Z-transformation graphs of maximum matchings of plane bipartite graphs

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Abstract

Let \(G\) be a plane bipartite graph. The \(Z\)-transformation graph \(Z(G)\) and its orientation \(\tilde{Z}(G)\) on the maximum matchings of \(G\) are defined. If \(G\) has a perfect matching, \(Z(G)\) and \(\tilde{Z}(G)\) are the usual \(Z\)-transformation graph and digraph. If \(G\) has neither isolated vertices nor perfect matching, then \(Z(G)\) is not connected. This paper mainly shows that some basic results for \(Z\)-transformation graph (digraph) of a plane elementary bipartite graph still hold for every nontrivial component of \(Z(G) (\tilde{Z}(G))\). In particular, by obtaining a result that every shortest path of \(Z(G)\) from a source of \(\tilde{Z}(G)\) corresponds to a directed path of \(\tilde{Z}(G)\), we show that every nontrivial component of \(\tilde{Z}(G)\) has exactly one source and one sink. Accordingly, it follows that the block graph of every nontrivial component of \(Z(G)\) is a path. In addition, we give a simple characterization for a maximum matching of \(G\) being a cut-vertex of \(Z(G)\).

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1. Introduction

The skeleton of carbon atoms in a benzenoid hydrocarbon is a hexagonal system. Its Kekulé structure coincides with a perfect matching of a graph. The number of perfect
matchings of a hexagonal system can be used to predict some physico-chemical properties of the corresponding molecule. Hence, perfect matchings of hexagonal systems have been studied extensively (cf. [1,2,3]). In connection with aromatic sextets of a benzenoid hydrocarbon, Zhang et al. [6,7] introduced a concept of Z-transformation graph on the set of perfect matchings of a hexagonal system: two vertices are adjacent provided that their corresponding perfect matchings only differ in the six edges of one hexagon (the boundary of an interior face). The degree sum of Z-transformation graph can be used to estimate the resonance energy of a benzenoid hydrocarbon [8]. It was shown that the connectivity of Z-transformation graph of a hexagonal system is equal to its minimum degree [7].

As hexagonal systems are a special type of plane bipartite graphs, Ref. [11] extended naturally the concept for Z-transformation graph to a general plane bipartite graph with perfect matching. For a polyomino graph (square system) with a perfect matching, it was shown that [5] its Z-transformation graph has the connectivity equal to the minimum degree except for two cases. For an “elementary” plane bipartite graph (that is 2-connected and each edge is contained in a perfect matching) [4,11], the Z-transformation graph is connected and bipartite, either a path or a graph of girth 4 different from cycle [11]; further, the Z-transformation graph may contain a cut-vertex and its block graph is a path [10]. To prove the latter, an orientation of Z-transformation graph was given in term of two ways [9] of an alternating cycle with respect to a perfect matching, and the directed Z-transformation graph was thus introduced. In fact, an utility of Z-transformation digraph enables one to easily understand undirected cases. Another example is referred to a simpler proof of Theorem 3.2 in this paper.

For a nonelementary plane bipartite graph with a perfect matching, its Z-transformation graph may be disconnected. The authors of [12] gave a complete characterization for the Z-transformation graph of perfect matchings of a plane bipartite graph being connected. Until now few properties of components of Z-transformation graph are revealed. In addition, general plane bipartite graphs do not admit necessarily a perfect matching. This paper gives a further extension of Z-transformation graph and digraph on the perfect matchings to those on the maximum matchings of a plane bipartite graph.

Let \( G \) be a plane bipartite graph. Denote by \( Z(G) \) and \( \tilde{Z}(G) \) the Z-transformation graph and digraph, respectively, of maximum matchings of \( G \). We show that \( Z(G) \) is bipartite and \( \tilde{Z}(G) \) has no directed cycles; \( Z(G) \) is not connected if \( G \) has neither isolated vertices nor perfect matchings. Further, this paper mainly shows that some basic results on the Z-transformation graph (digraph) of perfect matchings of plane elementary bipartite graph still hold for every nontrivial component of \( Z(G) \) (\( \tilde{Z}(G) \)). In particular, by obtaining a result that every shortest path of \( Z(G) \) from a source of \( \tilde{Z}(G) \) corresponds to a directed path of \( \tilde{Z}(G) \) we show that every nontrivial component of \( \tilde{Z}(G) \) has exactly one source and one sink. Accordingly, it follows that the block graph of each nontrivial component of \( Z(G) \) is a path. A nontrivial component of \( \tilde{Z}(G) \), however, is allowed to be a cycle of length 4; that is somewhat different from the previous (cf. Theorem 2.2(c)). Finally, we obtain a necessary and sufficient condition for a maximum matching of \( G \) being a cut-vertex of \( Z(G) \).
2. Definitions and related results

