

STABILITY OF THE ANISOTROPIC MAXWELL EQUATIONS WITH A CONDUCTIVITY TERM

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ABSTRACT. The dynamic Maxwell equations with a conductivity term are considered. Conditions for the exponential and strong stability of an initial-boundary value problem are given. The permeability and the permittivity are assumed to be 3×3 symmetric, positive definite tensors. A result concerning solutions of higher regularity is obtained along the way.

1. The problem. Consider the evolution of the electromagnetic field (e, h) in a bounded medium $\Omega \subset \mathbb{R}^3$ with electric permittivity ε , magnetic permeability μ , and conductivity σ , in the absence of external currents, that is the system of partial differential equations,

$$\begin{aligned} (\varepsilon e)_t - \nabla \times h + \sigma e &= 0 \\ (\mu h)_t + \nabla \times e &= 0 \end{aligned} \quad \text{in } Q := \mathbb{R}_+ \times \Omega, \quad (1)$$

supplemented with the initial conditions

$$e(0, x) = e^0(x), \quad h(0, x) = h^0(x), \quad \text{for } x \in \Omega, \quad (2)$$

and the boundary condition along the spatial boundary $\Gamma = \partial\Omega$

$$\nu \times e = 0, \quad \text{for } (t, x) \in \mathbb{R}_+ \times \Gamma. \quad (3)$$

Here ν is the exterior unit normal vector field of Ω along Γ . Throughout this work $\varepsilon, \mu \in L^\infty(\Omega, \mathbb{C}^{3 \times 3})$ are Hermitian and uniformly positive definite matrices. Furthermore, $\sigma \in L^\infty(\Omega, \mathbb{C}^{3 \times 3})$ is assumed to be Hermitian and positive semi-definite. The set Ω will be assumed to be open, non-empty, bounded, and connected with a Lipschitz boundary Γ . We introduce the Hilbert space

$$\mathcal{H} = L^2(\Omega, \mathbb{C}^3) \times \{h \in L^2(\Omega, \mathbb{C}^3) : \nabla \cdot (\mu h) = 0 \text{ in } \Omega, \langle \nu, \mu h \rangle = 0 \text{ in } \Gamma\}$$

equipped with the inner product

$$(u, v)_{\mathcal{H}} = \int_{\Omega} \langle \varepsilon u_1, v_1 \rangle dx + \int_{\Omega} \langle \mu u_1, v_2 \rangle dx, \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$

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where $\langle \cdot, \cdot \rangle$ is the standard scalar product in \mathbb{C}^3 . A linear, densely defined operator \mathcal{A} in \mathcal{H} is given by

$$\mathcal{D}(\mathcal{A}) = [H_0(\text{curl}, \Omega) \times H(\text{curl}, \Omega)] \cap \mathcal{H} \text{ and } \mathcal{A} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \varepsilon^{-1}(\nabla \times u_2) - \varepsilon^{-1}\sigma u_1 \\ -\mu^{-1}(\nabla \times u_1) \end{bmatrix}.$$

Semigroup theory provides the following existence and uniqueness result.

Proposition 1. *Suppose that Ω is a open, non-empty, and bounded subset of \mathbb{R}^3 with a Lipschitz boundary. Then the operator \mathcal{A} is the generator of a C_0 -semigroup of contractions $S(t)$. Given $u^0 = (e^0, h^0) \in \mathcal{H}$, there exists a unique mild solution $S(\cdot)u^0 \in C([0, \infty), \mathcal{H})$ to the initial-boundary value problem (1) - (3). Furthermore, for $u^0 \in \mathcal{D}(\mathcal{A})$ there exists a unique classical solution $S(\cdot)u \in C([0, \infty), \mathcal{D}(\mathcal{A})) \cap C^1([0, \infty), \mathcal{H})$ to the initial-boundary value problem (1) - (3).*

Proof. Using integration by parts one shows that the operator \mathcal{A} is dissipative, that is $\Re(\mathcal{A}u, u)_{\mathcal{H}} \leq 0$ for all $u \in \mathcal{D}(\mathcal{A})$. One computes

$$\begin{aligned} \Re(\mathcal{A}u, u)_{\mathcal{H}} &= \Re \int_{\Omega} [\langle \nabla \times u_2, u_1 \rangle - \langle \sigma u_1, u_1 \rangle - \langle \nabla \times u_1, u_2 \rangle] dx \\ &= - \int_{\Omega} \langle \sigma u_1, u_1 \rangle dx \leq 0, \end{aligned}$$

since σ is positive semi-definite. Note that there is no integral over the boundary because of $u \in \mathcal{D}(\mathcal{A})$. Next we will show that range of $I - \mathcal{A}$ is \mathcal{H} . Given $\begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{H}$, consider the following variational equation for u

$$\int_{\Omega} \langle (\varepsilon + \sigma)u, w \rangle dx + \int_{\Omega} \langle \mu^{-1}(\nabla \times u), \nabla \times w \rangle dx = \int_{\Omega} \langle \varepsilon f, w \rangle dx + \int_{\Omega} \langle g, \nabla \times w \rangle dx,$$

for all $w \in H_0(\text{curl}, \Omega)$. Note that the left-hand side is a bounded and coercive bilinear form on $H_0(\text{curl}, \Omega)$, considered as a Hilbert space with the norm $(\|u\|_{L^2(\Omega)} + \|\nabla \times u\|_{L^2(\Omega)})^{1/2}$. The right-hand side of the equation above defines a continuous linear functional on $H_0(\text{curl}, \Omega)$. Hence, by the Lax-Milgram lemma, there exists a unique solution $u \in H_0(\text{curl}, \Omega)$.

Define $v = g - \mu^{-1}\nabla \times u \in L^2(\Omega)$ and observe that

$$\int_{\Omega} \langle v, \nabla \times w \rangle dx = \int_{\Omega} \langle (\varepsilon + \sigma)u, w \rangle dx - \int_{\Omega} \langle \varepsilon f, w \rangle dx$$

for all $w \in H_0(\text{curl}, \Omega)$. This implies $\nabla \times v \in L^2(\Omega)$ and

$$\varepsilon u - \nabla \times v + \sigma u = \varepsilon f.$$

From above we know that

$$\mu v + \nabla \times u = \mu g.$$

These two equations can be written as $(I - \mathcal{A}) \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$. By the Lumer-Phillips Theorem [16, Theorem 4.3], the operator \mathcal{A} generates a contraction semigroup of linear operators $S(t)$. The statement regarding the solutions of the initial boundary value problem follow from semigroup theory, see for example [16, Chapter 4]. \square

We note that every mild solution is also a weak solution. The norm $\|u\|_{\mathcal{H}} = \sqrt{(u, u)_{\mathcal{H}}}$ serves as the energy functional. For mild solutions $u(t) = (e(t), h(t))$,

one has the identity

$$\|u(t_2)\|_{\mathcal{H}}^2 - \|u(t_1)\|_{\mathcal{H}}^2 = - \int_{t_1}^{t_2} \int_{\Omega} \langle \sigma(x)e(t, x), e(t, x) \rangle dx dt, \quad (4)$$

for any $0 \leq t_1 < t_2$. This formula is proved at first for classical solutions and then extended to mild solutions relying on the density of $\mathcal{D}(\mathcal{A})$ in \mathcal{H} and the fact that $S(t)$ is a contraction semigroup.

