

# Cosmological Horizon and the Quadrupole Formula in de Sitter Background

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## Abstract

An important class of observables for gravitational waves consists of the fluxes of energy, momentum and angular momentum carried away by them and are well understood for weak gravitational waves in Minkowski background. In de Sitter background, the future null infinity,  $\mathcal{J}^+$ , is space-like which makes the meaning of these observables subtle. A spatially compact source in de Sitter background also provides a distinguished null hypersurface, its *cosmological horizon*,  $\mathcal{H}^+$ . For sources supporting the short wavelength approximation, we adopt the Isaacson prescription to define an effective gravitational stress tensor. We show that the fluxes computed using this effective stress tensor can be evaluated at  $\mathcal{H}^+$ , match with those computed at  $\mathcal{J}^+$  and also match with those given by Ashtekar et al at  $\mathcal{J}^+$  *at a coarse grained level*.

PACS numbers: 04.30.-w

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## I. INTRODUCTION

Weak gravitational field of a spatially compact source is identified as a perturbation about a background space-time which is a solution of the Einstein equation in the source free region. In the presence of a positive cosmological constant, the background space-time is the de Sitter space-time. Unlike the Minkowski background for the zero cosmological constant, de Sitter space-time has different patches eg the global patch ( $\mathbb{R} \times S^3$ ), a Poincare patch and a static patch. In the cosmological context, a Poincare patch is appropriate which is what we focus on. A solution at the *linearized level*, valid throughout the Poincare patch and extending to the future null infinity  $\mathcal{J}^+$ , is available in [1–3]. However the space-like character of the  $\mathcal{J}^+$  poses challenges for defining energy, momentum and their fluxes.

Let us recall that the cleanest articulation of ‘infinity’ arises in the conformal completion of physical space-times. Conformal completion preserves the light cone structure of the physical space-time and naturally identifies boundary components,  $\mathcal{J}^\pm$  where time-like and null geodesics ‘terminate’. The causal nature of these boundary components is determined by the asymptotic form of ‘source-free’ equations:  $\mathcal{J}^\pm$  are null when  $\Lambda = 0$  and space-like for  $\Lambda > 0$  (time-like for  $\Lambda < 0$ ). These boundary components serve to define out-going (in-coming) fields as those solutions of the asymptotic equations that have suitably finite limiting values on  $\mathcal{J}^+(\mathcal{J}^-)$ . It is then a result that the Weyl tensor of out-going fields evaluated along out-going null geodesics, has a definite pattern of fall-off in inverse powers of an affine parameter along the geodesics (the peeling-off theorem) [4, 5]. This enables one to identify the leading term as representing gravitational radiation (far field of a source), in a coordinate invariant manner. It is conveniently described in terms of the Weyl scalars which are defined with respect to a suitable null tetrad. When  $\mathcal{J}^+$  is null, a null tetrad at a point  $p \in \mathcal{J}^+$  is uniquely determined (modulo real scaling and rotation) by the tangent vector  $\ell^\mu$  of an outgoing null geodesic reaching  $p$ , *and* the null normal  $n^\mu$ , satisfying  $\ell \cdot n = -1$ . Clearly as the null geodesic changes its direction,  $\ell$  changes but not  $n$  and hence the Weyl scalar  $\Psi_4$  ( $:= C_{\mu\nu\rho\sigma} n^\mu \bar{m}^\nu n^\rho \bar{m}^\sigma$ ) remains unchanged. Its non-zero value can be taken as showing the presence of gravitational radiation. This feature is lost when the  $\mathcal{J}^+$  is space-like. Now the null vector  $n^\mu$ , with  $\ell \cdot n = -1$ , is chosen to be in the plane defined by  $\ell^\mu$  and the (time-like) normal  $N^\mu$ . Clearly, as  $\ell^\mu$  changes, so does  $n^\mu$  and *none* of the Weyl scalars is invariant. An invariant characterization of gravitational radiation is no longer available [5].

The de Sitter space-time also has the so called observer horizons - boundary of the causal past of an observer's end point on  $\mathcal{J}^+$ . In particular, for a spatially extended but compact source, the worldlines of different components of the source, must reach the same point on  $\mathcal{J}^+$  to maintain a *finite* physical separation among them. A spatially compact source then defines (its) *cosmological horizon* as the past light cone of the common point on  $\mathcal{J}^+$  where the source world tube converges. Equally well, *any observer* who remains at a finite physical distance from the compact source for all times, must necessarily lie within the cosmological horizon i.e. within the static patch bounded by the cosmological horizon. Unlike the  $\mathcal{J}^+$ , the cosmological horizon is a null hypersurface but shares with  $\mathcal{J}^+$  the property, that whichever curve meets a point on it, can never causally intersect the world tube of the spatially compact source. In other words, once any energy/momentum/angular momentum is carried away across the cosmological horizon, it is 'lost' from the source forever. We would like to explore to what extent and under what conditions may we regard the cosmological horizon as a "substitute" for the *future null infinity*.

It is obvious at the outset that the out-going null geodesics emanating from the source intersect the cosmological horizon at a *finite* value of any affine parameter and it can be chosen to be 1 by a suitable normalization. Such a normalized affine parameter equals the ratio of the physical distance from the source to  $\sqrt{3/\Lambda} \sim 10^{26}m \sim 10Gpc$ . All spatially compact sources may be taken to lie within a sphere of radius  $\sim \Lambda^{-1/2}$ . Furthermore, only sources varying over cosmological time scales, will have comparable wavelengths. Thus, most sources producing gravitational waves would have wavelengths far smaller than  $\Lambda^{-1/2}$  and any wave crossing the horizon may be taken to be a 'far zone field'. Cosmological horizon being a null hypersurface, a  $\Psi_4$  can be defined on it, independent of the null geodesics meeting the horizon. A notion of radiation based on asymptotic behaviour of fields is physically useful, provided there are suitable definitions of fluxes of energy-momentum, and angular momentum in terms of these asymptotic fields. And there are many such definitions.

One of the definition of such conserved quantities is based on the covariant phase space framework [6, 7]. In the context of the linearised theory, it exploits the phase space structure of the space of solutions and defines a manifestly gauge invariant and conserved 'Hamiltonian' corresponding to each of the seven *isometries* of the Poincare patch. Although defined on each space-like hypersurface of the Poincare patch, the simplest expressions result for evaluation at  $\mathcal{J}^+$ . Thus, the conserved quantities are directly expressed in terms of the

asymptotic fields.

For sources which are sufficiently rapidly varying (relative to the scale set by the cosmological constant), there is an alternative identification of gravitational waves as *ripples on a background* within the so called *short wave approximation* [8, 9]. Furthermore, it is possible to define an *effective gravitational stress tensor*,  $t_{\mu\nu}$  for the ripples. For vanishing  $\Lambda$ , it is symmetric, conserved and gauge invariant. For non-zero  $\Lambda$  it is *not* gauge invariant but the gauge violations are suppressed by powers of  $\sqrt{\Lambda}$ . It is very convenient to have such a stress tensor to define and compute fluxes of energy and momenta carried by the ripples across *any* hypersurface.

We use the fluxes defined using the effective gravitational stress tensor and show that for the retarded solution given in [1–3], the fluxes of energy and momentum across the cosmological horizon exactly equal the corresponding fluxes across the  $\mathcal{J}^+$ . Furthermore, these fluxes computed at  $\mathcal{J}^+$  also equal the fluxes defined in the covariant phase space framework, [2] albeit at a coarse grained level (See equation (93)). The instantaneous power received at infinity matches with that crossing the horizon. This is our main result.

The paper is organised as follow.

