1) Method Maximum Likelihood (ML): The likelihood function of a sample bivariate observations $Y^1, Y^2, ..., Y^n$ of density defined in equation (5) is given by

$$ l(Y^i; \Pi) = -n s \ln(\Sigma_{11}^{-1}) - n \Sigma_{11}^{-1} \Sigma_{12}^{-1} + (s_1 - 1) \Sigma_{12}^{-1} \ln(y_1, y_2) \quad (9) $$

$$ - \ln \Gamma(s_1) - \frac{n}{2} \Sigma_{11}^{-1} \Sigma_{12}^{-1} + \sum_{i=1}^{n} \ln(f_s(y_i, y_i'), i)), $$

where $f_s = \frac{1}{n} \sum_{i=1}^{n} y_i^{1/2} y_i^{1/2}$.

By taking the differential, with respect to $\Sigma_{11}, \Sigma_{12}$ and $\Sigma_{22}$, one has

$$ -ns + \frac{n \Sigma_{11}^{-1} \Sigma_{12}^{-1} - n \Sigma_{11}^{-1} \Sigma_{12}^{-1} - 2 \Sigma_{12}^2}{\Sigma_{11} - \Sigma_{12}^2} H = 0 \quad (10) $$

$$ -ns + \frac{n \Sigma_{11}^{-1} \Sigma_{12}^{-1} - n \Sigma_{11}^{-1} \Sigma_{12}^{-1} - 2 \Sigma_{12}^2}{\Sigma_{11} - \Sigma_{12}^2} H = 0 \quad (11) $$

$$ ns - \frac{n \Sigma_{11}^{-1} \Sigma_{12}^{-1} - n \Sigma_{11}^{-1} \Sigma_{12}^{-1} + \Sigma_{11} - \Sigma_{12}^2}{\Sigma_{11} - \Sigma_{12}^2} H = 0 \quad (12) $$

with $H = \sum_{i=1}^{n} y_i^{1/2} f_s(y_i, y_i')$ and $f_{s} = f_s$.

From equations (10), (11) and (12), the ML estimators of $\Sigma_{11}$ and $\Sigma_{22}$ are defined by
\[ \hat{\Sigma}_{11} = \frac{\bar{Y}}{s}, \quad \hat{\Sigma}_{22} = \frac{\bar{Y}}{s^2} \]  

(13)

By replacing \( \Sigma_{11} \) and \( \Sigma_{22} \) by their estimators in equation (9), then the estimator of \( \Sigma_{12} \) is the root of the following function

\[ \phi(\hat{\Sigma}_{12}) = n - \frac{s}{\bar{Y}} \bar{Y}^2 - s^{-1} \sum_{i=1}^{n} Y_i Y_i' = 0, \]

(14)

where \( \hat{\delta} = \frac{s^4 \Sigma_{12}^2}{(\bar{Y} \Sigma_{12}^2 - s^3 \Sigma_{12}^2)^2} \).

A closed-form solution of equation (14) to determine \( \hat{\Sigma}_{12} \) does not exist. We can obtain in solution of equation (14) by using a Newton-Raphson procedure. The convergence of the Newton-Raphson procedure is generally obtained after few iterations.

2) Method of Moments (MM): The estimator of \( \Pi = (\Sigma_{11}, \Sigma_{22}, \Sigma_{12}) \) by the MM is the solution of the following system

\[ \left\{ \begin{array}{l}
\bar{Y}_1 = E[Y_1], \bar{Y}_2 = E[Y_2] \\
\frac{1}{n} \sum_{i=1}^{n} (Y_i^{(i)} - \bar{Y}_1)'(Y_i^{(i)} - \bar{Y}_2) = \text{cov}(Y_1, Y_2)
\end{array} \right. \]

Consequently,

\[ \hat{\Sigma}_{11} = \frac{\bar{Y}_1}{s}, \quad \hat{\Sigma}_{22} = \frac{\bar{Y}_2}{s} \]

(15)

\[ \hat{\Sigma}_{12} = \frac{1}{ns} \sum_{i=1}^{n} (Y_i^{(i)} - \bar{Y}_1)'(Y_i^{(i)} - \bar{Y}_2). \]

(16)

In this case, we constate that the parameters \( \Sigma_{11} \) and \( \Sigma_{22} \) have the same estimators obtained by the ML and the MM.

