Clique and chromatic number of circular-perfect graphs

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Abstract
A main result of combinatorial optimization is that clique and chromatic number of a perfect graph are computable in polynomial time (Grötschel, Lovász and Schrijver 1981). Circular-perfect graphs form a well-studied superclass of perfect graphs. We extend the above result for perfect graphs by showing that clique and chromatic number of a circular-perfect graph are computable in polynomial time as well. The results strongly rely upon Lovász’s Theta function.

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A k-coloring of a graph $G = (V, E)$ is a mapping $f : V \rightarrow \{1, \ldots, k\}$ with $f(u) \neq f(v)$ if $uv \in E$, i.e., adjacent vertices receive different colors. The minimum $k$ for which $G$ admits a $k$-coloring is called the chromatic number $\chi(G)$.

The clique number $\omega(G)$ of $G$ is the order of a largest clique of $G$, i.e., the maximum number of pairwise adjacent vertices of $G$. The clique number is a trivial lower bound for the chromatic number, but by far not always tight. The graphs where both parameters coincide for all induced subgraphs are called perfect.

To generalize perfect graphs, Zhu [10] introduced circular-perfect graphs based on a more general coloring concept introduced by Vince [8]. For positive integers $k \geq 2d$, a $(k, d)$-coloring of a graph $G$ is a mapping $f : V \rightarrow \{0, \ldots, k - 1\}$ such that for each edge $uv$ of $G$, $d \leq |f(u) - f(v)| \leq k - d$ holds. The circular-chromatic number is

$$\chi_c(G) = \min \left\{ \frac{k}{d} : G \text{ has a } (k, d)\text{-coloring} \right\}$$

and satisfies $\chi_c(G) \leq \chi(G)$, as a $(k, 1)$-circular coloring is a usual $k$-coloring. In addition, it is known by [8] that $\chi(G) = \lceil \chi_c(G) \rceil$ for any graph $G$.

As generalizations of cliques, circular-cliques $K_{k/d}$ having $k \geq 2d$ nodes $0, \ldots, k - 1$ and edges $ij$ if and only if $d \leq |i - j| \leq k - d$ are considered. The circular-clique number is

$$\omega_c(G) = \max \left\{ \frac{k}{d} : K_{k/d} \text{ is an induced subgraph of } G, \gcd(k, d) = 1 \right\}$$

and $\omega(G) \leq \omega_c(G)$ holds as $K_{k/1}$ is a usual clique. It is also known from [10] that $\omega(G) = \lfloor \omega_c(G) \rfloor$ for any graph $G$.

In addition, $\omega_c(G)$ is always a lower bound for $\chi_c(G)$ by [1] which implies

$$\omega(G) = \lfloor \omega_c(G) \rfloor \leq \omega_c(G) \leq \lceil \chi_c(G) \rceil = \chi(G). \tag{1}$$

A graph $G$ is circular-perfect [10] if each induced subgraph $G' \subseteq G$ satisfies $\chi_c(G') = \omega_c(G')$. By definition, every perfect graph is circular-perfect as we have everywhere equality in (1). All circular-cliques are circular-perfect as well by [10]. So circular-perfect graphs form a proper superclass of the class of perfect graphs.

Perfect graphs have been extensively studied and turned out to be an interesting and important class of graphs with a rich structure (see [7] for a recent survey). Most notably, both in general hard to compute graph parameters $\omega(G)$ and $\chi(G)$ can be determined in polynomial time if $G$ is perfect [4].

The latter result relies on the polyhedral characterization of perfect graphs. The stable set polytope $\text{STAB}(G)$ is defined as the convex hull of the incidence vectors of all stable sets of $G$ (in a stable set all nodes are mutually nonadjacent).
A canonical relaxation of STAB \((G)\) is the fractional stable set polytope

\[
QSTAB(G) = \{ x \in \mathbb{R}_{+}^{V} : \sum_{i \in Q} x_{i} \leq 1, \ Q \subseteq G \text{ clique} \}.
\]

We have \(\text{STAB}(G) \subseteq QSTAB(G)\) in general and equality for perfect graphs [2]. As stable sets of a graph \(G\) correspond to cliques in its complement \(\overline{G}\), the clique number \(\omega(G)\) equals the stability number \(\alpha(G)\). Thus, computing \(\omega(G)\) for a perfect graph \(G\) could be done by solving the linear program \(\max \mathbb{1}^{T} x, x \in QSTAB(\overline{G})\). However, maximizing a linear objective function over the fractional stable set polytope does not work directly [4]. For the class of perfect graphs, though, the problem can be solved in polynomial time via the following detour.