We only consider finite graphs without loops and multiple edges. For a graph $G = (V(G), E(G))$, $V(G)$ denotes its vertex-set and $E(G)$ its edge-set. A graph is bipartite if its vertex-set can be partitioned into two disjoint subsets so that any two vertices in each are not adjacent. The vertices of bipartite graph considered in this paper are always colored black and white so that any two adjacent vertices receive different colors.

A planar graph means a graph that can be embedded in the plane, i.e. it can be drawn in the plane (each vertex is represented by a point of a plane and each edge by a nonself-intersecting continuous curve), so that its edges intersect only at their ends. Such a planar embedding is referred to a plane graph $G$. The rest of the plane is partitioned into connected open sets, called faces of $G$. The unbounded face is called an exterior face of $G$ and the others interior faces of $G$. The boundary of an interior face of $G$ is called a ring if it is a cycle of $G$. A (plane) subgraph $G'$ of a plane graph $G$ is also a planar embedding restricted on the planar embedding $G$.

A set $M$ of some edges of a graph $G$ is called a matching of $G$ if no two edges of $M$ have an end-vertex in common. For a matching $M$ of $G$, the ends of an edge in $M$ are said to be $M$-saturated; The edges of $M$ are called $M$-double and the other edges of $G$ $M$-single, which are usually designated by double lines and single lines in the figure. Further, a matching $M$ is said to be perfect if every vertex of $G$ is $M$-saturated; $M$ is maximum if $G$ has no other matching whose size is larger than the size of $M$.

A connected graph is called elementary if the union of all perfect matchings forms a connected spanning subgraph. A bipartite graph is elementary if and only if $G$ is connected and every edge of $G$ belongs to a perfect matching of $G$ [4].

Let $G$ be a plane bipartite graph with a matching $M$. A cycle (resp. path $P$) of $G$ is called $M$-alternating if the edges of $C$ (resp. $P$) appear alternately in $M$ and $E(G) \setminus M$. A face $f$ of $G$ is said to be resonant if $G$ has a perfect matching $M$ such that the boundary of $f$ is an $M$-alternating cycle.

**Theorem 2.1** (Zhang and Zhang [11]). A plane bipartite graph $G$ is elementary if and only if every face of $G$ is resonant.

**Definition 2.1** (Zhang and Zhang [9]). Let $G$ be a plane bipartite graph with a matching $M$. An $M$-alternating cycle $C$ of $G$ is called proper if every edge of $C$ belonging to $M$ goes from white end-vertex to black end-vertex by the clockwise orientation of $C$; otherwise $C$ is known as improper (for example, see Fig. 1).

![Fig. 1. Proper $M$-alternating ring $s_1$ and improper $M$-alternating ring $s_2$.](image-url)
The symmetric difference of two finite sets $M_1$ and $M_2$ is defined as $M_1 \oplus M_2 := (M_1 \cup M_2) \setminus (M_1 \cap M_2)$. This binary operation is associative and commutative. For a plane bipartite graph with a matching $M$ and an $M$-alternating cycle $C$, $C \oplus M$ (here $C$ is viewed as $E(C)$) is also a matching of $G$ with the same size as $M$. Further $C$ is proper or improper with respect to $C \oplus M$ according as it is improper or proper with respect to $M$. We now define the $Z$-transformation graph and digraph on the maximum matchings of a plane bipartite graph as follows.

**Definition 2.2.** Let $G$ be a plane bipartite graph. The $Z$-transformation graph of maximum matchings of $G$, denoted by $Z(G)$, is defined as a simple graph in which the vertices are the maximum matchings of $G$ and two maximum matchings $M_1$ and $M_2$ are joined by an edge if and only if $M_1 \oplus M_2$ consists exactly of one ring of $G$.

**Definition 2.3.** Let $G$ be a plane bipartite graph with a given two-colour classes. The $Z$-transformation digraph of maximum matchings of $G$, denoted by $\tilde{Z}(G)$, is defined as an orientation of $Z(G)$ such that an edge $M_1M_2$ of $Z(G)$ is orientated from $M_1$ to $M_2$ if and only if $M_1 \oplus M_2$ is a proper $M_1$-alternating ring of $G$ (equivalently, it is an improper $M_2$-alternating ring).