In section II, we summarise various details needed to establish our result. Most are available in the cited literature and are collected here for self contained reading. It is divided in three subsection. In the subsection II A, we recall the solution at the linearised level [1–3] for which the fluxes will be evaluated. We specify and denote the (spatial components of) the *exact retarded* solution by  $\mathcal{X}_{ij}$ . This is approximated when the source dimension is much smaller than the distance to the source. The leading term is the *approximated retarded* solution and is denoted by  $\chi_{ij}$ . Physical solutions have to satisfy the gauge conditions imposed in simplifying the linearised equation. This is achieved by extracting the (spatial) *transverse and traceless* (TT) part of the solution which is denoted by  $\mathcal{X}_{ij}^{TT}$ . For the approximated solution, the TT part is conveniently extracted by an algebraic projection to the same level of approximation. The *algebraically projected transverse, traceless* part of the approximated solution is denoted by  $\chi_{ij}^{tt}$  and used throughout. We also collect relevant expressions for subsequent use. A table of notation is included at the end of this subsection. In subsection II B we summarise the covariant phase space framework and recall the definitions of the fluxes and quadrupole power from [2]. The energy momentum fluxes defined here are compared to those defined in the next subsection. In subsection II C, we discuss the Isaacson prescription

*adapted* to the presence of the cosmological constant and present the definition of the *ripple tensor* in eq. (46) which is used in the next section.

Section III is divided into three subsections. In the subsection III A, we present computations of the energy flux for the  $\chi_{ij}^{tt}$  across various hypersurfaces. In particular we show that the fluxes across the out-going null hypersurfaces are zero, implying for example, that the energy propagation is sharp. Subsection III B contains the fluxes for the momentum and the angular momentum. In the subsection III C we discuss how the computations can be extended to  $\chi_{ij}^{TT} := (\mathcal{X}_{ij}^{TT})_{approx}$ .

In section IV, we discuss applications of these flux computations and establish our main results. The final section V concludes with a discussion. An appendix is included to illustrate an averaging procedure.

## II. PRELIMINARIES

In this section we summarise and assemble already available relevant details needed for our main result, with the main citations included in the subsection headings.

### A. Weak gravitational field of interest [1–3]

Weak gravitational fields are understood as perturbations about a background specified in the form,  $g_{\mu\nu} := \bar{g}_{\mu\nu} + \epsilon h_{\mu\nu}$ . The background  $\bar{g}_{\mu\nu}$  is chosen to be a solution of the source free Einstein equation with a positive cosmological constant. The Einstein equation for  $g_{\mu\nu}$ , expanded to first order in  $\epsilon$ , gives the linearised Einstein equation for  $h_{\mu\nu}$ . The *physical perturbations* are understood as the equivalence classes of solutions  $h_{\mu\nu}$ , with respect to the *gauge transformations*:  $\delta h_{\mu\nu}(x) = \mathcal{L}_\xi \bar{g}_{\mu\nu}(x) = \bar{\nabla}_\mu \xi_\nu + \bar{\nabla}_\nu \xi_\mu$ . In terms of the trace reversed combination  $\tilde{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2} \bar{h}_{\mu\nu} (\bar{g}^{\alpha\beta} h_{\alpha\beta})$ , the linearised equation takes the form,

$$\frac{1}{2} \left[ -\bar{\square} \tilde{h}_{\mu\nu} + \{ \bar{\nabla}_\mu B_\nu + \bar{\nabla}_\nu B_\mu - \bar{g}_{\mu\nu} (\bar{\nabla}^\alpha B_\alpha) \} \right] + \frac{\Lambda}{3} \left[ \tilde{h}_{\mu\nu} - \tilde{h} \bar{g}_{\mu\nu} \right] = 8\pi T_{\mu\nu} \quad (1)$$

where,  $B_\mu := \bar{\nabla}_\alpha \tilde{h}^\alpha{}_\mu$ . The gauge freedom is exploited subsequently to simplify the equation.

In the present context, the background space-time is taken to be the *Poincare patch* of the de Sitter space-time (see figure 1) which admits a conformally flat form of the background

metric in coordinates  $(\eta, x^i)$ ,

$$ds^2 = \frac{1}{H^2\eta^2} \left[ -d\eta^2 + \sum_i (dx^i)^2 \right], \quad \eta \in (-\infty, 0), \quad x^i \in \mathbb{R}, \quad H := \sqrt{\frac{\Lambda}{3}}. \quad (2)$$

The future null infinity is approached as  $\eta \rightarrow 0_-$  while the  $\eta \rightarrow -\infty$  corresponds to the FLRW singularity. The conformal factor is  $a^2(\eta) := (H\eta)^{-2}$ .

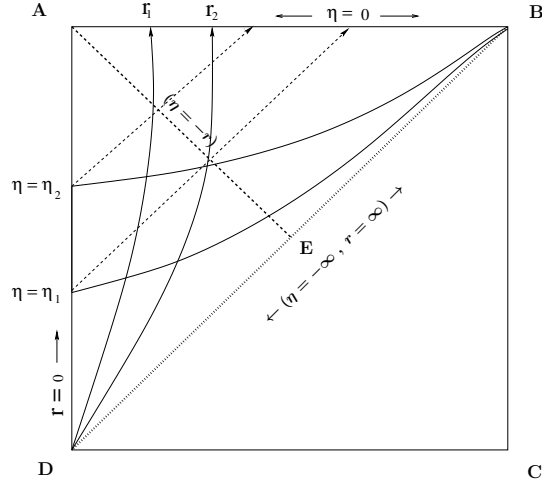


FIG. 1: The full square is the Penrose diagram of de Sitter space-time with generic point representing a 2-sphere. World tube of a spatially compact source is taken to be centred on the line DA. The corresponding Poincare patch is labeled ABD and is covered by the Poincare chart  $(\eta, r, \theta, \phi)$ . The line BD does not belong to the chart. The line AB denotes the *future null infinity*,  $\mathcal{J}^+$  while the line AE denotes the *cosmological horizon*,  $\mathcal{H}^+$  of the source. The region AED is the *static patch* admitting the stationary Killing vector,  $T$  of eqn. (21). Two constant  $\eta$  space-like hypersurfaces are shown with  $\eta_2 > \eta_1$ . The two constant  $r$ , time-like hypersurfaces have  $r_2 > r_1$  while the two dotted lines at 45 degrees, denote the out-going null hypersurfaces emanating from  $\eta = \eta_1, \eta_2$  on the world line at  $r = 0$ .

The linearised equation is simplified by imposing the *generalised transverse gauge* conditions:  $B_\mu = \frac{2\Lambda}{3}\eta\tilde{h}_{0\mu}$  The conformal factor can be scaled out by using the fields  $\tilde{\chi}_{\mu\nu} := a^{-2}\tilde{h}_{\mu\nu}$

and the linearised equation (with source included) then takes the form<sup>1</sup> [1],

$$-16\pi T_{\mu\nu} = \square\tilde{\chi}_{\mu\nu} + \frac{2}{\eta}\partial_0\tilde{\chi}_{\mu\nu} - \frac{2}{\eta^2}(\delta_\mu^0\delta_\nu^0\tilde{\chi}_\alpha{}^\alpha + \delta_\mu^0\tilde{\chi}_{0\nu} + \delta_\nu^0\tilde{\chi}_{0\mu}) . \quad (3)$$

$$0 = \partial^\alpha\tilde{\chi}_{\alpha\mu} + \frac{1}{\eta}(2\tilde{\chi}_{0\mu} + \delta_\mu^0\tilde{\chi}_\alpha{}^\alpha) \quad (\text{gauge condition}). \quad (4)$$

It is further shown in [1] that the residual gauge invariance is exhausted by imposing the additional gauge conditions:  $\tilde{\chi}_{0\alpha} = 0 = \hat{\chi}(\hat{:=} \tilde{\chi}_{00} + \tilde{\chi}_i{}^i)$ . The gauge condition (4) then implies that *physical perturbations* may be characterised solutions of (3) which satisfy the spatial transverse, traceless condition, or *spatial TT* for short:  $\partial^j\tilde{\chi}_{ji} = 0 = \tilde{\chi}^k{}_k$ . Thanks to the decoupled equations, it suffices to focus on the spatial components of the equation.

The *exact retarded solution* is given by,

$$\begin{aligned} \mathcal{X}_{ij}(\eta, x) = & 4 \int d^3x' \frac{\eta}{|x-x'|(\eta-|x-x'|)} T_{ij}(\eta', x')|_{\eta'=\eta-|x-x'|} \\ & + 4 \int d^3x' \int_{-\infty}^{\eta-|x-x'|} d\eta' \frac{T_{ij}(\eta', x')}{\eta'^2} \end{aligned} \quad (5)$$

The spatial integration is over the matter source confined to a compact region and is finite. The second term in the eqns. (5) is the *tail term*. This particular solution does *not* satisfy the spatial TT conditions. Using the transverse, traceless decomposition of the tensor fields, the TT part,  $\mathcal{X}_{ij}^{TT}$  is extracted which represents the *physical retarded field* due to the source.