The derivative $\partial_t u$ of a classical solution to the initial-boundary value problem (1)-(3) is a mild solution to the same initial-boundary value problem (but with different initial conditions). Hence, for classical solutions we have the identity

$$\|\partial_t u(t_2)\|_{\mathcal{H}}^2 - \|\partial_t u(t_1)\|_{\mathcal{H}}^2 = - \int_{t_1}^{t_2} \int_{\Omega} \langle \sigma(x)\partial_t e(t, x), \partial_t e(t, x) \rangle dx dt. \quad (5)$$

This work is dedicated to the stability of the this initial-boundary value problem. We will give sufficient conditions under which mild solutions converge to the zero solution at an exponential rate. This will work if the conductivity is positive definite on all of Ω , see Theorem 2.1 below. Furthermore, in Theorem 4.1 we will show that mild solutions converge to zero in norm, provided that the electric field is initially divergence free and that the conductivity is positive definite on an open non-empty subset of Ω while identical zero on the complement of ω in Ω .

Both results will require that Ω is simply connected. This is in contrast to the fairly general result on observability from the boundary where the boundary Γ must be simply connected [7]. For further discussions in this directions we refer to Belishev and Glasman [1].

Note that the stabilizing mechanism of this dynamical system is provided solely by the conductivity term. The energy identity (4) shows that the energy of the system is conserved if the conductivity vanishes everywhere on Ω .

The stability of many hyperbolic systems, in particular the wave equation, has been studied extensively. For a nice discussion we refer to the book by Rauch [19, Section 5.6]. Motivated by scattering theory the energy decay was initially studied for the exterior problem [13, 10]. The first result on energy decay for dissipative symmetric hyperbolic systems of first order on a bounded region was given by Iwasaki [9]. Another general result, due to Rauch and Taylor [20], gives a sufficient criterion for exponential stability for dissipative symmetric hyperbolic systems of first order. In both works, the dissipative term is supposed to act on all components of the vector function. This is not a natural assumption for Maxwell's equations: It is one of its peculiarities that the conductivity term applies only to the electric field. There is no term of lower order in the second equation of (1).

The first work to discuss the stability of the initial-boundary value problem above is the paper by Phung [17, Section 5]. This analysis is limited to the case with constant, isotropic coefficients. If the conductivity is strictly positive on a subset of Ω which controls Ω geometrically, then the energy decays at an exponential rate [17, Theorem 5.5]. Nicaise and Pignotti [14] extended this result to variable, isotropic coefficients. In section 2 we use the same approach as Phung and establish an observability inequality similar to Proposition 5.1 in [17].

The problem of strong stability of mild solutions to Maxwell's equations with a localized conductivity term is much more challenging than the same problem for the wave equation with damping term. Solutions to (1) - (3) with a local conductivity term do not approach the zero equilibrium as long as the initial data of the electric field are not divergence-free. Hence, we will assume divergence-free initial data. At

the core of the problem is that the domain of the operator \mathcal{A} is not compactly embedded in the state space \mathcal{H} . We overcome this by proving H^1 regularity of classical solutions in section 3. However, this result comes at a price: the regularity of the coefficients has to be increased from bounded to Lipschitz.

The H^1 -regularity of the classical solutions to this initial-boundary value problems with initial data in H^1 seems to be an interesting result in its own right. Since the lateral boundary of the space-time cylinder is characteristic, the result concerning higher regularity for hyperbolic system due to Rauch and Massey [21] does not apply. In their classical paper on the uniformly characteristic initial-boundary value problem, Majda and Osher, gave criteria for higher regularity [12, Theorem 3]. These criteria are difficult to verify, but the solution to the isotropic Maxwell equations with initial data in H^1 is in fact in H^1 at all times [12, Proposition 2.2]. We should also mention that both works cited above require boundary conditions of the Kreiss-Sakamoto type which our boundary condition (3) is not. Later, Ohkubo [15] gave a somewhat simpler criterion for higher regularity. Nevertheless, his criterion is not easy to verify in the case of anisotropic coefficients.

We chose to extend Maxwell's equations by two scalar equation which are generalizations of the divergence equations. However, since the conductivity precludes the electric field from being divergence free, one of the extended equation has an integral term. The resulting integro-differential equation can be treated using perturbation theory of semi-groups [5]. Higher regularity of solutions to initial-boundary value problems is of importance for non-linear Maxwell problems [23] where a result similar to our Theorem 3.1 is given.

For background material on the function spaces used, we refer to the books by Dautray and Lions [4, Chapter IX] and Cessenat [3, Section 2.1]. Note that all derivatives are understood in the sense of distributions and that the restriction of a function defined on Ω to the boundary Γ is understood in the sense of the trace in the appropriate Sobolev space. For the convenience of the reader, we include a list of the Sobolev-type spaces which are typically used in the analysis of Maxwell's equations.

$$\begin{aligned} H(\text{curl}, \Omega) &= \{u \in L^2(\Omega, \mathbb{C}^3) : \nabla \times u \in L^2(\Omega, \mathbb{C}^3)\} \\ H_0(\text{curl}, \Omega) &= \{u \in H(\text{curl}, \Omega) : \nu \times u = 0 \text{ on } \Gamma\} \\ H(\text{curl}0, \Omega) &= \{u \in L^2(\Omega, \mathbb{C}^3) : \nabla \times u = 0 \text{ in } \Omega\} \\ H_0(\text{curl}0, \Omega) &= \{u \in H(\text{curl}0, \Omega) : \nu \times u = 0 \text{ on } \Gamma\} \\ H(\text{div}, \Omega) &= \{u \in L^2(\Omega, \mathbb{C}^3) : \nabla \cdot u \in L^2(\Omega)\} \\ H_0(\text{div}, \Omega) &= \{u \in H(\text{div}, \Omega) : \langle u, \nu \rangle = 0 \text{ on } \Gamma\} \\ H(\text{div}0, \Omega) &= \{u \in L^2(\Omega, \mathbb{C}^3) : \nabla \cdot u = 0 \text{ in } \Omega\} \\ H_0(\text{div}0, \Omega) &= \{u \in H(\text{div}0, \Omega) : \langle u, \nu \rangle = 0 \text{ on } \Gamma\} \\ H_{\text{t}0}^1(\Omega) &= \{u \in H^1(\Omega, \mathbb{C}^3) : \nu \times u = 0 \text{ on } \Gamma\} = H_0(\text{curl}, \Omega) \cap H(\text{div}, \Omega) \\ H_{\text{n}0}^1(\Omega) &= \{u \in H^1(\Omega, \mathbb{C}^3) : \langle \nu, u \rangle = 0 \text{ on } \Gamma\} = H(\text{curl}, \Omega) \cap H_0(\text{div}, \Omega) \\ \mathbb{H}_1(\Omega) &= H(\text{curl}0, \Omega) \cap H_0(\text{div}0, \Omega) \\ \mathbb{H}_2(\Omega) &= H_0(\text{curl}0, \Omega) \cap H(\text{div}0, \Omega) \end{aligned}$$