For an example, see Fig. 2. For a subgraph $Z$ of $Z(G)$, denote $\tilde{Z}$ the corresponding orientation of $Z$ in $\tilde{Z}(G)$. From the above definitions, it is easily seen that for a plane bipartite graph $G$ with a perfect matching, $Z$-transformation graph and digraph on the maximum matchings of $G$ are also $Z$-transformation graph and digraph on the perfect matchings of $G$, which were defined and discussed previously in [5–7,10,11]. For plane elementary bipartite graphs, the following results were obtained.

**Theorem 2.2** (Zhang and Zhang [11]). Let $G$ be a plane elementary bipartite graph. Then

(a) $Z(G)$ is a connected bipartite graph,
(b) $Z(G)$ has at most two vertices of degree 1, and
(c) $Z(G)$ is either a path or a graph of girth 4 different from cycle.
Theorem 2.3 (Zhang and Zhang [10]). Let $G$ be a plane elementary bipartite graph. Then the block graph of $Z$-transformation graph of $G$ is a path (cf. Fig. 3).

We now list some basic facts on $Z$-transformation graph and digraph of maximum matchings of a plane bipartite graph. Similar to the proof of Theorem 3.2 in [11] we have

**Theorem 2.4.** Let $G$ be a plane bipartite graph. Then the $Z$-transformation graph $Z(G)$ of maximum matchings of $G$ is bipartite.

In an analogous manner as Lemma 10 in [10] we have

**Theorem 2.5.** Let $G$ be a plane bipartite graph. Then the $Z$-transformation digraph $\tilde{Z}(G)$ of maximum matchings of $G$ has no directed cycles.

**Proof.** By contrary, suppose that $\tilde{Z}(G)$ has a directed cycle $M_1M_2M_3 \cdots M_tM_1$ such that $s_i = M_i \oplus M_{i+1}$ is a proper $M_i$-alternating ring, $i = 1, 2, \ldots, t$ (the subscripts modulo $t$). For any face $f$ of $G$, the depth $d(f)$ of $f$ is defined as the length of a shortest path in the dual graph $G^*$ between two vertices corresponding to $f$ and the exterior face of $G$. If $f$ is an interior face of $G$, put $d(f) = d(s_i) = d(s_{i+1})$. For any face $f_0$ of $G$, put $d(f_0) = d(s_1) - 1$. We assert that $s_1$ is different from any $s_i$ for all $i = 2, \ldots, t$. Otherwise, there exists a subscript $j$, $2 \leq j \leq t$, such that $s_j = s_1$ and $s_1$ is different from any $s_i$ for all $2 \leq i < j - 1$. Then $s_1$ (i.e. $s_j$) is proper $M_1$- and $M_j$-alternating ring. Since $e$ belongs to $M_1$-alternating ring $s_1$, $e$ is either $M_1$-double or single edge. If $e$ is an $M_1$-double edge, then $e$ is $M_2$-single and $e$ goes from white end to black end by the clockwise direction of $s_1$. Since $e \notin s_i$, $2 \leq i < j - 1$ and $M_{i+1} = M_2 \oplus s_2 \oplus \cdots \oplus s_i$, the $e$ remains $M_{i+1}$-single. In particular, $e$ is an $M_j$-single edge. As $s_1$ is proper $M_j$-alternating, $e$ goes from black end to white end by the clockwise direction of $s_1$, a contradiction. If $e$ is an $M_1$-single edge, a similar contradiction would occur. So the assertion follows. Hence $e \notin E(s_i)$ for all $i = 2, \ldots, t$, which implies that $e \notin M_i \oplus M_{i+1}$ for all $i = 2, \ldots, t$. In the process of $M_2 \rightarrow M_3 \rightarrow M_4 \rightarrow \cdots \rightarrow M_t \rightarrow M_1$, the matched way of $e$ remains unchanged. Then $e \notin M_2 \oplus M_1$, which contradicts that $e \in M_1 \oplus M_2$. \qed
The following results are useful in next sections. Both proofs are obvious and similar to those of Lemma 4 in [9] and Lemma 3.7 in [11], respectively.

**Lemma 2.6.** Let \( G \) be a plane bipartite graph with a matching \( M \). Then any two proper (resp. improper) \( M \)-alternating rings are disjoint.

**Lemma 2.7.** Let \( G \) be a plane bipartite graph with a matching \( M \). If \( G \) have three different \( M \)-alternating rings, then there are two of them which are disjoint.