For  $|\vec{x}| \gg |\vec{x}'|$ , we can approximate  $|\vec{x} - \vec{x}'| \approx r := |\vec{x}|$ . This allows us separate out the  $\vec{x}'$  dependence from the  $\eta - |x - x'|$ . The so *approximated retarded solution*,  $\chi_{ij}$ , is given by,

$$\begin{aligned} \mathcal{X}_{ij} &= \chi_{ij}(\eta, x) + o(r^{-1}) \quad , \quad \text{with} \\ \chi_{ij}(\eta, x) &:= 4 \frac{\eta}{r(\eta-r)} \int d^3x' T_{ij}(\eta', x') \Big|_{\eta'=\eta-r} + 4 \int_{-\infty}^{\eta-r} d\eta' \frac{1}{\eta'^2} \int d^3x' T_{ij}(\eta', x') \end{aligned} \quad (6)$$

We will work with the approximated solution. Note that  $\chi_{ij}$  depends on  $\vec{x}$  only through  $r = |\vec{x}|$ . The spatial integral of  $T_{ij}$  can be simplified using moments. This is done through the matter conservation equation.

To define these moments, introduce the orthonormal tetrad  $f_{\underline{a}}{}^\alpha := -H\eta\delta_{\underline{a}}{}^\alpha$  and denote the frame components of the source stress tensor as:  $\rho := H^2\eta^2 T_{\underline{0}\underline{0}}$ ,  $\pi := H^2\eta^2 T_{\underline{i}\underline{j}}\delta^{ij}$ . Define moment variable  $\bar{x}^{\underline{i}} = f^{\underline{i}}{}_\alpha x^\alpha = -(\eta H)^{-1}\delta^{\underline{i}}{}_j x^j := a(t)x^{\underline{i}}$ . Two sets of moments are defined

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<sup>1</sup> From now on in this subsection, the tensor indices are raised/lowered with the Minkowski metric.

as functions of  $\eta$ , (or of  $t$  defined through  $\eta := -H^{-1}e^{-Ht}$ ) as,

$$Q^{ij}(t) := \int_{Source(t)} d^3x a^3(t) \rho(t, \vec{x}) \bar{x}^i \bar{x}^j \quad , \quad \bar{Q}^{ij}(t) := \int_{Source(t)} d^3x a^3(t) \pi(t, \vec{x}) \bar{x}^i \bar{x}^j . \quad (7)$$

In terms of these, the approximated retarded solution is given by,

$$\chi_{ij}(\eta, \vec{x}) = \frac{1}{r} f_{ij}(\eta_{ret}) + g_{ij}(\eta_{ret}) + \hat{g}_{ij} \quad \text{with,} \quad (8)$$

$$f_{ij}(\eta_{ret}) := \frac{2}{a(\eta_{ret})} [\mathcal{L}_T^2 Q_{ij} + 2H \mathcal{L}_T Q_{ij} + H \mathcal{L}_T \bar{Q}_{ij} + 2H^2 \bar{Q}_{ij}] \quad , \quad (9)$$

$$g_{ij}(\eta_{ret}) := -2H [\mathcal{L}_T^2 Q_{ij} + H \mathcal{L}_T Q_{ij} + H \mathcal{L}_T \bar{Q}_{ij} + H^2 \bar{Q}_{ij}] \quad \text{and,} \quad (10)$$

$$\hat{g}_{ij} := -2H^2 [\mathcal{L} Q_{ij} + H \bar{Q}_{ij}]|_{-\infty} \quad (11)$$

All moments are evaluated at the retarded  $\eta_{ret} := (\eta - r)$ ,  $a(\eta_{ret}) := -(H\eta_{ret})^{-1}$  and  $\mathcal{L}_T$  denotes the Lie derivative with respect to the time translation Killing vector defined in equation (21) below. On the moments, it is given by,

$$\mathcal{L}_T Q_{ij} = -H(\eta \partial_\eta + r \partial_r) Q_{ij} - 2H Q_{ij} = -H(\eta - r) \partial_\eta Q_{ij} - 2H Q_{ij} = \partial_t Q_{ij}|_{t_{ret}} - 2H Q_{ij} \quad (12)$$

In equation (8), the first term is the contribution of the so called sharp term while the second and the third terms denote the tail contributions. The tail contribution has separated into a term which depends on retarded time,  $(\eta - r)$  only, just as the sharp term does, and the contribution from the history of the source is given by the limiting value at  $\eta = -\infty$ . This expression is valid *as the leading term for*  $|\vec{x}| \gg |\vec{x}'|$ . There is no TT label on these expressions. While the solution has tail term, it will turn out that the energy propagation is sharp.

For future use in section III A, we display the derivatives of  $\chi_{ij}$ . Since  $\chi_{ij}$  depends on  $\vec{x}$  only through  $r$ , we need only the derivatives with respect to  $\eta$  and  $r$ . On functions of  $\eta_{ret}$ ,  $\partial_r = -\partial_\eta$  and we can replace the  $r$ -derivatives in favour of  $\eta$ -derivatives. Hence,

$$\partial_\eta \chi_{ij} = \frac{1}{r} \partial_\eta f_{ij} + \partial_\eta g_{ij} \quad , \quad \partial_r \chi_{ij}(\eta, r) = -\partial_\eta \chi_{ij} - \frac{f_{ij}}{r^2} . \quad (13)$$

There is a well known *algebraic projection* method to construct spatial tensors which satisfy the spatial TT condition *to the leading order in*  $r^{-1}$ . Since the approximated solution is also valid to  $o(r^{-1})$ , we may use this convenient method.

For the unit vectors  $\hat{x}$  denoting directions, define the projectors

$$P_i^j := \delta_i^j - \hat{x}_i \hat{x}^j \quad , \quad \Lambda_{ij}^{kl} := \frac{1}{2} (P_i^k P_j^l + P_i^l P_j^k - P_{ij} P^{kl}) \quad , \quad \chi_{ij}^{tt} := \Lambda_{ij}^{kl} \chi_{kl} \quad (14)$$



We have used the notation of ‘tt’ to refer to the algebraically projected transverse, traceless part as in [2]. Noting that on  $\chi_{ij}$  the spatial derivative is  $\partial^j = \hat{x}^j \partial_r$ , it follows that,

$$\partial_\eta(\chi_{ij}^{tt}) = (\partial_\eta \chi_{ij})^{tt}, \quad \partial_r(\chi_{ij}^{tt}) = (\partial_r \chi_{ij})^{tt}, \quad (15)$$

$$\partial_m(\chi_{ij}^{tt}) = (\partial_m \Lambda_{ij}{}^{kl}) \chi_{kl} + \hat{x}_m (\partial_r \chi_{ij})^{tt} \quad (16)$$

$$\therefore \partial^j(\chi_{ij}^{tt}) = \hat{x}^j \Lambda_{ij}{}^{kl} \partial_r \chi_{ij} + (\partial^j \Lambda_{ij}{}^{kl}) \chi_{kl} = 0 + o(r^{-1}); \quad \text{where we used,} \quad (17)$$

$$\partial_m \Lambda_{ij}{}^{kl} = -\frac{1}{r} [\hat{x}_i \Lambda_{mj}{}^{kl} + \hat{x}_j \Lambda_{mi}{}^{kl} + \hat{x}^k \Lambda_{ijm}{}^l + \hat{x}^l \Lambda_{ijm}{}^k] = o(r^{-1}). \quad (18)$$

The tracelessness of  $\chi_{ij}^{tt}$  is manifest and hence  $\chi_{ij}^{tt}$  satisfies the spatial TT condition to  $o(r^{-1})$ .