An orthonormal representation of \(G = (V, E)\) is a sequence \((u_{i} : i \in V)\) of \(|V|\) unit-length vectors \(u_{i} \in \mathbb{R}^{N}\), where \(N\) is some positive integer, such that \(u_{i}^{T} u_{j} = 0\) for all \(i j \not\in E\). For any orthonormal representation of \(G\) and any additional vector \(c \in \mathbb{R}^{N}\) of unit length, the corresponding orthonormal representation constraint is \(\sum_{i \in V} (c^{T} u_{i})^{2} x_{i} \leq 1\). \(\text{TH}(G)\) denotes the convex set of all vectors \(x \in \mathbb{R}_{+}^{V}\) satisfying all orthonormal representation constraints for \(G\) and we have \(\text{STAB}(G) \subseteq \text{TH}(G) \subseteq QSTAB(G)\).

The key property of \(\text{TH}(G)\) for linear programming is that, for any graph \(G\) and any weight vector \(w \in \mathbb{Q}_{+}^{V}\), the optimization problem

\[
\vartheta(G, w) = \max w^{T} x, \ x \in \text{TH}(G)
\]

can be solved in polynomial time [4]. For perfect graphs, \(\text{STAB}(G)\) and \(\text{TH}(G)\) coincide and, thus, the (weighted) stability number equals \(\alpha(G, w) = \vartheta(G, w)\). This also allows to compute the (weighted) clique number \(\omega(G, w) = \alpha(\overline{G}, w)\) and the (weighted) chromatic number \(\chi(G, w) = \omega(G, w)\) in polynomial time.

Our aim is to address the question which properties of perfect graphs extend to circular-perfect graphs. As (1) implies for each circular-perfect graph \(G\) that

\[
\omega(G) \leq \chi(G) \leq \omega(G) + 1,
\]

circular-perfect graphs have coloring properties almost as nice as perfect graphs. However, in contrary to perfect graphs, it was not known so far whether clique and chromatic number of a circular-perfect graph (also the weighted or circular versions) are computable in polynomial time. In this paper, we establish that computing the (weighted) clique and the chromatic number of circular-perfect graphs can be done in polynomial time.
The paper is organized as follows: In Section 1, we briefly recall the known inequalities involving the studied graph parameters. In Section 2, we show that the weighted clique number of a circular-perfect graph $G$ is computable in polynomial time and that $\omega(G) = \lceil \vartheta(G) \rceil$ holds. In Section 3, we show that the chromatic number of a circular-perfect graph $G$ is computable in polynomial time as $\chi(G) = \lceil \vartheta(G) \rceil$ holds. The last section is devoted to some concluding remarks.

1 Relating the involved graph parameters

The clique polytope $\text{CLI}(G)$ of a graph $G$ is the stable set polytope of its complement $\overline{G}$, and the fractional clique polytope $\text{SCLI}(G)$ is the fractional stable set polytope of $\overline{G}$. Hence, $\text{CLI}(G) = \text{STAB}(\overline{G}) \subseteq \text{QSTAB}(\overline{G}) = \text{SCLI}(G)$ implies $\omega(G, w) = \max \{ w^T x : x \in \text{CLI}(G) \} \leq \max \{ w^T x : x \in \text{SCLI}(G) \} = \omega_f(G, w)$ and recalling that $\text{TH}(G)$ is sandwiched between $\text{STAB}(G)$ and $\text{QSTAB}(G)$ yields $\omega(G, w) \leq \vartheta(G, w) \leq \omega_f(G, w)$.

In [6], a suitable polytope $\text{CLI}_c(G)$ was introduced such that for the (weighted) circular-clique number $\omega_c(G, w) = \max \{ w^T x : x \in \text{CLI}_c(G) \}$ holds. By

$$\omega_c(G, w) = \max \left\{ \frac{1}{\alpha(K)} \left( \sum_{v \in K} w(v) \right) : K \subseteq G \text{ prime circular-clique} \right\}$$

we immediately see that the circular-clique polytope

$$\text{CLI}_c(G) = \text{conv} \{ 1/\alpha(K) | \chi_K : K \subseteq G \text{ prime circular-clique} \}$$

is the studied polytope. In [6], it is shown that $\text{CLI}_c(G)$ is sandwiched between the clique polytope and the fractional clique polytope.

**Lemma 1.1 ([6])** For all graphs $G$, we have $\text{CLI}(G) \subseteq \text{CLI}_c(G) \subseteq \text{SCLI}(G)$ and, thus, $\omega(G, w) \leq \omega_c(G, w) \leq \omega_f(G, w)$ holds.