3. Components of \( Z \)-transformation graphs

In general, the \( Z \)-transformation graph of a plane bipartite graph with perfect matchings is not necessarily connected (Fig. 2). In [12], a simple characterization was given for \( Z \)-transformation graph of a plane bipartite graph with perfect matchings being connected.

**Theorem 3.1.** Let a plane bipartite graph \( G \) have no isolated vertex. If \( G \) has no perfect matching, then \( Z(G) \) is disconnected.

**Proof.** Since \( G \) has no perfect matching, then \( G \) has an unsaturated vertex for every maximum matching. Let \( M \) be a maximum matching of \( G \) and \( u \) its \( M \)-unsaturated vertex. As \( G \) has no isolated vertex, \( G \) has an edge \( uv \) incident with \( u \). Then \( v \) is an \( M \)-saturated vertex; Otherwise, \( \{uv\} \cup M \) is also a matching of \( G \), contradicting that \( M \) is a maximum matching of \( G \). So let \( vw \) be an \( M \)-double edge incident with \( v \) (\( w \neq u \)). Put \( M' := (M \setminus \{vw\}) \cup \{uv\} \), which is another maximum matching of \( G \). It is obvious that the saturated vertices of \( G \) are the same for all maximum matchings of \( G \) belonging to one component of \( Z(G) \). But \( u \) is both unsaturated for \( M \) and saturated for \( M' \). This implies that \( M \) and \( M' \) belong to distinct components of \( Z(G) \). \( Z(G) \) is thus not connected. \( \square \)

Let \( P \) be a polyomino graph with 13 vertices as shown in Fig. 4. Its \( Z \)-transformation graph \( Z(P) \) are the disjoint-union of graphs in Fig. 4(a), (b) and (d) and four copies of Fig. 4(c). Obviously \( Z(P) \) has eight components. The removal of any one of black vertices \((v_1 \text{ to } v_7)\) from \( P \) results in a graph with a perfect matching. Every maximum matching of \( P \) is its near perfect matching (not covering exactly one black vertex). Thus, \( Z(P) \) is the union of its induced subgraphs by the perfect matchings of \( P - v_i \) for \( i = 1, \ldots, 7 \). Note that the induced subgraphs by the perfect matchings of \( P - v_i \) is not necessarily isomorphic to \( Z(P - v_i) \). For example, the perfect matchings of \( P - v_5 \) induce two copies of \( K_2 \) (cf. Fig. 4(b)); \( Z(P - v_3) \), however, is a path of length 3.

In what follows we shall mainly consider various properties on components of \( Z(G) \) and \( \overline{Z}(G) \).

**Theorem 3.2.** Let \( G \) be a plane bipartite graph. Then every component of \( Z(G) \) is either a path or a graph of girth 4.
Proof. Let $Z_1(G)$ be a component of $Z(G)$. If $Z_1(G)$ is a single vertex, it is a trivial path. We now suppose that $Z_1(G)$ contains at least two vertices.

If $Z_1(G)$ has a vertex $M$ whose degree is no less than 3, by Lemma 2.7 $G$ has two disjoint $M$-alternating rings, say $s_1$ and $s_2$. Then $Z(G)$ contains a cycle of length 4: $M_1 = M_1 \oplus s_1$, $M_2 = M_2 \oplus s_2$, $M_3 = M_3 \oplus s_1$, and $M_4 = M_4 \oplus s_2$. Furthermore, the girth of $Z_1(G)$ is 4 as $Z_1(G)$ is a bipartite graph.

Next, we assume that the degree of every vertex in $Z_1(G)$ is no more than 2. Then $Z_1(G)$ is either a path or a cycle. It suffices to consider the latter case. If $G$ has two disjoint $M$-alternating rings for some vertex $M$ of $Z_1(G)$, by the same method as above it is shown that $Z_1(G)$ contains a cycle of length 4. This implies that $Z_1(G)$ is a cycle of length 4. Otherwise, for every vertex $M$ of $Z_1(G)$ the two $M$-alternating rings intersect; one is proper and the other one is improper with respect to $M$. So every vertex of the digraph $\tilde{Z}_1(G)$ is of in-degree 1 and out-degree 1, which implies that $\tilde{Z}_1(G)$ is a directed cycle, contradicting Theorem 2.5. So the theorem follows.