Using the derivatives of  $\chi_{ij}$  given in (13), we can write (the right hand sides denote row vectors of the  $\mu = \eta$  and  $\mu = m$  components),

$$\partial_\mu \chi_{ij}^{tt} = (\partial_\eta \chi_{ij}^{tt}, \hat{x}_m \partial_r \chi_{ij}^{tt} + (\partial_m \Lambda_{ij}{}^{kl}) \chi_{kl}) \quad (19)$$

$$= (\partial_\eta \chi_{ij}^{tt}) (1, -\hat{x}_m) - \left( \frac{f_{ij}^{tt}}{r^2} \right) (0, \hat{x}_m) - \frac{1}{r} (0, [\hat{x}_i \Lambda_{mj}{}^{kl} + \hat{x}_j \Lambda_{mi}{}^{kl} + \hat{x}^k \Lambda_{ijm}{}^l + \hat{x}^l \Lambda_{ijm}{}^k] \chi_{kl}). \quad (20)$$

The first term is proportional to a null vector. The second term is proportional to the space-like, radial vector. The third is again a space-like vector. Both the second and the third terms are down by a power of  $r$  relative to  $\chi_{ij}$  and therefore also relative to the first term. We will see later in the calculation of the fluxes that for energy and momentum, the second and the third terms can be neglected. However for flux of angular momentum, the third term is crucial. *When* the second and the third terms can be neglected, the effective gravitational stress tensor turns out to correspond to an *out-going null dust* with energy density proportional to  $\langle \partial_\eta \chi_{mn} \partial_\eta \chi^{mn} \rangle$ .

Finally, we note the isometries of the Poincare patch. There are seven globally defined Killing vectors on the Poincare chart, corresponding to energy, 3 momenta and 3 angular momenta [2, 10]. They are given by (up to constant scaling):

$$\text{Generator of time translation : } T = -H(\eta \partial_\eta + x^i \partial_i) \quad (21)$$

$$\text{Generators of space translation : } S_{(i)} = \partial_i \quad (22)$$

$$\text{Generators of space rotations : } L_{(j)} = \epsilon_{jk}{}^i x^k \partial_i. \quad (23)$$

We focus on the *time translation vector field*, which is time-like in the static patch, null on the cosmological horizon and space-like beyond it. In particular it is space-like and tangential to  $\mathcal{J}^+$ .

Many different symbols used for the retarded solution, its approximations, their TT parts and different ‘radiation fields’ are summarised below.

$\tilde{\chi}_{ij}$	: generic solution of linearised eqn.	
$\mathcal{X}_{ij}$	: exact, retarded solution	eqn. (5)
$\mathcal{X}_{ij}^{TT}$	: TT part of exact, retarded solution	eqn. (86)
$\chi_{ij}$	: approximated, retarded solution	eqn. (6)
$\chi_{ij}^{tt}$	: $\Lambda$ –projection of approximated, retarded solution	eqn. (14)
$\chi_{ij}^{TT}$	: approximation of $\mathcal{X}_{ij}^{TT}$ for $ \vec{x}'  \ll  \vec{x} $	Above eqn. (87)
	: $\chi_{ij}^{TT}$ does <i>not</i> denote TT part of $\chi_{ij}$	See footnote 3.
$\mathcal{R}_{ij}^{TT}$	: Radiation field defined in [2]	eqns. (33, 34)
$\mathcal{Q}_{ij}^{tt}$	: Radiation field used throughout	eqns. (58, 59)
$\mathcal{Q}_{ij}^{TT}$	: defined to equal $2\partial_{\eta}\mathcal{M}_{ij}^{TT}$	eqn. (87)

## B. Covariant phase space framework [2, 6, 7]

Traditionally, the conserved energy, momentum etc are defined through pseudo-tensors which have their shortcoming of not being covariant. The framework of covariant phase space provides manifestly gauge invariant definitions of the conserved quantities and is briefly recalled below.

Consider the space  $\mathcal{C}$  of a class of solutions of the full Einstein equation, satisfying stipulated boundary condition. At each point of this space, the linearised solutions provide tangent vectors. Under certain conditions, it is possible to define a *pre-symplectic form* on the tangent spaces. Every infinitesimal diffeomorphism of the space-time, with suitable asymptotic behaviour, induces a vector field on  $\mathcal{C}$ . Some of these lie in the kernel of the pre-symplectic form and constitute ‘gauge directions’ while the remaining ones constitute (asymptotic) symmetries shared by the stipulated class of solutions. Modding out by the gauge directions (null space of the pre-symplectic form), one imparts a symplectic structure to the space of solutions, now denoted as  $\Gamma \sim \mathcal{C}/\text{gauge}$ . Under favourable conditions, the vector fields on  $\mathcal{C}$  corresponding to the asymptotic symmetries descend to  $\Gamma$  and generate infinitesimal *canonical transformations*. Their generating functions, or ‘Hamiltonians’, are candidates for representing energy, momenta, angular momenta etc [6].

In [2, 7], this strategy is applied to the space of fully gauge fixed solutions of the linearised

equation and we summarise it below. Isometries of the background, leave the covariant phase space itself invariant and constitute canonical transformations. In the present context, the Hamiltonians corresponding to the 7 isometries are the proposed definitions of energy, linear momentum and angular momentum.

Explicitly,  $\mathcal{C}$  denotes the solutions of the equation (24) together with the gauge fixing conditions (25):

$$\square \tilde{\chi}_{ij} + \frac{2}{\eta} \partial_\eta \tilde{\chi}_{ij} = 0 \quad (24)$$

$$\partial^i \tilde{\chi}_{ij} = 0 \quad , \quad \tilde{\chi}_{ij} \delta^{ij} = 0 \quad (25)$$

A symplectic form is defined by an integral over a cosmological slice  $\Sigma_\eta$ . A definition which has a smooth limit to  $\mathcal{J}^+$  ( $\eta \rightarrow 0_+$ ) is defined in terms of the *electric part* of the perturbed Weyl tensor,  $\mathcal{E}_{ij} := -(H\eta)^{-1} [{}^{(1)}C^0_{j0i}] = \frac{1}{2H\eta^2} (\partial_\eta \tilde{\chi}_{ij} + \eta \nabla^2 \tilde{\chi}_{ij}) = \frac{1}{2H\eta} (\partial_\eta^2 - \frac{1}{\eta} \partial_\eta) \tilde{\chi}_{ij}$ . For two elements  $\tilde{\chi}, \underline{\tilde{\chi}} \in \mathcal{C}$ , the symplectic form is defined by[2],

$$\omega(\tilde{\chi}, \underline{\tilde{\chi}}) = \frac{1}{16\pi H} \int_{\Sigma_\eta} d^3x (\tilde{\chi}_{ij} \underline{\mathcal{E}}_{kl} - \underline{\tilde{\chi}}_{kl} \mathcal{E}_{ij}) \delta^{ik} \delta^{jl} \quad (26)$$

The  $TT$  label on the  $\tilde{\chi}$ 's is suppressed.

A Killing vector  $K$  of the de Sitter background, defines a vector field  $h_{ij}^{(K)} := \mathcal{L}_K h_{ij} = a^2 (\mathcal{L}_K \tilde{\chi}_{ij} + 2(a^{-1} \mathcal{L}_K a) \tilde{\chi}_{ij}) =: a^2 \tilde{\chi}_{ij}^{(K)}$ , on the space  $\mathcal{C}$ . This vector field generates a canonical transformation and the corresponding Hamiltonian function is given by,

$$H_K := -\frac{1}{2} \omega(h, h^{(K)}) = -\frac{1}{2} \omega(\tilde{\chi}, \tilde{\chi}^{(K)}) \quad (27)$$

For the time translation Killing vector  $T$ ,  $H_T (= E_T)$ , is obtained as,

$$E_T := -\frac{1}{2} \omega(\tilde{\chi}, \tilde{\chi}^{(T)}) = -\frac{1}{32\pi H} \int_{\Sigma_\eta} d^3x (\tilde{\chi}_{ij} \mathcal{E}_{kl}^{(T)} - \tilde{\chi}_{kl}^{(T)} \mathcal{E}_{ij}) \delta^{ik} \delta^{jl} \quad (28)$$

$$= -\frac{1}{32\pi H} \int_{\Sigma_\eta} d^3x (\tilde{\chi}_{ij} \mathcal{L}_T \mathcal{E}_{kl} - \mathcal{E}_{kl} \mathcal{L}_T \tilde{\chi}_{ij} - 3H \tilde{\chi}_{ij} \mathcal{E}_{kl}) \delta^{ik} \delta^{jl} \quad (29)$$