The last two spaces are referred to as cohomology spaces. They are finite dimensional and characterize the topological properties of Ω . We give the following characterizations of these spaces. Let $\Gamma_1, \dots, \Gamma_N$ be the components of $\Gamma = \partial\Omega$. The set

Ω is made simply connected by a finite number of cuts $\Sigma_1, \dots, \Sigma_M$ which are regular hypersurfaces which are non-tangential to Γ . Then the set $\dot{\Omega} = \Omega \setminus \cup_{j=1}^M \Sigma_j$ is simply connected. We have

$$\mathbb{H}_1(\Omega) = \{u \in L^2(\Omega)^3 : u = (\nabla q) \in \dot{\Omega}, q \in H^1(\dot{\Omega}), \Delta q = 0 \text{ in } \dot{\Omega}, \\ \partial_\nu u|_\Gamma = 0, [q]|_{\Sigma_j} = C_j, [\partial_\nu q]|_{\Sigma_j} = 0, j = 1, 2, \dots, M\}$$

where the $C_j, j = 1, 2, \dots, M$ are free constants and $[v]|_{\Sigma_j}$ denotes the jump of the function v across the surface Σ_j . Furthermore,

$$\mathbb{H}_2(\Omega) = \{u \in L^2(\Omega)^3 : u = \nabla q, q \in H^1(\Omega), \Delta q = 0, q|_{\Gamma_j} = C_j, j = 1, 2, \dots, N\}. \tag{6}$$

In particular $\mathbb{H}_1 = \{0\}$ for a simply connected region Ω and $\dim \mathbb{H}_2 = N - 1$.

In the next section we will also need the following result which provides orthogonal decompositions in $L^2(\Omega, \mathbb{C}^3)$, see [3, Chapter 2, Theorem 8, 10] or [4, Chapter IX, §1, Propositions 1-4]. In the literature these orthogonal decompositions are also known as Helmholtz or Hodge decompositions.

Theorem 1.1. *If $f \in H(\text{curl}0, \Omega)$, then $f = \nabla p + h_1$ with $p \in H^1(\Omega)$ and $h_1 \in \mathbb{H}_1(\Omega)$. In this decomposition the function p is unique up to a constant.*

If $f \in H_0(\text{curl}0, \Omega)$, then exists unique $p \in H_0^1(\Omega)$ and $h_2 \in \mathbb{H}_2(\Omega)$ such that $f = \nabla p + h_2$.

If $g \in H(\text{div}0, \Omega)$, then $g = \nabla \times a + h_2$ with $a \in H^1(\Omega, \mathbb{C}^3)$ and $h_2 \in \mathbb{H}_2(\Omega)$. In this decomposition the function a is unique under the additional conditions

$$\nabla \cdot a = 0 \text{ in } \Omega, \quad \langle a, \nu \rangle = 0 \text{ in } \Gamma, \quad \int_{\Sigma_j} \langle a, \nu \rangle d\sigma = 0 \text{ for } j = 1, 2, \dots, M. \tag{7}$$

If $g \in H_0(\text{div}0, \Omega)$, then $g = \nabla \times a + h_1$ with $a \in H_{t0}^1(\Omega)$ and $h_1 \in \mathbb{H}_1(\Omega)$. The function a is unique if subject to the conditions

$$\nabla \cdot a = 0 \text{ in } \Omega, \quad \int_{\Gamma_j} \langle a, \nu \rangle d\gamma = 0 \text{ for } j = 1, 2, \dots, N. \tag{8}$$

Every $u \in L^2(\Omega, \mathbb{C}^3)$ has the Helmholtz decomposition

$$u = \nabla p + \nabla \times a + h_2 \quad \text{with } p \in H_0^1(\Omega), a \in H_{n0}^1(\Omega), h_2 \in \mathbb{H}_2(\Omega).$$

If a is subject to the conditions (7), then the a, p, h_2 are unique. Another Helmholtz decomposition is

$$u = \nabla p + \nabla \times a + h_1 \quad \text{with } p \in H^1(\Omega), a \in H_{t0}^1(\Omega), h_1 \in \mathbb{H}_1(\Omega).$$

If a is subject to conditions (8), then a, h_1 are unique and p is unique up to constants on each component of Ω .

Remark 1. In the following section this theorem will be applied to vector fields in the space $C([0, \infty), L^2(\Omega, \mathbb{C}^3))$. The Helmholtz decompositions will then be continuous in t since they are obtained as solutions to elliptic boundary problems with a time parameter. Elliptic estimates guarantee then the continuity of the solution with respect to time.

2. Exponential stability.

Theorem 2.1. *Suppose σ is uniformly positive definite on the simply connected region Ω . Then there exists a positive constants α such that for all mild solutions $u(t) = (e(t), h(t))$ to (1 - 3) in the sense of Proposition 1,*

$$\|u(t)\|_{\mathcal{H}} \lesssim e^{-\alpha t} \|u^0\|_{\mathcal{H}} . \tag{9}$$

Here and henceforth we write $a \lesssim b$ for $a \leq Cb$ where the constant C does depend only on $\Omega, \varepsilon, \mu,$ and σ . This theorem is proved by establishing an observability inequality, see Proposition 3 below. But first we need a special Helmholtz decomposition of the magnetic and the electric field which will be established with the aid of Theorem 1.1.

Proposition 2. *Suppose that the region Ω is simply connected and that $u(t) = (e(t), h(t))$ is a mild solution to (1) - (3) in the sense of Proposition 1. Then there exist vector fields $a \in C([0, \infty), H_{t_0}^1(\Omega)) \cap C^1([0, \infty), L^2(\Omega, \mathbb{C}^3)), h_2 \in C([0, \infty), \mathbb{H}_2(\Omega)),$ and a scalar function $p \in C([0, \infty), H_0^1(\Omega))$ such that*

$$\mu h = \nabla \times a \quad \text{and} \quad e = -\partial_t a + \nabla p + h_2.$$

Furthermore, $\|a(t)\|_{L^2(\Omega)} \lesssim \|\nabla \times a(t)\|_{L^2(\Omega)},$ uniformly on compact time intervals.