B. Case 2: \( s_2 > s_1 \)

1) Method Maximum Likelihood (ML): The log-likelihood function of \( Y \) is given by

\[ l(Y; \Pi) = -ns_i \ln(\Sigma_{i,i}) + n(s_i - s_1) \ln(\Sigma_{i,2}) - \frac{n \Sigma_{i,2} \bar{Y}_i}{\Sigma_{i,i}} \]

\[ + (s_i - s_1) \sum_{j=1}^{n} \ln(y_j') - \ln(\Gamma(s_i)) - \frac{n \Sigma_{i,2} \bar{Y}_i}{\Sigma_{i,i}} + (s_i - s_1) \sum_{j=1}^{n} \ln(y_j') \]

\[ - \ln(\Gamma(s_2)) + \sum_{j=1}^{n} \ln(\Phi_i(y_j), y_j') - \sum_{j=1}^{n} \ln(\Phi_i(y_j), y_j'). \]

(17)

where \( \bar{Y}_j = \frac{1}{n} \sum_{i=1}^{n} Y_i, f = 1,2, \delta = \frac{\Sigma_{i,2}}{\Sigma_{i,i}}, \) and \( h = \delta \Sigma_{i,2}. \)

By taking the differential, with respect to \( \Sigma_{11}, \Sigma_{22} \) and \( \Sigma_{12}^2 \), one has

\[ -ns_1 + \frac{n \Sigma_{i,2} \bar{Y}_i}{\Sigma_{i,i}} + n \Sigma_{i,2} \bar{Y}_i \]

\[ - (s_2 - s_1) \frac{\Sigma_{i,2} \bar{Y}_i}{\Sigma_{i,i}} = \frac{2 \Sigma_{i,2} \Phi_{i,3}}{s_2 \Sigma_{i,i}} \Phi_{i,3} = 0, \]

(18)

Next, the same estimators obtained by the ML and the MM.

V. SIMULATION

In order to compare the performance of the MM estimator and the ML estimator, we propose any simulations.

A. Case 1: \( s_2 = s_1 = s \).

In this case, we generate a random vector \( Y \) according to a BGD with different parameters:

i) \( s = 3, \Sigma_{11} = 0.3, \Sigma_{22} = 1, \)

ii) \( s = 3, \Sigma_{11} = 10, \Sigma_{22} = 5. \)

The comparative study of the MM estimator and ML estimator is characterized by the determination of the values of the MSE as a function of \( n \), where \( n \) is the size of the sample. The number of resampling is \( N = 1000 \). We present respectively in Fig.1 and Fig.2 the MSE of the estimator of the parameter \( \Sigma_{12}^2 \) in two case (\( \Sigma_{12}^2 = 0.105 \) and \( \Sigma_{12}^2 = 45 \)). The circle curves correspond to the estimator of MM whereas the triangle curves correspond to the estimator of ML. We observe from the two figures that the ML method is more efficient than the MM method.
Fig. 1. MSE versus $n$ for parameters $\Sigma_{12}$ ($s = 3, \Sigma_{11} = 0.3, \Sigma_{22} = 1$).

Fig. 2. MSE versus $n$ for parameters $\Sigma_{12}$ ($s = 3, \Sigma_{11} = 10, \Sigma_{22} = 5$).

B. Case 2: $s_2 > s_1$

In this case, we generate two random vectors $Z$ and $T$. The random vector $Z = (Z_1, Z_2)$ follows the BGD with parameters $s_1, \Sigma_{11}, \Sigma_{22}$ and $\Sigma_{12}$. The random vector $T$ follows the univariate gamma distribution with parameters $s_2 - s_1$, and $\Sigma_{22}$. This two variables $Z$ and $T$ are independent.

The random variable $Y$ according to the diagonal of the modified Riesz distribution is obtained from the two variables $Z$ and $T$, where $Y = (Z_1, Z_2 + T)$.

The comparative study of the MM estimator and ML estimator is characterized by the determination of the values of the MSE as a function of $n$, where $n$ is the size of the sample. The number of resampling is $N = 1000$. In Fig.3, Fig.4 and Fig.5 we present the variation of the MSE depending of $n$ for different case of the parameters $(s_1, s_2, \Sigma_{11}, \Sigma_{22}, \Sigma_{12})$.

Fig.3 presents the variation of the MSE for different case of the parameter $\Sigma_{12}$ where the parameters $s_1$, $s_2$, $\Sigma_{11}$, and $\Sigma_{22}$ are fixed. The circle curves correspond to the estimator of MM whereas the triangle curves correspond to the estimator of ML. We observe from the three figures that the ML method is more efficient than the MM method.
VI. Conclusion

In this paper, a diagonal of the modified Riesz distribution is developed and different estimators of parameters of this distribution are introduced. These estimators are obtained by using the Moment method and the Maximum Likelihood method. A comparison criter of performance of these estimators is based to the Mean Square Error. Several simulations studied in our work shows that the Maximum Likelihood estimator is more perform then the Moment estimator. In the future work we will generalize the diagonal of the modified Riesz distribution to the mixture model.

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