Lemma 3.3. Let $G$ be a plane bipartite graph. If $\tilde{Z}(G)$ have two arcs from $M_1$ and $M_2$ to $M_0$ (resp. from $M_0$ to $M_1$ and $M_2$), $M_1 \neq M_2$, then $\tilde{Z}(G)$ has a vertex $M'_0$ that is incident with two out-arcs to $M_1$ and $M_2$ (resp. in-arcs from $M_1$ and $M_2$).

Proof. Put $s_1 := M_1 \oplus M_0$ and $s_2 := M_2 \oplus M_0$. If $(M_1,M_0)$ and $(M_2,M_0)$ are arcs, both $s_1$ and $s_2$ are improper $M_0$-alternating rings. So $s_1$ and $s_2$ are disjoint. Put $M'_0 := M_0 \oplus s_1 \oplus s_2$. It is obvious that $s_1$ and $s_2$ are proper $M'_0$-alternating rings. But $M'_0 \oplus M_1 = (M_0 \oplus s_1 \oplus s_2) \oplus (M_0 \oplus s_1) = s_2$, $M'_0 \oplus M_2 = (M_0 \oplus s_1 \oplus s_2) \oplus (M_0 \oplus s_2) = s_1$. Thus, $(M'_0,M_1)$ and $(M'_0,M_2)$ are arcs of $\tilde{Z}(G)$. For the other case the result can be proven similarly.

Lemma 3.4. Let $G$ be a plane bipartite graph and $Z_1(G)$ a nontrivial component (at least two vertices) of $Z(G)$. Let $M_0$ and $M_1$ be a source (its in-degree 0) and a sink (its out-degree 0) of $\tilde{Z}_1(G)$, respectively. Then a path starting at a source $M_0$ (resp. terminating at a sink $M_1$) is a shortest path of $Z(G)$ if and only if it corresponds to a directed path of $\tilde{Z}_1(G)$ from $M_0$ (resp. to $M_1$).
Proof. Necessity. Let \( P := M_0M_1M_2 \cdots M_k \) (\( k \geq 1 \)) be a shortest path of \( Z_1(G) \) from a source \( M_0 \) of \( Z_1(G) \). We shall prove that \( \tilde{P} \) is a directed path in \( \tilde{Z}_1(G) \) by induction on the length \( k \) of \( P \). If \( k = 1 \), \((M_0,M_1)\) is an arc of \( \tilde{Z}_1(G) \) and the result is obvious. In what follows let \( k \geq 2 \). Suppose that such a path of length less than \( k \) corresponds to a directed path. Since \( P' := M_0M_1M_2 \cdots M_{k-1} \) is also a shortest path of \( Z_1(G) \). By the induction hypothesis \( \tilde{P}' \) is a directed path of \( \tilde{Z}_1(G) \) (cf. Fig. 5). We assert that \((M_{k-1},M_k)\) is an arc of \( \tilde{Z}_1(G) \). Then the result is true. In the following we shall verify this assertion.

By contrary, suppose that \((M_k,M_{k-1})\) is an arc of \( \tilde{Z}_1(G) \). Then both \((M_k,M_{k-1})\) and \((M_{k-2},M_{k-1})\) are arcs of \( \tilde{Z}_1(G) \). By Lemma 3.3 \( \tilde{Z}_1(G) \) has a vertex \( N_1 \) such that both \((N_1,M_{k-2})\) and \((N_1,M_k)\) are arcs of \( \tilde{Z}_1(G) \). If \( N_1 \) is some \( M_i \) (\( 0 \leq j \leq k - 3 \)), \( M_0M_1 \cdots M_j \) is a path of \( Z_1(G) \) between \( M_0 \) and \( M_k \), which is shorter than \( P \), a contradiction. So \( N_1 \) is different from any \( M_i \) (\( 0 \leq i \leq k - 3 \)) and both \((M_{k-3},M_{k-2})\) and \((N_1,M_{k-2})\) are arcs. By repeating the above procedure we arrive in that \( \tilde{Z}_1(G) \) has a sequence of mutually distinct vertices \( N_1,N_2,\ldots,N_{k-1} \) such that \((N_{k-1},N_{k-2},N_{k-1})\) is a directed path of \( \tilde{Z}_1(G) \) and every \( N_i \) (\( 1 \leq i \leq k -1 \)) is different from any \( M_j \) (\( 0 \leq j \leq k - i - 2 \)) and \( M_{k-i}, \) the \((N_i,M_{k-i})\) are all arcs of \( \tilde{Z}_1(G) \) (cf. Fig. 5). In particular, \((N_{k-1},M_0)\) is an arc of \( \tilde{Z}_1(G) \), contradicting that \( M_0 \) is a source of \( \tilde{Z}_1(G) \). So the assertion is verified. Similarly, it is shown that a shortest path of \( Z_1(G) \) terminating at a sink corresponds to a directed path of \( \tilde{Z}_1(G) \).