Proof. Using Theorem 1.1 we have,

$$\mu h = \nabla \times a$$

where $a \in C([0, \infty), H_{t_0}(\Omega)), \nabla \cdot a = 0$ in $\Omega,$ and $\int_{\Gamma_j} \langle a, \nu \rangle d\gamma = 0$ for $j = 1, 2, \dots, N.$ There is no function in $\mathbb{H}_1(\Omega)$ here, since Ω is simply connected. Using an orthogonal decomposition from Theorem 1.1 for the electric field, we know that

$$e = \nabla \times b + \nabla p + h_2$$

for some $b \in C([0, \infty), H_{n_0}^1(\Omega)), p \in C([0, \infty), H_0^1(\Omega, \mathbb{C}^3)),$ and $h_2 \in C([0, \infty), \mathbb{H}_2(\Omega)).$ Since (e, h) is a weak solution to the Maxwell system (1) we have

$$\int_Q \langle \psi_t, \nabla \times a \rangle dt dx = \int_Q \langle \nabla \times \psi, e \rangle dt dx$$

for all $\psi \in C_0^1((0, \infty), L_2(\Omega, \mathbb{C}^3)) \cap C_0((0, \infty), H^1(\Omega, \mathbb{C}^3)).$ Choose some $\delta_0 > 0.$ Then, for $\psi \in C_0^1((\delta_0, \infty), L_2(\Omega, \mathbb{C}^3)) \cap C_0((\delta_0, \infty), H^1(\Omega, \mathbb{C}^3))$ we have

$$\int_Q \langle \psi_t, \nabla \times a_\delta \rangle dt dx = \int_Q \langle \nabla \times \psi, e_\delta \rangle dt dx , \quad \delta < \delta_0,$$

where $a_\delta(t, x) = \int_0^\infty \rho_\delta(t-s)a(s, x) ds$ and $e_\delta(t, x) = \int_0^\infty \rho_\delta(t-s)e(s, x) ds$ are the regularization of a and e in the time variable, respectively. Here $\rho_\delta \in C_0^\infty(\mathbb{R})$ is an approximate identity, that is $\int \rho_\delta(t) dt = 1$ and $\text{supp } \rho \subset (-\delta, \delta).$ After integration by parts in time and space, the identity above can be written as

$$\int_Q \langle \nabla \times \psi, (a_\delta)_t + e_\delta \rangle dt dx = 0 .$$

This shows that $(a_t)_\delta + e_\delta \in H_0(\text{curl}0, \Omega)$ for all $t > \delta_0$ and another application of Theorem 1.1 gives

$$(a_t)_\delta + e_\delta = \nabla q_\delta + k_\delta \quad \text{when } t > \delta_0 > \delta \tag{10}$$

for some $q_\delta \in C^\infty((\delta_0, \infty), H_0^1(\Omega))$ and $k_\delta \in C^\infty((\delta_0, \infty), \mathbb{H}_2(\Omega)).$ As a matter of fact, equation (10) is a Helmholtz decomposition of e_δ due to the properties of $(a_t)_\delta.$

Finally, we use that for any $T > 0$, $e_\delta \rightarrow e$ in $C([0, T], L^2(\Omega))$ for $\delta \rightarrow 0$ and that the Helmholtz decomposition is orthogonal. Thus

$$\begin{aligned} \|e(t) - e_\delta(t)\|_{L^2(\Omega)}^2 &= \|\nabla q_\delta(t) - \nabla p(t)\|_{L^2(\Omega)}^2 \\ &\quad + \|\nabla \times b(t) + (a_t)_\delta(t)\|_{L^2(\Omega)}^2 + \|h_2(t) - k_\delta(t)\|_{L^2(\Omega)}^2, \end{aligned}$$

for $\delta_0 < t < T$ which gives

$$(a_t)_\delta \rightarrow -\nabla \times b \in L^\infty((\delta_0, T), L^2(\Omega, \mathbb{C}^3)) \quad \delta \rightarrow 0,$$

and $a_t = -\nabla \times b$ in $L^2(\Omega, \mathbb{C}^3)$ for all $t \in (\delta_0, T)$. Since $\nabla \times b \in C([0, \infty), L^2(\Omega, \mathbb{C}^3))$ we conclude $a_t \in C([0, \infty), L^2(\Omega, \mathbb{C}^3))$ for δ_0 and T are arbitrary positive reals.

For the second statement we argue by contradiction. Suppose the inequality does not hold. Then there exists a sequence $\{a_n\} \in H_{t_0}^1(\Omega)$ with $\nabla \cdot a_n = 0$ for all n such that

$$\|a_n\|_{L^2(\Omega)} = 1 \quad \text{and} \quad \|\nabla \times a_n\|_{L^2(\Omega)} \rightarrow 0 \text{ for } n \rightarrow \infty.$$

Hence the $H^1(\Omega)$ norm of a_n is uniformly bounded and by Rellich's Theorem the sequence $\{a_n\}$ must have a strongly convergent subsequence in $L^2(\Omega, \mathbb{C}^3)$. The limit function a satisfies $a \in \mathbb{H}_2(\Omega)$ and $\|a\|_{L^2(\Omega)} = 1$.

This will lead to a contradiction when we use the characterization of the cohomology space $\mathbb{H}_2(\Omega)$ given in (6). There exists a function $\phi \in H^1(\Omega)$ such that $a = \nabla \phi$, $\phi|_{\Gamma_j} = C_j$ and from the beginning of the proof we know that $\int_{\Gamma_j} \langle \nu, a \rangle d\gamma = 0$. Hence, using integration by parts

$$\|a\|_{L^2(\Omega)}^2 = \int_{\Omega} \langle a, \nabla \phi \rangle dx = \int_{\Gamma} \langle \nu, a \rangle \phi d\gamma = 0,$$

which contradicts $\|a\|_{L^2(\Omega)} = 1$. □

The next proposition states an observability inequality which will be crucial for the proof of Theorem 2.1.