Weighted circular-chromatic and chromatic numbers have been studied by Deuber and Zhu [3] who observed that $\chi_f(G, w) \leq \chi_c(G, w) \leq \chi(G, w)$ holds for every graph $G$. Together with LP-duality, this finally implies:

$$\omega(G, w) \leq \omega_c(G, w) \leq \omega_f(G, w) = \chi_f(G, w) \leq \chi_c(G, w) \leq \chi(G, w). \quad (3)$$
2 Computing the clique number of circular-perfect graphs

Xuding Zhu proved the following necessary condition for circular-perfectness:

**Theorem 2.1 (Zhu (2005) [10])** If $G$ is circular-perfect then $N[x]$ is perfect for every vertex $x$.

Recall that for a perfect graph, the weighted clique number is computable in polynomial time. Since for every clique $Q$ of a graph $G$, there is a vertex $v$ such that $Q \subseteq N[v]$ holds, this implies that for a circular-perfect graph $G = (V, E)$ and any vector $w \in \mathbb{Q}^{|V|}$ of vertex weights,

$$\omega(G, w) = \max\{\vartheta(G', w) : G' = N[x], x \in V\}$$

holds. This shows:

**Corollary 2.2** For a circular-perfect graph $G$ and every vector $w$ of vertex weights, the weighted clique number $\omega(G, w)$ is computable in polynomial time.

For the unweighted clique number, we further have the following:

**Theorem 2.3** If $G$ is circular-perfect, then $\omega(G) = \lfloor \vartheta(G) \rfloor$.

**Proof.** For all graphs $G$, we have

$$\omega(G) = \lfloor \omega_c(G) \rfloor \leq \omega_c(G) \leq \omega_f(G) = \chi_f(G) \leq \chi_c(G) = \lfloor \chi_c(G) \rfloor = \chi(G).$$

For a circular-perfect graph $G$, we have in addition $\chi(G) \leq \omega(G) + 1$. Thus, one of the following cases occurs. Either, we have $\omega_c(G) \not\in \mathbb{Z}$, then

$$\omega(G) < \omega_c(G) = \omega_f(G) = \chi_f(G) = \chi_c(G) < \chi(G) = \omega(G) + 1$$

follows by combining the above facts. Or $\omega_c(G) \in \mathbb{Z}$ holds and we obtain

$$\omega(G) = \omega_c(G) = \omega_f(G) = \chi_f(G) = \chi_c(G) = \chi(G)$$

as $\omega(G) = \omega_c(G)$ follows from $\omega(G) = \lfloor \omega_c(G) \rfloor$ and $\omega_c(G) = \chi_c(G) \in \mathbb{Z}$ implies also $\chi_c(G) = \chi(G)$ by $\lfloor \chi_c(G) \rfloor = \chi(G)$.

In both cases, $\omega_c(G) = \omega_f(G)$ and $\chi(G) \leq \omega(G) + 1$ implies that

$$\omega(G) = \lfloor \omega_f(G) \rfloor.$$

As we generally have $\omega(G) \leq \vartheta(G) \leq \omega_f(G)$ for all graphs, this proves $\omega(G) = \lfloor \vartheta(G) \rfloor$ and, thus, the assertion. $\square$
3 Computing the chromatic number of circular-perfect graphs

Since a circular-perfect graph is $(\omega(G) + 1)$-colorable, computing $\chi(G)$ amounts to testing whether $\chi(G) = \omega(G)$. We show in this section that this can be done in polynomial time for the class of circular-perfect graphs.

From the above result, $\chi(G)$ can be approximated in polynomial time with a gap of at most 1. If $\vartheta(G) \not\in \mathbb{Z}$ holds, then $\chi(G) = \lceil \vartheta(G) \rceil$ clearly follows. However, it remains to clarify which of the possible values $\chi(G) \in \{\vartheta(G), \vartheta(G) + 1\}$ is attained if $\vartheta(G) \in \mathbb{Z}$.

For that, the following lemma is crucial. We say that a graph $G$ is homomorphic to a graph $H$ if there is a map $f$ from the vertex set of $G$ to the vertex set of $H$, preserving adjacency: if $ij$ is an edge of $G$ then $f(i)f(j)$ is an edge of $H$.