This integral is independent of the choice of  $\Sigma_\eta$  and is conveniently performed on  $\mathcal{J}^+ = \Sigma_0$ . The Killing vector  $T$  also has a smooth limit to  $\mathcal{J}^+$ ,  $T|_{\mathcal{J}^+} = -H(x\partial_x + y\partial_y + z\partial_z)$ . The equation (29) simplifies to,

$$E_T = \frac{1}{16\pi H} \int_{\mathcal{J}^+} d^3x \mathcal{E}_{kl} (\mathcal{L}_T \tilde{\chi}_{ij} + 2H \tilde{\chi}_{ij}) \delta^{ik} \delta^{jl} \quad (30)$$

Now using  $(\mathcal{L}_T \tilde{\chi}_{ij} + 2H\dot{\tilde{\chi}}_{ij})|_{\mathcal{J}^+} = T^m \partial_m \tilde{\chi}_{ij}$ ,

$$E_T = \frac{1}{16\pi H} \int_{\mathcal{J}^+} d^3x \mathcal{E}_{kl} (T^m \partial_m \tilde{\chi}_{ij}) \delta^{ik} \delta^{jl} \quad (31)$$

$$= \frac{1}{32\pi H^2} \int_{\mathcal{J}^+} d^3x \left[ \frac{1}{\eta} (\partial_\eta^2 - \frac{1}{\eta} \partial_\eta) \tilde{\chi}_{kl} \right]^{TT} (T^m \partial_m \tilde{\chi}_{ij})^{TT} \delta^{ik} \delta^{jl} \quad (32)$$

In the last line we have used equation of motion and restored the  $TT$  label [2]. Both  $\mathcal{E}_{kl}$  and  $T^m \partial_m \tilde{\chi}_{ij}$  have smooth limit on  $\mathcal{J}^+$ .

When evaluated at the approximated solution given in (6), the energy flux turns out to be given by [2],

$$E_T = \frac{1}{8\pi} \int_{\mathcal{J}^+} d\tau d^2s [\mathcal{R}_{kl} \mathcal{R}_{ij}^{TT}] \delta^{ik} \delta^{jl}, \quad (33)$$

where,  $\mathcal{R}_{ij}$  denotes the ‘radiation field’ on  $\mathcal{J}^+$ , expressed in terms of source moments and is given by

$$\mathcal{R}_{mn}^{TT} := \left[ \ddot{Q}_{mn} + 3H\dot{Q}_{mn} + 2H^2\dot{Q}_{mn} + H\ddot{Q}_{mn} + 3H^2\dot{Q}_{mn} + 2H^3\bar{Q}_{mn} \right]^{TT} (t_{ret}), \quad (34)$$

with the overdot denoting the Lie derivative  $\mathcal{L}_T$ .

The instantaneous *power* received on  $\mathcal{J}^+$  at ‘ $\tau$ ’ is given by,

$$P(\tau) := \frac{1}{8\pi} \int_{S^2} d^2s [\mathcal{R}^{ij} \mathcal{R}_{ij}^{TT}] (-r(\tau)). \quad (35)$$

This expression is not manifestly positive. Manifestly positive expressions for the flux and the power are given by [2],

$$E_T = \frac{1}{2\pi} \int_{\mathcal{J}^+} d\tau d^2s [\partial_r \mathcal{M}_{ij}^{TT}] [\partial_r \mathcal{M}_{TT}^{ij}], \quad \text{where} \quad (36)$$

$$\mathcal{M}_{ij}^{TT}(\eta - r) := \int d^3x' T_{ij}^{TT'}(\eta - r, \vec{x}'); \quad (37)$$

$$P(\tau) = \frac{1}{2\pi} \int_{S^2} d^2s [\partial_r \mathcal{M}_{ij}^{TT}] [\partial_r \mathcal{M}_{TT}^{ij}] (-r(\tau)) \quad (38)$$

In the definition of  $\mathcal{M}_{ij}^{TT}$ , the  $TT'$  on the stress tensor on the right hand side denotes transversality with respect to the  $\vec{x}'$  argument. The  $\mathcal{M}_{ij}^{TT}$  has no simple relation to the various source moments and its radial derivative is distinct from the  $\mathcal{R}_{ij}^{TT}$ . For completeness, the momentum and angular momentum fluxes are given by [2],

$$P_j = \frac{1}{16\pi H} \int_{\mathcal{J}^+} d^3x \mathcal{E}^{mn} \mathcal{L}_{\xi_j} \tilde{\chi}_{mn}^{TT} = 0; \quad (39)$$

$$\begin{aligned} J_j &= -\frac{1}{8\pi H} \int_{\mathcal{J}^+} d^3x \mathcal{E}^{mn} \mathcal{L}_{L_j} \tilde{\chi}_{mn}^{TT} \\ &= \frac{1}{4\pi} \int_{\mathcal{J}^+} d\tau d^2s \epsilon_{jmn} \mathcal{R}^{nl} \left[ \ddot{Q}_l^m + H\dot{Q}_l^m + H\dot{Q}_l^m + H^2\bar{Q}_l^m \right]^{TT} \end{aligned} \quad (40)$$

The momentum flux is zero because the integrand is linear in  $x_j$  (parity odd) and in the angular momentum flux, the second factor is proportional to the *tail term*.

### C. Isaacson Prescription [8]

In the previous subsection we saw a definition of *total energy* of radiation field of compactly supported sources in equation (33). The radiated power, received at infinity, is given in equation (35). In this subsection we recall an alternative framework, based on a ‘short wavelength expansion’ [8, 9], for a restricted class of sources but with the benefit of a symmetric, conserved, suitably gauge invariant *effective gravitational stress tensor*.

Conceptually, the framework is somewhat different from perturbation about a *fixed*, given background solution. It is designed to construct a class of solutions for which there exists a coordinate system in which the metric components display two widely separated temporal/spatial scales of variation. The slowly varying (or long wavelength  $\sim L$ ) component is taken as the *background* component and the fast (or short wavelength  $\sim \lambda$ ) component whose amplitude is small compared to that of the background, is identified as the *ripple* component<sup>2</sup>. These statements are manifestly coordinate dependent, but existence of a coordinate system with sufficiently large domain admitting such an identification, itself is a physical property. The calculational scheme is again iterative but now allows for both the background and the ripple components to be corrected. To make such a separation, an *averaging scheme* is introduced. It splits the Einstein equation into two separate, coupled equations for the background and the ripple. These equations provide a definition of the effective gravitational stress tensor.

For the metric of the form  $g_{\mu\nu} = \bar{g}_{\mu\nu} + \epsilon h_{\mu\nu}$ , the Einstein equation to  $o(\epsilon^2)$  takes the form,

$$\begin{aligned}
 R_{\mu\nu}(\bar{g} + \epsilon h) &= \Lambda(\bar{g}_{\mu\nu} + \epsilon h_{\mu\nu}) + 8\pi\epsilon(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}T_{\alpha\beta}) \\
 \therefore R_{\mu\nu}^{(0)}(\bar{g}) + \epsilon R_{\mu\nu}^{(1)}(\bar{g}, h) + \epsilon^2 R_{\mu\nu}^{(2)}(\bar{g}, h) &= \Lambda(\bar{g}_{\mu\nu} + \epsilon h_{\mu\nu}) + 8\pi \left\{ \epsilon T_{\mu\nu} - \frac{1}{2}(\bar{g}_{\mu\nu} + \epsilon h_{\mu\nu}) \right. \\
 &\quad \left. (\bar{g}^{\alpha\beta} - \epsilon h^{\alpha\beta} + \epsilon^2 h^{\alpha\rho} h_{\rho}^{\beta}) (\epsilon T_{\alpha\beta}) \right\} \quad (41)
 \end{aligned}$$

Introduce an averaging over an intermediate scale  $\ell$ ,  $\lambda \ll \ell \ll L$  which satisfies the

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<sup>2</sup> In the present context,  $L \sim \Lambda^{-1/2}$  while  $\lambda$  could be taken as the inverse of the characteristic frequency. The length scale  $R$  denoting the extent of a spatially compact source satisfies,  $R \ll \lambda$ .

properties: (i) average of odd powers of  $h$  vanish and (ii) average of space-time divergence of tensors are sub-leading [9, 11]. The average of course leaves the  $L$ -scale variations intact, in particular average of  $g_{\mu\nu}$  equals  $\bar{g}_{\mu\nu}$ . For simplicity, we will assume that the average of matter stress tensor is zero i.e. it has only  $\lambda$ -scale variations.

