# ACTIVE ELECTROLOCATION IN WEAKLY ELECTRIC FISH: MULTI-FREQUENCY TARGET DETECTION

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**ABSTRACT**

We present a mathematical model for the problem of active electrolocation. This ability is shared by fishes that are able to recognize an object by means of emitting an electric field and recording it at the surface of their skin. For the forward problem, an electro-quasistatic formulation is carried on, and multi-scale analysis of the problem is performed for the boundary conditions. Then, the imaging techniques presented involve multi-frequency measurements, leading to an adaptation of the so-called MUSIC algorithm.

## 1. INTRODUCTION

In turbid waters of South America and Africa, there can be found small fishes that are called *weakly electric*. Indeed, they generate an electric field that does not exceed 1 mV. Thus, such a small voltage cannot be generated for defense purpose. This enigma worried Charles Darwin about his theory of evolution [4] and was resolved by Lissmann in 1958 [5]: this electric machinery is a location mechanism. In other words, knowing the distortion of the field induced by an object, these fishes are able to recognize this latter.

In the last decades, behavioral studies have shown that they are able to recognize the location, the shape, and the electrical parameters of any object located in their vicinity (for a review, see [6]). Hence, in a mathematical point of view, studying this ability - called *active electrolocation* - is a great opportunity for the understanding of inverse problems.

This summary is built as follows. In a first part, a mathematical model will be formulated for the forward problem, *i.e.* for the calculus of the electric field. Then, a second part formulates an algorithm that locates a small object situated far away from the fish, using the multi-frequency aspect of the measurements.

The results presented in this abstract are available in [1].

## 2. MATHEMATICAL MODEL

In $\mathbb{R}^d$ where $d = 2$ or 3, let us denote $\Omega$ the body of the fish, and $D \subset \mathbb{R}^d \setminus \overline{\Omega}$ an object to recognize. This section is devoted to the equations governing the electric field generated and recorded on the skin $\partial\Omega$ of the fish.

Analysis of the electroreceptors have shown that the multi-frequency content of the measurements are very important for these fishes [6]. Thus, a static electric field cannot model precisely the forward problem. Neither does the Maxwell system, because the typical wavelength is much greater than the space scale. Indeed, from measurements of the electric organ [6], it is known that the frequency does not exceed 10 kHz, so the typical wavelength is about 3 km. Moreover, the size of the body is not greater than 1 m, and the range of electrolocation is about twice the size of the fish [6]. Thus, the *electro-quasistatic* (or EQS) approximation is best suited for this problem. That is, the electric field $\mathbf{E}$ depends on the time but is still considered to be irrotationnal, so that there exists an electric potential $u$ such that $\nabla u = \mathbf{E}$ and

$$\nabla \cdot (\sigma + i\varepsilon\omega)\nabla u = f, \qquad (1)$$

where $\sigma$ and $\varepsilon$ are the electric parameters of the medium (respectively, the conductivity and the permittivity), $\omega$ is the frequency, and $f$ is the source of current (which is here the electric organ of the fish : $\operatorname{supp} f \subset \Omega$). Moreover, the electric parameters will be constant by part: they will be denoted $\sigma_0$, $\varepsilon_0$ in the water, $\sigma_b$, $\varepsilon_b$ in the body of the fish, $\sigma_s$, $\varepsilon_s$ inside the skin of the fish, and $\sigma_1$, $\varepsilon_1$ in the anomaly $D$.

The equation 1 with a constant by part complex-valued conductivity function $k := \sigma + i\varepsilon\omega$ is simplified to the Laplace equation with the following jump relations across the surfaces of discontinuity for $k$

$$[u] = 0, \quad \text{and} \quad \left[k\frac{\partial u}{\partial \nu}\right] = 0,$$

where $\nu$ is the outward normal unit vector of the surface and $[\cdot] = \cdot|_+ - \cdot|_-$. Thus, it only remains to write the boundary conditions across the skin, which is very thin and very resistive. In this purpose, let us mention that the permittivites $\varepsilon_i$ for $i \in \{0, s, b\}$ can be neglected compared to their respective conductivities, and the conductivity in the body is very high : $\sigma_b \gg 1$. Then, from a multi-scale analysis of the equations in terms of layer potentials developed in [8, chap. 3], we have shown that on $\partial\Omega$, one has [1]

$$[u] - \xi \left.\frac{\partial u}{\partial \nu}\right|_+ = 0, \quad \text{and} \quad \left.\frac{\partial u}{\partial \nu}\right|_- = 0, \qquad (2)$$

where $\xi$ is called the *effective thickness* by Assad [3], who derived formally these equations during his Ph.D. thesis. It can be seen as the ratio between the small conductivity of the skin $\sigma_s$ and its small thickness.

Finally, for the sake of unicity of the solution, behavior at infinity has to be made precise : the electric potential must goes to 0 at infinity.

## 3. ALGORITHM OF DETECTION

In this section, an algorithm that locates the anomaly $D$ will be presented. It uses multi-frequency measurements : if $\omega_1, \dots, \omega_M$ are the frequencies of the emitted signal, and $x_1, \dots, x_N$ are the locations of the receptors on the skin $\partial\Omega$, the following *space-frequency response matrix* can be built

$$\mathcal{M} = \left[ \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega^+} (x_p, \omega_q) - \left.\frac{\partial U}{\partial \nu}\right|_{\partial\Omega^+} (x_p) \right]_{1 \le p \le N,\, 1 \le q \le M}, \qquad (3)$$

where $U$ is the background electric field (which does not depend on the frequency because the permittivity of the living tissues of the fish is negligible). In the case where $D = z + \delta B$, $B$ being an open set of radius 1, $\mathrm{dist}(z, \partial\Omega) \gg 1$ and $\delta \ll 1$, a dipolar approximation of this measurements gives us [1]

$$\mathcal{M}_{pq} \approx -\delta^2 \nabla U(z) \cdot M(k_q, B) \cdot \nabla_z \frac{\partial G}{\partial \nu_x}(z, x_p), \qquad (4)$$

where $G$ is the Green's function associated to the boundary conditions 2 and $M(k_q, B)$ is the first-order polarization tensor (PT) associated to $B$ with the ratio $k_q := (\sigma_1 + i\varepsilon_1\omega_q)/\sigma_0$. Compared to the real case [2], this complex PT has some more properties investigated in [1].

From 4, we deduce that the columns of the matrix $\mathcal{M}$ are linear combination of partial derivatives of $G$ ; we can then use a modification of the classical MUSIC algorithm (standing for *MUltiple SIgnal Classification*). This modification was proposed by Scholz for breast tumor imaging in [7] and adapted to the data 3 in [1]. In this purpose, let us define the *illumination vector* for a search point $z_s \in \mathbb{R}^d \setminus \overline{\Omega}$

$$g(z_s) := \left[ \frac{\partial G}{\partial z_i}(z, x_p) \right]_{1 \le p \le N, 1 \le i \le d}.$$

Then, there exists $N_0 \in \mathbb{N}$ such that, if $N \ge N_0$, we have

$$g(z_s) \in \mathrm{Range}(\mathcal{M}) \iff z = z_s.$$

Thus, denoting $\Pi$ the projector onto the range of $\mathcal{M}$, the imaging functional

$$\mathcal{I}(z_s) = \frac{1}{|(I - \Pi) g(z_s)|}$$

will have a large peak at $z = z_s$. The effect of post-processing the data is analyzed in [1].

Numerical simulations involving integral equations will be presented, and the stability of the algorithm with respect to measurement noise will be investigated.

## 4. REFERENCES


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