Proposition 3. *Suppose that all the assumptions of Theorem 2.1 are met. Then, for all $\delta > 0$ and $T > 2\delta$, we have the estimate*

$$\int_{\delta}^{T-\delta} \int_{\Omega} \langle \mu h, h \rangle dxdt \lesssim \int_0^T \int_{\Omega} \langle \sigma e, e \rangle dxdt. \tag{11}$$

Proof. Note that

$$\int_0^T \int_{\Omega} \langle \psi_t, \varepsilon e \rangle + \langle \nabla \times \psi, h \rangle - \langle \psi, \sigma e \rangle dxdt = 0$$

for all $\psi \in C([0, T], H_{t_0}^1(\Omega)) \cap C^1([0, T], L^2(\Omega, \mathbb{C}^3))$ satisfying $\psi(0) = \psi(T) = 0$. Choosing $\varphi \in C_0^\infty(0, T)$ with $\varphi \equiv 1$ on $(\delta, T - \delta)$ and $\psi = \varphi^2 a$, where a is the function introduced in the previous proposition, gives for any $\alpha > 0$

$$\begin{aligned} \int_{\delta}^{T-\delta} \int_{\Omega} \langle \mu h, h \rangle dxdt &\leq \int_0^T \int_{\Omega} \varphi^2(t) \langle \nabla \times a, h \rangle dxdt \\ &= - \int_0^T \int_{\Omega} \varphi^2(t) \langle a_t, \varepsilon e \rangle dxdt - \int_0^T \int_{\Omega} 2\varphi'(t) \varphi(t) \langle a, \varepsilon e \rangle dxdt \\ &\quad + \int_0^T \int_{\Omega} \varphi^2(t) \langle a, \sigma e \rangle dxdt \\ &\leq C_\alpha \int_0^T \|e(t)\|_{L^2(\Omega)}^2 dt + \int_0^T \|a_t(t)\|_{L^2(\Omega)}^2 dt + \alpha \int_0^T \int_{\Omega} \varphi^2(t) |a|^2 dxdt \end{aligned}$$

where the Cauchy-Schwarz inequality was used in the last step. Using now the orthogonal decomposition of e from Proposition 2 gives

$$\int_0^T \int_{\Omega} \varphi^2(t) \langle \mu h, h \rangle dx dt = \int_0^T \int_{\Omega} \varphi^2(t) \langle \nabla \times a, \mu^{-1}(\nabla \times a) \rangle dx dt \lesssim \int_0^T \int_{\Omega} \langle e, e \rangle dx dt.$$

Finally, the integral on the right-hand side can be estimated by some constant times

$$\int_0^T \int_{\Omega} \langle \sigma e, e \rangle dx dt$$

since σ is uniformly positive definite. \square

Theorem 2.1 is now a corollary of the observability inequality (11). For the convenience of the reader we include its proof. Combining inequalities (11) and (4) gives

$$\begin{aligned} \|u(T)\|_{\mathcal{H}}^2 &= \|u(0)\|_{\mathcal{H}}^2 - \int_0^T \int_{\Omega} \langle \sigma e, e \rangle dx dt \\ &\leq \|u(0)\|_{\mathcal{H}}^2 - C \int_{\delta}^{T-\delta} \|u(t)\|_{\mathcal{H}}^2 dt \leq \|u(0)\|_{\mathcal{H}}^2 - C(T-2\delta)\|u(T)\|_{\mathcal{H}}^2, \end{aligned}$$

and hence,

$$\|u(T)\|_{\mathcal{H}} \leq \frac{\|u(0)\|_{\mathcal{H}}}{\sqrt{1 + C(T-2\delta)}}.$$

Now one chooses $T > 2\delta$ and obtains by translating the time interval from $(0, T)$ to $((n-1)T, nT)$ for $n = 1, 2, 3, \dots$ and using the inequality above in each of these intervals

$$\|u(nT)\|_{\mathcal{H}} \leq B^n \|u(0)\|_{\mathcal{H}}, \quad \text{with} \quad B = \frac{1}{\sqrt{1 + C(T-2\delta)}} < 1, \quad n = 1, 2, \dots$$

Since the energy is decreasing, one obtains for all $t \in [nT, (n+1)T)$

$$\|u(t)\|_{\mathcal{H}} \leq \|u(nT)\|_{\mathcal{H}} \leq B^n \|u(0)\|_{\mathcal{H}} \leq \frac{B^{t/T}}{B} \|u(0)\|_{\mathcal{H}}$$

and choosing $\alpha = (-\log B)/T$ gives (9).

Remark 2. It is known that our topological assumption, as well the assumption on initial data for the magnetic field cannot be relaxed.

If Ω is not simply connected, then with initial data $u^0 = (0, h^0)$ with $h^0 \in \mathbb{H}_1(\Omega)$, then the solution to our initial-boundary value problem is constant in time, that is $u(t) = S(t)u^0 = u^0$ for all $t > 0$, and does not decay as long as h^0 is not identical zero.

Furthermore, the boundary condition on the initial condition for the magnetic field cannot be relaxed. If $u^0 = (0, \nabla g)$ for some non-constant $g \in H^1(\Omega, \mathbb{C}^3)$ satisfying the elliptic equation $\nabla \cdot (\mu \nabla g) = 0$, then the solution is $u(t) = S(t)u^0 = (0, \nabla g)$ for all $t > 0$ and does not decay.

3. On the regularity of classical solutions. Here we will show that the classical solutions discussed in Proposition 1 are differentiable provided some additional assumptions are imposed.

Theorem 3.1. *Suppose that the assumptions of Proposition 1 are satisfied and that, in addition, the coefficients satisfy $\varepsilon, \mu, \sigma \in W^{1,\infty}(\Omega, \mathbb{C}^{3 \times 3})$ and that the initial data $e^0, h^0 \in \mathcal{D}(\mathcal{A})$ satisfy $\varepsilon e^0 \in H(\operatorname{div}0, \Omega)$. Then the classical unique solution has the regularity $(e, h) \in C([0, \infty), H^1(\Omega, \mathbb{C}^6))$.*

Proof. Following the approach given in [2, Section 2] (see also [18, p.172]), we extend the initial boundary value problem (1) - (3) to an initial boundary value problem for the 8×8 system

$$\begin{aligned} (\varepsilon e)_t - \nabla \times h - \varepsilon \nabla w_2 + \sigma e &= 0 \\ \partial_t w_1 + \nabla \cdot (\mu h) &= 0 \\ (\mu h)_t + \nabla \times e + \mu \nabla w_1 &= 0 \quad \text{in } Q := \mathbb{R}_+ \times \Omega, \\ \partial_t w_2 - \nabla \cdot (\varepsilon e) - \int_0^t \nabla \cdot (\sigma e) ds &= 0 \end{aligned} \tag{12}$$

$$\begin{aligned} e(0, x) = e^0(x), \quad h(0, x) = h^0(x), \quad w_1(0, x) = w_2(0, x) = 0 & \quad x \in \Omega \\ (\nu \times e)(t, x) = 0, \quad w_1(t, x) = w_2(t, x) = 0 & \quad (t, x) \in \mathbb{R}_+ \times \Gamma. \end{aligned}$$

This system is of interest since a solution to this system contains always a solution to the original problem (1) - (3). This follows from the observation that the functions w_1, w_2 both solve a second-order hyperbolic equation with zero initial data (position and velocity) and zero Dirichlet boundary data. For example, if one takes the divergence of the first equation and the time derivative of the last equation, one has

$$\begin{aligned} \nabla \cdot (\varepsilon e)_t + \nabla \cdot (\varepsilon \nabla w_2) + \nabla \cdot (\sigma e) &= 0 \\ \partial_{tt}^2 w_2 - \nabla \cdot (\varepsilon e)_t - \nabla \cdot (\sigma e) &= 0 \end{aligned}$$

Adding these two equations gives $\partial_{tt}^2 w_2 - \nabla \cdot (\varepsilon \nabla w_2) = 0$, and from the last equation of the system one infers $\partial_t w_2(0, x) = 0$ because of $\nabla \cdot (\varepsilon e^0) = 0$ in Ω . The uniqueness theorem of the Cauchy-Dirichlet problem for second-order hyperbolic equations tells us that $w_1 = w_2 \equiv 0$.