Taking average of the above equation and noting that  $\langle R_{\mu\nu}^{(0)} \rangle = R_{\mu\nu}^{(0)}$  and  $R_{\mu\nu}^{(2)} - \langle R_{\mu\nu}^{(2)} \rangle \approx (R_{\mu\nu}^{(2)})_{\lambda\text{-scale}}$ , the equation (41) can be separated into equation (43) for the background and equation (42) for the ripple:

$$8\pi T_{\mu\nu} = G_{\mu\nu}^{(1)} + \Lambda h_{\mu\nu} = R_{\mu\nu}^{(1)} - \frac{1}{2} (\bar{g}_{\mu\nu} R^{(1)} - \bar{g}_{\mu\nu} h^{\alpha\beta} \bar{R}_{\alpha\beta} + h_{\mu\nu} \bar{R}) + \Lambda h_{\mu\nu} \quad (42)$$

$$8\pi t_{\mu\nu} = \bar{R}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{R} + \Lambda \bar{g}_{\mu\nu} \quad \text{with,} \quad (43)$$

$$t_{\mu\nu}(\bar{g}, h) := -\frac{\epsilon^2}{8\pi} \left[ \langle R_{\mu\nu}^{(2)} \rangle - \frac{1}{2} \bar{g}_{\mu\nu} \bar{g}^{\alpha\beta} \langle R_{\alpha\beta}^{(2)} \rangle \right] \quad (44)$$

The equation (42) is exactly the same linearised Einstein equation we had before for the weak field  $h_{\mu\nu}$  and every term of it has a scale of variation  $\lambda$ . However, the equation (43) for the background is different. Although it has terms of order  $\epsilon^2$ , every term has a scale of variation  $L$ . If we now recognise that for  $\lambda$ -scale variation,  $\partial h \sim \lambda^{-1} h$  and  $\epsilon' := \lambda/L$  is taken to be of the same order as  $\epsilon$ , then the effective stress tensor which has a leading term of the form  $(\partial h)^2$ , is of the order  $(\epsilon/\epsilon')^2 \sim o(1)$  and is thus included in the equation.

The effective stress tensor defined in equation (44) is manifestly symmetric and is covariantly conserved w.r.t. the background covariant derivative, since divergence of the right hand side of (43) vanishes identically. For ripples over the Minkowski background, it is gauge invariant and the energy momentum computed using it, agrees with the quadrupole formula obtained by other methods, thereby strengthening its interpretation as *gravitational* stress tensor. An averaging procedure constructing a tensor has been given in [8, 12] and an explicit illustrative computation is given in the appendix.

At the zeroth iteration, we choose the Poincare patch of the de Sitter space-time as the solution of (43), ignoring the effective gravitational stress tensor. Let us quickly verify gauge invariance of  $t_{\mu\nu}$  under  $\delta_\xi h_{\mu\nu} = \mathcal{L}_\xi \bar{g}_{\mu\nu}$ , to leading order in  $\epsilon$ . Recall that the gauge transformation involves derivatives of  $\xi_\mu$  and for a consistency with the background plus ripple split, the gauge transformation should also be restricted to preserve it. There are two possibilities for the generator: (i)  $\xi$  is comparable with  $h$  and slowly varying, and (ii)  $\xi$  is order  $\epsilon h$  but is rapidly varying so that its derivative becomes order  $h$ . The gauge

transformation of  $t_{\mu\nu}$ , after dropping space-time divergences in the averaging, has left over terms of the form  $\Lambda\langle h\nabla\xi\rangle$ . These vanish identically for Minkowski background making the  $t_{\mu\nu}$  gauge invariant. For the  $\xi$  of type (i), the average vanishes since the enclosed quantity is rapidly varying and for  $\xi$  of type (ii), the averaged quantity is order  $\epsilon$ . But  $t_{\mu\nu}$  itself is  $o(1)$  and hence *gauge invariance of  $t_{\mu\nu}$  is ensured to the leading order* [8].

Using the properties of averaged quantities, the effective gravitaional stress tensor for the gauges fixed solution of the ripple equation evaluates to,

$$-8\pi t_{\mu\nu} = \epsilon^2 \left\langle \left[ -\frac{1}{4}\bar{\nabla}_\mu\tilde{h}^{\alpha\beta}\bar{\nabla}_\nu\tilde{h}_{\alpha\beta} + \frac{\Lambda}{3}\tilde{h}_\mu{}^\alpha\tilde{h}_{\alpha\nu} - \frac{\Lambda}{4}\bar{g}_{\mu\nu}\left(\tilde{h}^{\alpha\beta}\tilde{h}_{\alpha\beta}\right) \right] \right\rangle \quad (45)$$

This expression reduces to the stress tensor for the Minkowski background by taking  $\bar{\nabla}_\mu \rightarrow \partial_\mu$  and dropping the last two terms. However, for the ripple,  $\partial\tilde{h} \sim \lambda^{-1}\tilde{h} \sim \epsilon^{-1}\tilde{h}$ . The connection terms in the covariant derivatives are order  $\tilde{h}$ . Hence, to the leading order in  $\epsilon \sim \lambda/L$ , *all terms without derivatives of the ripple*, can be dropped and we are back to the same expression for the Minkowski background. Notice that the leading term has no  $\epsilon$ .

In the conformal coordinates, substituting  $\tilde{h}_{\alpha\beta} = a^2\tilde{\chi}_{\alpha\beta}$  and once again, keeping only the terms with derivatives of the ripple, the stress tensor for the fully gauge fixed solutions of the (42) becomes,

$$t_{\mu\nu} = \frac{1}{32\pi} \langle \partial_\mu\tilde{\chi}_{ij}^{TT} \partial_\nu\tilde{\chi}_{TT}^{ij} \rangle. \quad (46)$$

We will refer to this as the *ripple stress tensor*. We will compute this for the ‘tt’ projected, approximated retarded solution,  $\chi_{ij}^{tt}$ . In the subsection III C, we will discuss how the computations change when  $\chi_{ij}^{tt} \rightarrow \mathcal{X}_{ij}^{TT}$ .

### III. CONSERVED QUANTITIES

Given *any* symmetric, conserved stress tensor, for every Killing vector of the background space-time,  $\xi^\mu$ , the current  $J_\xi^\mu \sim T^\mu{}_\nu\xi^\nu$ , is covariantly conserved. In order that for a future directed time-like Killing vector, the corresponding energy-momentum current is also time-like and future directed, we define  $J_\xi^\mu := -T^\mu{}_\nu\xi^\nu$ . We adopt this definition for the time translation Killing vector  $T = -H(\eta\partial_\eta + x^i\partial_i)$ .