Sufficiency: Let \( \tilde{P} := (M_0,M_1,M_2,\ldots,M_k) \) be a directed path of \( \tilde{Z}_1(G) \). We proceed by induction on the length \( k \) of \( P \). For \( k = 1 \) it is trivial. Let \( k \geq 2 \). Suppose that \( P_0 := M_0M_1M_2 \cdots M_{k-1} \) is a shortest path of \( Z(G) \) between \( M_0 \) and \( M_{k-1} \) and its length is \( k-1 \). If \( P \) is not a shortest path of \( Z_1(G) \) between \( M_0 \) and \( M_k \), let \( P' := N_0N_1N_2 \cdots N_iM_k \) be a shortest path between \( M_0 \) and \( M_k \). Hence \( l_0 + 1 < k \). It is known that the lengths of all paths in a bipartite graph between two given vertices are of the same parity. Since \( Z(G) \) is bipartite, then \( l_0 + 1 \leq k - 2 \), i.e. \( l_0 \leq k - 3 \). It is obvious that \( P'_0 := M_0N_1N_2 \cdots N_{l_0}M_k \) is a path of \( Z(G) \) between \( M_0 \) and \( M_{k-1} \). As the length of a shortest path between \( M_0 \) and \( M_{k-1} \) is \( k-1 \), the length \( l_0 + 2 \) of \( P'_0 \) is no less than \( k - 1 \). Then \( l_0 \geq k - 3 \). Together with \( l_0 \leq k - 3 \), we have that \( l_0 = k - 3 \). So \( P'_0 \) is also a shortest path of length \( l_0 + 2 = k - 1 \). By the verified necessity \( \tilde{P}_0 \) is a directed path of \( \tilde{Z}(G) \) from \( M_0 \). This implies that \((M_k,M_{k-1})\) is an arc, which contradicts that \((M_{k-1},M_k)\) is an arc of \( \tilde{Z}_1(G) \). Therefore, \( P \) is a shortest path of \( Z_1(G) \). In an analogous method, it can be shown that a directed path terminating at a sink of \( \tilde{Z}_1(G) \) corresponds to a shortest path of \( Z_1(G) \). □
Theorem 3.5. Let $G$ be a plane bipartite graph. Then every nontrivial component of $\tilde{Z}(G)$ has exactly one source and one sink.

Proof. Let $\tilde{Z}_1(G)$ be a nontrivial component of $\tilde{Z}(G)$. As $\tilde{Z}(G)$ has no directed cycles, $\tilde{Z}_1(G)$ has at least one source. If $\tilde{Z}_1(G)$ has two distinct sources $M_0$ and $M'_0$, then $Z_1(G)$ has a shortest path $P$ between $M_0$ and $M'_0$. By Lemma 3.4 $\tilde{P}$ is a directed path of $\tilde{Z}_1(G)$ from $M_0$ to $M'_0$, which contradicts that $M'_0$ is a source of $\tilde{Z}_1(G)$. Then $\tilde{Z}_1(G)$ has exactly one source. It can be shown similarly that $\tilde{Z}_1(G)$ has exactly one sink. \qed

As an immediate consequence, we have the following result.

Corollary 3.6. Let $G$ be a plane bipartite graph. In every nontrivial component of $\tilde{Z}(G)$ there is a directed path from the source to any vertex and a directed path from any vertex to the sink.

Corollary 3.7. Let $G$ be a plane bipartite graph. Then all directed paths between any two vertices in $\tilde{Z}(G)$ are of the same length.