We will show that this initial-boundary value problem has a unique classical solution $(e, w_1, h, w_2) \in C([0, \infty), H^1(\Omega, \mathbb{C}^8))$. In case of a zero conductivity, this is fairly straightforward and follows from semigroup theory applied to the unbounded operator A on $H = L^2(\Omega, \mathbb{C}^8)$ defined by

$$\begin{aligned} \mathcal{D}(A) &= \{(e, w_1, h, w_2) \in H^1(\Omega, \mathbb{C}^8) : \nu \times e = 0, w_1 = w_2 = 0 \text{ on } \Gamma\}, \\ A \begin{bmatrix} e \\ w_1 \\ h \\ w_2 \end{bmatrix} &= \begin{bmatrix} \varepsilon^{-1}(\nabla \times h) + \nabla w_2 \\ -\nabla \cdot (\mu h) \\ -\mu^{-1}(\nabla \times e) - \nabla w_1 \\ \nabla \cdot (\varepsilon e) \end{bmatrix}. \end{aligned}$$

Here is the point where the higher regularity of ε and μ is required. Note that the vector-valued functions under consideration have 8 components and that the splitting (e, w_1, h, w_2) involves two scalar functions w_1, w_2 and two vector fields e and h . On the state space H one introduces the inner product

$$(U, V)_H = \int_{\Omega} \langle \varepsilon e, E \rangle dx + \int_{\Omega} w_1 \overline{W}_1 dx + \int_{\Omega} \langle \mu h, H \rangle dx + \int_{\Omega} w_2 \overline{W}_2 dx,$$

for

$$U = \begin{bmatrix} e \\ w_1 \\ h \\ w_2 \end{bmatrix}, V = \begin{bmatrix} E \\ W_1 \\ H \\ W_2 \end{bmatrix} \in L^2(\Omega, \mathbb{C}^8).$$

The operator A operator is densely defined, closed and skew-symmetric. Actually, it is even skew-adjoint. In order to prove this we note at first that A defines an elliptic operator in Ω . Its symbol is the 8×8 matrix

$$a(x, \xi) = \begin{bmatrix} 0 & p_{\varepsilon, \mu}(x, \xi) \\ -p_{\mu, \varepsilon}(x, \xi) & 0 \end{bmatrix}$$

with

$$p_{\varepsilon, \mu}(x, \xi) = i \begin{bmatrix} \varepsilon^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -\xi_3 & \xi_2 & \varepsilon_{1j}\xi_j \\ \xi_3 & 0 & -\xi_1 & \varepsilon_{2j}\xi_j \\ -\xi_2 & \xi_1 & 0 & \varepsilon_{3j}\xi_j \\ -\mu_{1j}\xi_j & -\mu_{2j}\xi_j & -\mu_{3j}\xi_j & 0 \end{bmatrix},$$

where the summation convention is used and one computes

$$\det a(x, \xi) = -\frac{\langle \varepsilon \xi, \xi \rangle^2 \langle \mu \xi, \xi \rangle^2}{\det \varepsilon \det \mu}.$$

This proves the ellipticity of the symbol since the matrices ε and μ are positive definite.

Now we establish the skew-adjointness of A . For $U, V \in \mathcal{D}(A)$ one has

$$(AU, V)_H = -(U, AV)_H$$

and hence $\mathcal{D}(A) \subset \mathcal{D}(A^*)$. Now, suppose that $V \in \mathcal{D}(A^*)$. Then, there exists $F \in H$ such that

$$(AU, V)_H = (U, F)_H$$

for all $U \in \mathcal{D}(A)$. Choosing $U \in H_0^1(\Omega, \mathbb{C}^8)$ shows that V is a weak solution to $-AV = F \in H$. Since A is an elliptic operator one infers that $V \in H^1(\Omega, \mathbb{C}^8)$. To show that $V = (E, W_1, H, W_2)^T$ has to satisfy the same homogeneous boundary conditions as the elements in $\mathcal{D}(A)$ one chooses $U = (e, 0, 0, 0)^T$ and computes

$$\begin{aligned} (AU, V)_H &= \int_{\Omega} [-\langle \nabla \times e, H \rangle + [\nabla \cdot (\varepsilon e)] \overline{W}_2] dx \\ &= \int_{\Omega} [-\langle e, \nabla \times H \rangle - \langle \varepsilon e, \nabla W_2 \rangle] dx + \int_{\Gamma} \langle \varepsilon e, \nu \rangle \overline{W}_2 d\gamma \\ &= -(U, AV)_H + \int_{\Gamma} \langle \varepsilon e, \nu \rangle \overline{W}_2 d\gamma. \end{aligned}$$

Hence, the boundary integral has to vanish which requires $W_2 = 0$ on Γ . Similarly, with other choices for the function U , one establishes that $\nu \times E = 0$ and $W_1 = 0$ on Γ . Hence, $\mathcal{D}(A^*) = \mathcal{D}(A)$.

By Stone's Theorem the operator A generates a strongly continuous unitary group of operators on H . The operator $A_{\sigma} = A - \Sigma$, where Σ is the 8×8 Hermitian matrix with the 3×3 matrix $\varepsilon^{-1}\sigma$ in the upper left corner and zeros otherwise, is a bounded perturbation of A and generates a C_0 -semigroup of operators on H .

In the case of a non-zero conductivity an integral term arises in the system and one obtains the existence and uniqueness of a classical solution by relying

on a perturbation argument established by Desch, Grimmer and Schappacher [5, Corollary p.228]. The system (12) can be written in the form

$$U_t(t) = A_\sigma U(t) + \int_0^t B(t - \tau)U(\tau) d\tau, \quad U = (e, w_1, h, w_2)^T,$$

where $B(t)U = (0, 0, 0, \nabla \cdot (\sigma e))^T$. This operator satisfies conditions (H1)-(H3) imposed in [5, p. 220]. \square

Remark 3. The operator $P_\varepsilon(x, \partial)v = (\nabla \times v, \nabla \cdot (\varepsilon v))$ is an (overdetermined) elliptic operator in Ω and the boundary operator $\nu \times v$ on Γ satisfies the Lopatinskii condition. Hence, $P_\varepsilon(x, \partial)v \in L^2(\Omega, \mathbb{C}^4)$ and $\nu \times v \in H^{1/2}(\Gamma)$ imply $v \in H^1(\Omega, \mathbb{C}^3)$ and the estimate

$$\|v\|_{H^1(\Omega)} \lesssim \|\nabla \times v\|_{L^2(\Omega)} + \|\nabla \cdot (\varepsilon v)\|_{L^2(\Omega)} + \|\nu \times v\|_{H^{1/2}(\Gamma)}.$$

Hence, the condition $e^0 \in H^1(\Omega, \mathbb{C}^3)$ in Theorem 3.1 can be weakened to $\varepsilon e^0 \in H(\text{div}, \Omega)$.