The time translation Killing vector field,  $T^\nu\partial_\nu$  involves only the  $\eta$  and  $r$  derivatives since  $x^i\partial_i = r\partial_r$  and these pass through the  $\Lambda$ -projector. For the space translation along  $j^{th}$  direction, we have  $\partial_j$  which does act on the  $\Lambda$ -projector. In the present context where

derivatives of the ripple dominate over (ripple/r), the derivative of the projector can be neglected and we write,  $\partial_j \chi_{mn}^{tt} \approx \hat{x}_j \partial_r \chi_{mn}^{tt}$ . For generators of rotation however the situation is different. Once again we get two term from the  $\partial_i$ , but now the  $\epsilon_{jki} x^k \hat{x}^i \partial_r \chi^{mn} = 0!$  and we can no longer neglect the derivative of the projector. With these understood, we write the the corresponding currents,  $J_\xi^\mu = -\frac{a^{-2}}{32\pi} \langle \partial^\mu \chi_{tt}^{mn} \partial_\nu \chi_{mn}^{tt} \rangle \xi^\nu$ . Note that the ripple stress tensor has been defined as a covariant rank 2 tensor and hence there is the factor of  $a^{-2} = H^2 \eta^2$  since the index  $\mu$  has been raised. The currents are given by,

$$a^2 J_T^\eta = -\frac{H}{32\pi} \{ \eta \langle \partial_\eta \chi_{tt}^{mn} \partial_\eta \chi_{mn}^{tt} \rangle + r \langle \partial_\eta \chi_{tt}^{mn} \partial_r \chi_{mn}^{tt} \rangle \} \quad (47)$$

$$a^2 J_T^i = \frac{H}{32\pi} \{ \eta \langle \hat{x}^i \partial_r \chi_{tt}^{mn} \partial_\eta \chi_{mn}^{tt} \rangle + r \langle \hat{x}^i \partial_r \chi_{tt}^{mn} \partial_r \chi_{mn}^{tt} \rangle \} \quad (48)$$

$$a^2 J_{\xi_j}^\eta = \frac{1}{32\pi} \langle \partial_\eta \chi_{tt}^{mn} \hat{x}_j \partial_r \chi_{mn}^{tt} \rangle \quad , \quad a^2 J_{\xi_j}^i = -\frac{1}{32\pi} \langle \hat{x}^i \partial_r \chi_{tt}^{mn} \hat{x}_j \partial_r \chi_{mn}^{tt} \rangle \quad (49)$$

$$a^2 J_{L_j}^\eta = -\frac{1}{16\pi} \epsilon_{jmn} \hat{x}^m \langle \partial_\eta \chi_{tt}^{nl} \chi_{lk} \hat{x}^k \rangle \quad , \quad a^2 J_{L_j}^i = \frac{1}{16\pi} \epsilon_{jmn} \hat{x}^m \langle \hat{x}^i \partial_r \chi_{tt}^{nl} \chi_{lk} \hat{x}^k \rangle \quad (50)$$

The unit vectors within the angular brackets have come from the spatial derivatives while those outside the brackets come from the Killing vector. It is shown in the appendix (eq. A9) that for the averaging regions far away from the source, the *unit vectors can be taken across the angular brackets and we will do so in the subsequent expressions.*

Notice that for the energy and momentum currents (48, 49), both fields have the ‘tt’ label whereas for the angular momentum current (50), the second factor does *not* have the tt label. The entire contribution to the angular momentum current comes from the derivative of the  $\Lambda$ -projector. The contribution from the derivative of the field vanishes since the field (without the projector) is spherically symmetric. In all these equations we may use  $\partial_r \chi_{mn} = -\partial_\eta \chi_{mn} - \frac{f_{mn}}{r^2}$  from (13).

We note in passing that *if* the  $\frac{f_{mn}}{r^2}$  can be neglected compared to  $\partial_\eta \chi_{mn}$ , then the currents corresponding to the generators of time and space translations, both become *proportional* to the vector  $(1, x^i/r)$  which is a null vector. Both energy and momentum propagate along this direction.

Let  $\mathcal{V}$  denote a space-time region with a boundary  $\partial\mathcal{V}$ . Then it follows that,

$$0 = \int_{\mathcal{V}} d^4x \sqrt{\bar{g}} \bar{\nabla}_\mu J_\xi^\mu = \int_{\mathcal{V}} d^4x \partial_\mu (\sqrt{\bar{g}} J_\xi^\mu) = \int_{\partial\mathcal{V}} d\sigma_\mu J_\xi^\mu, \quad (51)$$

where  $d\sigma_\mu$  is the oriented volume element of the boundary  $\partial\mathcal{V}$ .

In the next subsection we evaluate the *energy flux*,  $\mathcal{F}_\Sigma := \int_\Sigma d\sigma_\mu J_T^\mu$ , for various hypersurfaces,  $\Sigma$ 's. These, together with the conservation equation (51) will be used to relate



power received at  $\mathcal{J}^+$  to that crossing the cosmological horizon. In the following subsection, we will present the fluxes for momentum and angular momentum.

### A. Flux computations

We present flux calculations for three classes of hypersurfaces: (a) hypersurfaces of constant physical radial distance, (b) space-like hypersurfaces of constant  $\eta$  and (c) the out-going and in-coming null hypersurfaces.

The solution  $\chi_{ij}$  in this and the next subsection stands for  $\chi_{ij}^{tt}$ .

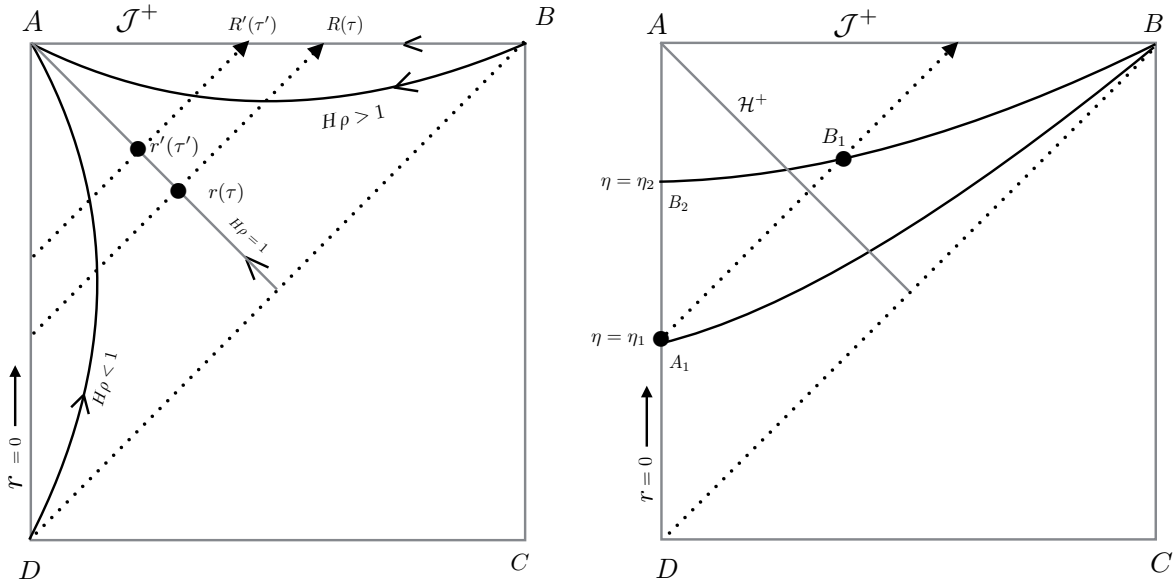


FIG. 2: The figure on the left shows the  $\rho = \text{constant}$  hypersurfaces which are time-like for  $H\rho < 1$ , null for  $H\rho = 1$  and space-like for  $H\rho > 1$ . The two 45 degree out-going null hypersurfaces intersecting the  $\mathcal{H}^+$  and  $\mathcal{J}^+$  in the spheres at  $r(\tau), r'(\tau'), R(\tau), R'(\tau')$ , bound a space-time region. The figure on the right shows the space-like hypersurfaces with constant value of  $\eta$ . The fluxes across the out-going null hypersurfaces turn out to be zero signifying sharp propagation of energy-momentum and angular momentum. Hence the energy flux across the portion of the horizon bounded by the spheres at  $r(\tau), r'(\tau')$  equals the flux across the portion of the future infinity bounded by the spheres at  $R(\tau), R'(\tau')$ .

1. *Hypersurface of constant physical radial distance:*

These hypersurfaces are time-like, null and space-like according as the physical distance being less than, equal to and greater than the physical distance to the cosmological horizon, namely  $H^{-1}$ . They are spanned by the integral curves of the Killing vector  $T$ .