Proof. Let $\tilde{Z}_1(G)$ be a nontrivial component of $\tilde{Z}(G)$. Let $\tilde{P}$ be a directed path of $\tilde{Z}_1(G)$ from vertices $M$ to $M'$. By Corollary 3.6 $\tilde{Z}_1(G)$ has a directed path, denoted by $\tilde{P}^s$, from the source $M_0$ to $M$ and a directed path, denoted $\tilde{P}^t$, from $M'$ to the sink $M_t$. Connecting the three directed paths together, we get a directed path $\tilde{P}^s\tilde{P}\tilde{P}^t$ from the source to the sink, which corresponds to the shortest path of $Z_1(G)$ between $M_0$ and $M_t$ by Lemma 3.4. Hence $P$ is a shortest path of $Z_1(G)$ between given vertices $M$ and $M'$. \qed

4. Cut-vertices of $Z$-transformation graphs

For a plane bipartite graph, a component of its $Z$-transformation graph may contain a cut-vertex. A necessary condition (cf. [10, Lemma 9]) was given for a perfect matching of a plane elementary bipartite graph being a cut-vertex of its $Z$-transformation graph. In this section, we show that such a necessary condition is also sufficient in a more general way. We now describe the following extension to the result as mentioned above and its proof is completely similar to the previous one.

Lemma 4.1. Let $G$ be a plane bipartite graph and $Z_1(G)$ a nontrivial component of $Z(G)$. If $M$ is a cut-vertex of $Z_1(G)$, then

(i) $M$ belongs exactly to two blocks $B_1$ and $B_2$ of $Z_1(G)$,
(ii) $M$ must be the source of one and the sink of the other one in $\bar{B}_1$ and $\bar{B}_2$ and
(iii) Each proper $M$-alternating ring of $G$ intersects each improper $M$-alternating ring of $G$.

In an analogous manner as [10], by Theorem 3.5 and Lemma 4.1 we obtain the following result.
Theorem 4.2. Let $G$ be a plane bipartite graph. Then the block graph of every component of $Z(G)$ is a path.

In the following, we shall give a characterization for a maximum matching of a plane bipartite graph being a cut-vertex of its Z-transformation graph.

Let $Z_1(G)$ be a nontrivial component of $Z(G)$. Let $M_0$ be the source of $Z_1(G)$ and $M'_0$ its sink. For each vertex $M$ of $Z_1(G)$, let $l(M)$ denote the length of a directed path from $M_0$ to $M$ in $\tilde{Z}_1(G)$. Since all directed paths between two vertices of $\tilde{Z}_1(G)$ have same length (Corollaries 3.6 and 3.7), these labels are well defined. Along any directed path from the source the labels of vertices are 0, 1, 2, ..., respectively. Let $V_k := \{M \in V(\tilde{Z}(G)) : l(M) = k\}$ for all integers $0 \leq k \leq t$, where $t$ denotes the length of directed path from the source to the sink in $\tilde{Z}_1(G)$. Then $V_0 = \{M_0\}$, $V_t = \{M'_0\}$ and $\bigcup_{k=0}^{t} V_k$ is a partition of the vertex-set of $Z_1(G)$.

Lemma 4.3. Suppose that $|V_k| \geq 2$ for some integer $1 \leq k \leq t - 1$. Then for every $M_k \in V_k$ there exist $M'_k \in V_k$ other than $M_k$. There are directed paths $P$ and $Q$ from the source $M_0$ to the sink $M_t$ passing through $M_k$ and $M'_k$, respectively. Let $M_{k-i} \in V_{k-i}$ $(0 \leq i \leq k)$ denote the last vertex of $P$ on $Q$ preceding $M_k$; let $M_{k+j} \in V_{k+j}$ $(1 \leq j \leq t-k)$ denote the first vertex of $P$ on $Q$ after $M_k$. For convenience, denote $M_{k-i}M_{k-i+1}\cdots M_{k+j-1}M_{k+j}$ and $M_{k-i}M'_{k-i+1}\cdots M'_{k+j-1}M'_{k+j}$ the subpaths of $P$ and $Q$ from $M_{k-i}$ to $M_{k+j}$, respectively.