In the case that ε is the identity matrix, the proof of this remark can be found in [22]. Alternatively, the proof can be established by extending the operator P_ε to a 4×4 system by adding a gradient of a scalar function to the curl (similar to the operator A above). The resulting system can be analyzed using elliptic theory.

4. Strong stability.

Theorem 4.1. *Suppose that $\varepsilon, \mu \in W^{1,\infty}(\Omega, \mathbb{C}^{3 \times 3})$, and that $\sigma \in W^{1,\infty}(\Omega, \mathbb{C}^{3 \times 3})$ is positive definite on the open and non-empty subset ω of Ω and that $\sigma = 0$ on $\Omega \setminus \omega$. For mild solutions $u(t)$ of (1) - (3) in the sense of Proposition 1 with the additional assumption $\varepsilon e^0 \in H(\text{div}0, \Omega)$ one has $\lim_{t \rightarrow \infty} \|u(t)\|_{\mathcal{H}} = 0$.*

For the proof we need the following result, which can be understood as a Helmholtz decomposition in the case of variable coefficients.

Lemma 4.2. *Let $(e(t), h(t))$ be a classical solution to (1) - (3). Then, the electric field $e(t)$ has the Helmholtz type decomposition*

$$\varepsilon e(t) = w(t) + \varepsilon \nabla g(t) \quad \text{for } t \in [0, \infty).$$

Here $w \in C([0, \infty), H^1(\Omega)) \cap C^1([0, \infty), H(\text{div}0, \Omega))$ and $g \in C^1([0, \infty), H_0^1(\Omega))$.

Proof. Fix $t \in [0, \infty)$ and consider the elliptic boundary value problem

$$\begin{aligned} \nabla \cdot (\varepsilon \nabla g) &= \nabla \cdot (\varepsilon e) & \text{in } \Omega, \\ g &= 0 & \text{in } \Gamma. \end{aligned} \tag{13}$$

Since $\nabla \cdot (\varepsilon e) \in H^{-1}(\Omega)$, there exists a unique weak solution $g \in H_0^1(\Omega)$. Set now $w = \varepsilon e - \varepsilon \nabla g$ which satisfies $w \in H(\text{div}0, \Omega)$. In addition, from $\nabla \times e \in L^2(\Omega)$ one infers $\nabla \times (\varepsilon^{-1}w) \in L^2(\Omega)$. Furthermore, the boundary condition on g and $\nu \times e = 0$ on $\partial\Omega$ imply $\nu \times (\varepsilon^{-1}w) = 0$ on $\partial\Omega$. Hence, the function $f = \varepsilon^{-1}w$ is a solution to the elliptic boundary value problem

$$\nabla \times f = \nabla \times e, \quad \nabla \cdot (\varepsilon f) = 0 \quad \text{in } \Omega, \quad \nu \times f = 0 \quad \text{in } \partial\Omega,$$

and elliptic estimates show that $w \in H^1(\Omega)$, see Remark 3.

The continuity in t of the functions g and w follows from the linearity of (13) as well as elliptic estimates. \square

Remark 4. A similar argument as used near the end of the proof shows that $h(t) \in H^1(\Omega, \mathbb{C}^3)$ for all $t \in [0, \infty)$. More specifically, the boundary operator $\langle \mu v, \nu \rangle$ on Γ satisfies the Lopatinskii condition for elliptic operator $P_\mu(x, \partial)$ from Remark 3. Hence, $P_\mu(x, \partial)v \in L^2(\Omega, \mathbb{C}^4)$ and $\langle \mu v, \nu \rangle \in H^{1/2}(\Gamma)$ implies $v \in H^1(\Omega, \mathbb{C}^3)$ and the estimate

$$\|v\|_{H^1(\Omega)} \lesssim \|\nabla \times v\|_{L^2(\Omega)} + \|\nabla \cdot (\mu v)\|_{L^2(\Omega)} + \|\langle \mu v, \nu \rangle\|_{H^{1/2}(\Gamma)}.$$

If μ is the identity matrix, the proof of this statement can be found in [22]. The general case can be handled with elliptic regularity theory, see the comment made at the end of the previous section.

Now we can prove theorem 4.1.

Proof. Using the density of $\mathcal{D}(\mathcal{A})$ in \mathcal{H} as well as the energy identity (4), it will suffice to prove the result for classical solutions only. Fix $u^0 = (e^0, h^0) \in \mathcal{D}(\mathcal{A})$ such that $\operatorname{div}(\varepsilon e^0) = 0$ in Ω . As before, let $u(t) = S(t)u^0$ be the solution to the initial-boundary value problem (1) - (3) and according to Theorem 3.1, we have $u \in C([0, \infty), H^1(\Omega, \mathbb{C}^6))$.

From the energy identity (4) one infers that

$$\int_0^t \int_\omega \langle \sigma(x)e(t, x), e(t, x) \rangle dxdt = \int_0^t \int_\Omega \langle \sigma(x)e(t, x), e(t, x) \rangle dxdt \leq \|u^0\|_{\mathcal{H}}^2$$

for all $t \in \mathbb{R}_+$. Hence, the improper integral

$$\int_0^\infty \int_\omega \langle \sigma e(t, x), e(t, x) \rangle dxdt$$

is convergent and consequently $\|e(t, \cdot)\|_{L^2(\omega)} \rightarrow 0$ as $t \rightarrow \infty$.

Recall that the function $\varphi(t) = \|u(t)\|_{\mathcal{H}}$ is decreasing and positive, see formula (4). Suppose that $\lim_{t \rightarrow \infty} \varphi(t) = \beta > 0$.

Note that we have

$$\varepsilon e_t - \nabla \times h = 0 \quad \text{in } (0, \infty) \times (\Omega \setminus \omega).$$

Hence, $\nabla \cdot (\varepsilon e)_t(t, x) = 0$ for $t > 0$ and $x \in \Omega \setminus \omega$ and the divergence free initial condition implies $\nabla \cdot (\varepsilon e)(t, x) = 0$ for all $t > 0$ and $x \in \Omega \setminus \omega$.