**Lemma 3.1** If $G$ is homomorphic to $H$ then $\vartheta(G) \leq \vartheta(H)$.

Since every circular-clique $K_{p/q}$ is homomorphic to every circular-clique $K_{p'/q'}$ such that $p/q \leq p'/q'$ [1], we get:

**Corollary 3.2** If $p/q \leq p'/q'$ then $\vartheta(K_{p/q}) \leq \vartheta(K_{p'/q'})$.

We are now ready to prove the main result of this paper:

**Theorem 3.3** If $G$ is circular-perfect then $\chi(G) = \lceil \vartheta(G) \rceil$.

**Proof.** For a circular-perfect graph $G$, we have $\omega(G) \leq \vartheta(G) \leq \chi(G) \leq \omega(G) + 1$. Thus, if $\vartheta(G) \not\in \mathbb{Z}$ then clearly $\chi(G) = \lceil \vartheta(G) \rceil$ follows. Proving the assertion of the theorem, therefore, means to ensure that $\chi(G) = \vartheta(G)$ in the case $\vartheta(G) \in \mathbb{Z}$.

Assume in contrary that

$$\omega(G) = \vartheta(G) < \chi(G) = \omega(G) + 1$$

holds (recall that we have $\omega(G) = \vartheta(G)$ by Theorem 2.3 if $\vartheta(G)$ is integer). Then $\omega(G) < \omega_c(G)$ follows and there are integers $p$ and $q \neq 0$, $\gcd(p, q) = 1$ such that $K_{p/q} \subseteq G$ with $\omega_c(G) = p/q > \omega(G) = \omega$, i.e., $K_{p/q}$ is homomorphic to $G$.

Let $p = \omega q + r$ where $1 \leq r < q$ and let $p' = \omega q + 1$. Then

$$\frac{p'}{q} = \frac{\omega q + 1}{q} \leq \frac{\omega q + r}{q} = \frac{p}{q}$$

holds and, thus,

$$\vartheta(K_{(\omega q + 1)/q}) \leq \vartheta(K_{(\omega q + r)/q}) \leq \vartheta(G)$$
follows from Lemma 3.1 and Corollary 3.2. Notice that $K_{(ωq+1)/q}$ is a partitionable graph, and so we have $\vartheta(K_{(ωq+1)/q}) > ω + 1/p'$ by [5]. This implies

$$\omega < ω + 1/p' < \vartheta(K_{(ωq+1)/q}) ≤ \vartheta(G),$$

a contradiction to $ω = ω(G) = \vartheta(G)$. Thus $χ(G) = ω(G)$ follows if $\vartheta(G)$ is integer. This verifies the assertion that $χ(G) = \lceil ϑ(G) \rceil$.

In order to use the fact that $χ(G) = \lceil ϑ(G) \rceil$ holds to compute $χ(G)$ for a circular-perfect graph in polynomial time, the difficulty is to decide algorithmically whether $\vartheta(G)$ is integer or not. However, this can be done as follows. Let $G$ be a circular-perfect graph with $n$ vertices and $ε = 1/n^2$.

- Compute $\vartheta(G)$ with an error of at most $ε/4$ (polynomial in $n$ since the encoding length of $ε$ is $n \log_2 n$ bits) and denote by $\overline{ϑ}$ this value.
- If $\overline{ϑ} − [\overline{ϑ}] < ε/2$, then return $[\overline{ϑ}]$ else return $[\overline{ϑ}] + 1$ (polynomial since the test can be done in $n \log_2 n$ operations).

Note that one can show similarly as in the proof of Theorem 3.4 that $\vartheta(G)$ is integer if $\overline{ϑ} − [\overline{ϑ}] < ε/2$ holds. This finally implies:

**Corollary 3.4** If $G$ is circular-perfect then $χ(G)$ is computable in polynomial time.

4 Conclusion

In this paper we proved that, for every circular-perfect graph, clique and chromatic number can be computed in polynomial time. Thus, the nice coloring properties indeed extend from perfect graphs to the larger class of circular-perfect graphs.

It remains open whether also circular-clique and circular-chromatic number of circular-perfect graphs can be computed in polynomial time. Partial results in this direction were already obtained in [6]. There it is shown that the strong optimization problem over $QSTAB(G)$ can be solved in polynomial time whenever $G$ has the property that $N[x]$ is perfect for any vertex $x$. For every circular-perfect graph $G$ with the additional property that in its complement, $N[x]$ is perfect for any vertex $x$, we can thus compute $ω_f(G)$ in polynomial time and also $ω_c(G) = ω_f(G)$ and $χ_c(G) = ω_c(G)$. This applies to all circular-perfect graphs where the complement is circular-perfect [10] or $a$-perfect [9]. Here, a graph is $a$-perfect if its stable set polytope has rank constraints $\sum_{i \in K_{k/d}} x_i ≤ d$ associated with its induced circular-cliques $K_{k/d}$ as only nontrivial facets.
References