We choose such vertex $M'_k$, paths $P$ and $Q$ so that the distance $i+j$ from $M_{k-i}$ to $M_{k+j}$ is minimum. We assert that $i = j = 1$. If $i = 1$ or $j = 1$, by Lemma 3.3 the assertion holds. Otherwise both $i$ and $j$ are larger than 1. Since both $(M'_{k+j-1}, M_{k+j})$ and $(M_{k+j-1}, M_{k+j})$ are arcs of $\tilde{Z}_1(G)$, by Lemma 3.3 there is an $N_{k+j-2} \in V_{k+j-2}$ such that both $(N_{k+j-2}, M_{k+j-1})$ and $(N_{k+j-2}, M'_{k+j-1})$ are arcs of $\tilde{Z}_1(G)$. Further, $N_{k+j}$ is neither $M_{k+j-2}$ nor $M'_{k+j-2}$; otherwise, it would contradict our choice. By the similar manner, we can easily show that there exist $N_{k+s} \in V_{k+s}$ different from $M_{k+s}$ and $M'_{k+s}$ for all $0 \leq s \leq j - 2$ such that $N_{k+j-2}N_{k+j-3}\cdots N_{k+s+1}M'_{k+j-1}$ is a directed path and the $(N_{k+s}, M_{k+s+1})$ are arcs (cf. Fig. 6). By Lemma 3.3 there is an $N_{k-1} \in V_{k-1}$ such that both $(N_{k-1}, N_k)$ and $(N_{k-1}, M_k)$ are arcs. Thus, $N_{k-1}N_kM_{k-1}$ and $N_{k-1}M_kM_{k-1}$ are directed paths from $N_{k-1}$ to $M_{k-1}$, which contradicts our choice that $i+j$ is minimum. So the assertion follows. □
Lemma 4.4. Let $G$ be a plane bipartite graph and $Z_1(G)$ a nontrivial component of $Z(G)$. Then every $V_k$ ($0 < k < t$) is both independent set and minimal cut-set of $Z_1(G)$. Further, $M$ is a cut-vertex of $Z_1(G)$ if and only if $0 < l(M) < t$ and $|V_l(M)|=1$.

Proof. For any adjacent vertices $M$ and $M'$ in $Z_1(G)$, we assert that $|l(M') - l(M)|=1$. Without loss of generality, assume that $(M, M')$ is an arc of $\tilde{Z}_1(G)$. Then there is a directed path from $M_0$ to $M'$ consisting of a directed path from $M_0$ to $M$ and an arc from $M$ to $M'$. Thus, $l(M') = l(M) + 1$ and the assertion holds. Further, every $V_k$ is an independent set. Since no vertices in $V_i$ are adjacent to some vertex in $V_j$ whenever $j \geq i+1$, every $V_k$ is a cut-set of $Z_1(G)$. It is obvious that $Z_1(G) - V_k$ has exactly two components, which are deduced by $\bigcup_{i=0}^{k-1} V_i$ and $\bigcup_{i=k+1}^{t} V_i$ separately. Since every vertex of $V_k$ has a neighbor in both $V_{k-1}$ and $V_{k+1}$, any proper subset of $V_k$ is not cut-set of $Z_1(G)$; namely, $V_k$ is a minimal cut-set. Accordingly, the latter of the lemma follows.

Theorem 4.5. Let $G$ be a plane bipartite graph and $Z_1(G)$ a nontrivial component of $Z(G)$. Then a vertex $M$ of $Z_1(G)$ is a cut-vertex if and only if $G$ has both proper and improper $M$-alternating rings and every proper $M$-alternating ring of $G$ intersects every improper $M$-alternating ring of $G$.

Proof. If $M$ is a cut-vertex of $Z_1(G)$, by Lemma 4.4 $V_k = \{M\}$ for some integer $1 \leq k \leq t-1$. Then $\tilde{Z}_1(G)$ has a directed path from the source to the sink passing through $M$. Thus, $G$ has both proper and improper $M$-alternating rings. Further by Lemma 4.1(iii) the necessity follows.

Conversely, for a maximum matching $M$ of $G$ belonging to $Z_1(G)$, suppose that $G$ has both proper and improper $M$-alternating rings and every proper $M$-alternating ring of $G$ intersects every improper $M$-alternating ring of $G$. Thus $M \in V_k$ for some $1 \leq k \leq t-1$. If $M$ is not cut-vertex of $Z_1(G)$, by Lemma 4.4 we have that $|V_k| \geq 2$. By Lemma 4.3 there exist $M' \in V_k$ other than $M$, $M_{k-1} \in V_{k-1}$ and $M_{k+1} \in V_{k+1}$ such that $M_{k-1}M'M_{k+1}$ and $M_{k-1}MM_{k+1}$ are directed paths from $M_{k-1}$ to $M_{k+1}$ in $\tilde{Z}_1(G)$. Let $s = M_{k-1} \oplus M$ and $s' = M_{k-1} \oplus M'$. Since both $s$ and $s'$ are proper $M_{k-1}$-alternating rings and thus disjoint by Lemma 2.6. However, $s$ and $s'$ are improper and proper $M$-alternating rings respectively, which is a contradiction.

Finally, we would like to point out that the concept for $Z$-transformation graph can be extended on all matchings with any given size in a plane bipartite graph. The results obtained in this paper still hold.

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References