From the lemma and the remark above we know that $(w(t), h(t)) \in H^1(\Omega, \mathbb{C}^6)$ for all $t \in [0, \infty)$. Using the energy identities (4),(5) and Maxwell's equations one infers that the norms $\|(\nabla \times e)(t, \cdot)\|_{L^2(\Omega)}^2$ and $\|(\nabla \times h)(t, \cdot)\|_{L^2(\Omega)}$ are bounded for all t . Since $\nabla \cdot (\mu h) = 0$ in Ω and $\langle \mu h, \nu \rangle = 0$ in Γ , Remark 4 can be applied to show that the H^1 -norm of h is bounded in time. Since $\nabla \times \varepsilon^{-1}w = \nabla \times e$ in Ω , $w \in H(\operatorname{div}0, \Omega)$, and $\nu \times \varepsilon^{-1}w = 0$ in Γ , Remark 3 shows that the H^1 -norm of w is bounded for all $t > 0$. By the Rellich selection theorem, there exists a sequence $t_n \rightarrow \infty$ such that $(\varepsilon^{-1}w(t_n), h(t_n)) \rightarrow v$ in \mathcal{H} .

Elliptic estimates imply that $\|g(t)\|_{H^1(\Omega)} \lesssim \|\nabla \cdot (\varepsilon e)(t)\|_{H^{-1}(\Omega)}$. Furthermore, since $\nabla \cdot (\varepsilon e) \in L^2(\Omega)$ and $\nabla \cdot (\varepsilon e) = 0$ in $\Omega \setminus \omega$, we have

$$\|\nabla \cdot (\varepsilon e)(t)\|_{H^{-1}(\Omega)} = \sup_{\|\varphi\|_{H_0^1(\Omega)}=1} \left| \int_\Omega \nabla \cdot (\varepsilon e) \varphi dx \right| = \sup_{\|\varphi\|_{H^1(\omega)}=1} \left| \int_\omega \nabla \cdot (\varepsilon e) \varphi dx \right|.$$

For $\varphi \in H^1(\omega)$ and a sequence $\chi_n \in C_0^\infty(\omega)$ converging to the characteristic function of ω pointwise,

$$\int_\omega \nabla \cdot (\varepsilon e) \chi_n \varphi dx \rightarrow \int_\omega \nabla \cdot (\varepsilon e) \varphi dx,$$

by the dominated convergence theorem. This implies that $\|\nabla \cdot (\varepsilon e)(t)\|_{H^{-1}(\Omega)} = \|\nabla \cdot (\varepsilon e)(t)\|_{H^{-1}(\omega)}$ and results in

$$\|g(t)\|_{H^1(\Omega)} \lesssim \|\nabla \cdot (\varepsilon e)(t)\|_{H^{-1}(\omega)} \lesssim \|e(t)\|_{L^2(\omega)},$$

where all the constants are independent of t . Hence, $\|\nabla g(t)\|_{L^2(\Omega)} \rightarrow 0$ as $t \rightarrow \infty$ and

$$\|u(t_n) - v\|_{\mathcal{H}} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Note that $\|v\|_{\mathcal{H}} = \beta$. Using properties of the semigroup $S(t)$ we have for each $t \in \mathbb{R}_+$

$$(E(t), H(t)) := S(t)v = S(t) \lim_{n \rightarrow \infty} S(t_n)(e^0, h^0) = \lim_{n \rightarrow \infty} S(t + t_n)(e^0, h^0)$$

which gives $\|S(t)v\|_{\mathcal{H}} = \beta$ for all $t \in \mathbb{R}_+$. From the energy identity (4) we infer that $E(t) = 0$ on ω for all $t \geq 0$.

In the final step of the proof we will show that $E(t) = H(t) = 0$ in Ω for all $t > 0$. First observe that (E, H) is a mild solution to the initial-boundary value problem (1) - (3) with zero conductivity. Furthermore, the fields εE and μH are divergence free. Hence, the vector function $U = (E, 0, H, 0)$ of 8 components satisfies the differential equation $\partial_t U = AU$ where A is the operator introduced in Section 3. The initial data $U^0 = U(0)$ is the vector v extended to a vector with eight components by inserting a zero after the third component and another zero after the last component.

We know that $(A - I)^{-1}$ is a bounded operator from $L^2(\Omega, \mathbb{C}^8)$ into $H^1(\Omega, \mathbb{C}^8)$, and hence by Rellich's selection theorem, a compact operator on $L^2(\Omega, \mathbb{C}^8)$. Since it is also normal, there exists a basis of orthonormal eigenfunctions $\{u_k\}_{k=1}^\infty$ for the operator $(A - I)^{-1}$ in the state space H . These functions are also eigenfunctions of the operator A whose eigenvalues $\lambda_k, k = 1, 2, \dots$ are purely imaginary (or zero) because of its skew-adjointness. Hence, the vector U can be expanded into an orthogonal series of the form

$$U(t, x) = \sum_{k=1}^\infty e^{\lambda_k t} U_k, \quad \text{with } (U_k, U_l)_H = 0 \text{ and } \lambda_k \neq \lambda_l \text{ for } k \neq l,$$

where $U^0 = \sum_{k=1}^\infty U_k$. Introduce now

$$W_k(t, x) = \frac{1}{t} \int_0^t U(s, x) e^{-\lambda_k s} ds, \quad \text{for } k = 1, 2, \dots$$

and observe that the first three components of this function vanish on ω for $t \geq 0$. Using the series expansion for U one has

$$W_k(t, x) = U_k(x) + \frac{1}{t} \sum_{l \neq k} \frac{e^{(\lambda_l - \lambda_k)t} - 1}{(\lambda_l - \lambda_k)} U_l(x).$$

Letting $t \rightarrow \infty$ proves that the first three components of U_k vanish on ω for $k = 1, 2, 3, \dots$. By the same argument the fourth and last component of U_k is identically zero for all k . Each eigenfunction U_k solves the differential equation

$$\lambda_k U_k = AU_k$$

which is a time harmonic anisotropic Maxwell system (considering that the 4th and 8th component of U_k are identically zero). We infer that the last three components of U_k vanish on ω . Using now the unique continuation property of this system [8]

we conclude that all functions U_k have to vanish. Hence, $v \equiv 0$ in Ω and thus $\lim_{t \rightarrow \infty} \varphi(t) = 0$. \square

Remark 5. The first work pertaining to the unique continuation of the time-harmonic anisotropic Maxwell equations with non-analytic coefficient is the one of Leis [11]. He works with coefficients of class C^2 . The paper by Eller and Yamamoto [8] weakens this assumption to C^1 coefficients. With the aid of paradifferential operators the assumption on the coefficients can be further relaxed to Lipschitz continuous coefficients.

We like to emphasize that the assumptions made in Theorem 4.1 are most likely not necessary. While the positive definiteness of σ on some open subset $\omega \subset \Omega$ is indispensable, we believe that the condition of being identical zero on the complement of ω is far from necessary.

Another question is whether a rate of decay can be computed. Using Carleman estimates Eller and Toundykov [6] showed logarithmic decay of solutions to Maxwell's equations with isotropic coefficients and a local dissipative term acting on both the electric and the magnetic field.

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