This Killing vector is special because in the static patch, it is time-like and its integral curves represent Killing observers. Denoting  $x^i := r\hat{x}^i$ ,  $\hat{x}^i\hat{x}^j\delta_{ij} = 1$ , in general, its integral curves are given by  $\eta(\tau) = \eta_*e^{-H\tau}$ ,  $r(\tau) = r_*e^{-H\tau}$ ,  $\hat{x}^i = \hat{x}_*^i$ . Evidently, along each curve,  $\rho := r/(-H\eta) = r_*/(-H\eta_*)$  is constant. This also represents the *physical radial distance*,  $r_{phy} := |\Omega|r$ . Each particular curve is labelled by  $\rho$  and the two angular coordinates  $\hat{x}_*^i$ . We compute the flux across the hypersurface  $\Sigma_\rho$ , defined by  $r_{phy} = \rho$ . This surface is coordinatized by the Killing parameter  $\tau$  and the usual spherical angles  $\theta, \phi$  represented by the unit vectors  $\hat{x}^i$ . These hypersurfaces are topologically  $\Sigma_\rho \sim \Delta\tau \times S^2$  and their embedding is given by,

$$\eta(\tau, \theta, \phi) = \eta_*e^{-H\tau} , \quad x = r_*e^{-H\tau} \sin\theta \cos\phi , \quad y = r_*e^{-H\tau} \sin\theta \sin\phi , \quad z = r_*e^{-H\tau} \cos\theta ,$$

with  $r_* + H\rho\eta_* = 0$ .

The induced metric is given by  $h_{ab} = \text{diag}(H^2\rho^2 - 1, \rho^2, \rho^2\sin^2\theta)$ . This has Lorentzian signature for  $H\rho < 1$  (inside the static patch), is degenerate for  $H\rho = 1$  (the cosmological horizon) and Euclidean signature for  $H\rho > 1$  (beyond the cosmological horizon). The measure factor for the non-null cases is given by  $\sqrt{|\det h_{ab}|} = \sqrt{|1 - H^2\rho^2|}\rho^2\sin\theta$  while on the cosmological horizon it is given by  $\sqrt{h_{ab}} = \rho^2\sin\theta$ . Here  $\underline{a}, \underline{b}$  denote the ‘transverse’ coordinates  $\theta, \phi$ . In the non-null case, the unit normal is given by  $n_\mu = \frac{\epsilon}{|H\eta|}|1 - H^2\rho^2|^{-1/2}(H\rho, x_i/r) \leftrightarrow n^\mu = \epsilon|H\eta||1 - H^2\rho^2|^{-1/2}(-H\rho, x^i/r)$ . Here  $\epsilon = +1$  for *time-like*  $\Sigma_\rho$  ( $H\rho < 1$ ) and  $\epsilon = -1$  for *space-like*  $\Sigma_\rho$  ( $H\rho > 1$ ). On the cosmological horizon, we *choose* the normal to be:  $n_\mu = -|H\eta|^{-1}(H\rho, x_i/r) \leftrightarrow n^\mu = -|H\eta|(-H\rho, x^i/r)$ , so that  $n^\mu = T^\mu$  is future directed. Introduce  $N^\mu := (-H\rho, \hat{x}^i)$ , so that the normal for non-null cases is expressed as  $n^\mu = \epsilon|H\eta||1 - H^2\rho^2|^{-1/2}N^\mu$ . Note that the  $n^\mu$  is the same for the space-like and the null hypersurfaces,  $\Sigma_{\rho \geq H^{-1}}$ . For the time-like hypersurface, the  $n^\mu$  points in the opposite direction. However, the induced orientation on  $\Sigma_\rho$  is also reversed as the hypersurface changes from being space-like to being time-like. Hence, in *all cases*,  $H\rho > 0$ ,  $n^\mu\sqrt{h} = -|H\eta|N^\mu\rho^2\sin\theta$

and the hypersurface integral is expressed as,

$$\begin{aligned} \int_{\Sigma_\rho} d\Sigma_\alpha J_T^\alpha &= - \int_{\tau_1}^{\tau_2} d\tau \int_{S^2} d^2s \rho^2 (-|H\eta(\tau)| N^\mu) (-t_{\mu\nu} T^\nu) \quad \text{with} \quad (52) \\ N^\mu t_{\mu\nu} T^\nu &= -Hr(\tau) \{t_{\eta\eta} + \hat{x}^i t_{ij} \hat{x}^j - \hat{x}^i t_{i\eta} ((H\rho)^{-1} + H\rho)\} \end{aligned}$$

The minus sign in front of the hypersurface integral is because the orientation defined by the Killing parameter and the angles, is *negative* relative to that defined by the  $r$  and the angles. The  $\sin\theta$  is absorbed in  $d^2s$ . The minus sign in the last parentheses is due to the definition  $J_\mu = -t_{\mu\nu} T^\nu$ . In the second line, we have also used  $-H\rho\eta = r$  valid on  $\Sigma_\rho$ .

Substituting for the ripple stress tensor, and taking the unit vectors  $\hat{x}$  outside of the angular bracket as mentioned before, the expression within the braces becomes,

$$\{\} = \frac{1}{32\pi} \left\{ \langle \partial_\eta \chi_{mn} \partial_\eta \chi^{mn} + \partial_r \chi_{mn} \partial_r \chi^{mn} \rangle - \frac{1 + H^2 \rho^2}{H\rho} \langle \partial_r \chi_{mn} \partial_\eta \chi^{mn} \rangle \right\} \quad (53)$$

The (implicit)  $tt$  projection introduces angle dependence in the  $\chi_{ij}^{tt}$ , however equation (52) needs only  $\eta$  and  $r$  derivatives.

Eliminating  $\partial_r \chi_{ij}$  using equation (13), we write,

$$\begin{aligned} \{\} &= \frac{1}{32\pi} \left[ \langle \partial_\eta \chi_{mn} \partial_\eta \chi^{mn} \rangle \frac{(1 + H\rho)^2}{H\rho} + \left\langle \frac{f_{mn}}{r^2} \partial_\eta \chi^{mn} \right\rangle \frac{(1 + H\rho)^2}{H\rho} + \left\langle \frac{f_{mn} f^{mn}}{r^4} \right\rangle \right] \\ &= \frac{1}{32\pi} \frac{(1 + H\rho)^2}{H\rho} \left[ \left\langle \partial_\eta \chi_{mn} \partial_\eta \chi^{mn} + \frac{f_{mn}}{r^2} \partial_\eta \chi^{mn} \right\rangle + \frac{H\rho}{(1 + H\rho)^2} \left\langle \frac{f_{mn} f^{mn}}{r^4} \right\rangle \right] \\ \therefore \int_{\Sigma_\rho} d\Sigma_\alpha J_T^\alpha &= \int_{\tau_1}^{\tau_2} d\tau \int_{S^2} d^2s \left[ -\rho^2 H^2 \eta(\tau) r(\tau) \right] \{\} \quad (54) \end{aligned}$$

The approximated solution  $\chi_{ij}$ , is valid for (source dimension)/(distance to the source)  $\ll 1$ . This is consistent with the assumption that the  $\lambda/r \ll 1$ . Furthermore, the source being rapidly changing,  $\lambda H \ll 1$ , it follows that  $f_{mn}/r^2 \ll \dot{f}_{mn}/r$ . Hence we drop  $f_{mn}/r^2$  terms. With this,  $\{\}$  takes a simple quadratic form  $\frac{1}{32\pi} (1 + H\rho)^2 (H\rho)^{-1} \langle \partial_\eta \chi_{mn} \partial_\eta \chi^{mn} \rangle$ .

To compute  $\partial_\eta$  we recall,  $\eta_{ret} = \eta - r := -H^{-1} e^{-Ht_{ret}} := -(Ha(t_{ret}))^{-1}$  and use,

$$\partial_\eta f_{ij}(\eta_{ret}) = \partial_{\eta_{ret}} f_{ij}(\eta_{ret}) = a(\eta_{ret}) \partial_{t_{ret}} f_{ij}(t_{ret}) = a(\eta_{ret}) (\mathcal{L}_T + 2H) f_{ij}(t_{ret})$$

This leads to (overdot denoting  $\mathcal{L}_T$ ),

$$\begin{aligned}\partial_\eta \chi_{mn}(\eta_{ret}) &= \frac{1}{r} \partial_\eta f_{mn}(\eta_{ret}) + \partial_\eta g_{mn}(\eta_{ret}) \\ &= \frac{a(t_{ret})}{r} (\mathcal{L}_T f_{mn} + 2H f_{mn}) + a(t_{ret}) (\mathcal{L}_T g_{mn} + 2g_{mn})\end{aligned}\quad (55)$$

$$\mathcal{L}_T f_{mn} = \frac{2}{a(t_{ret})} \left[ \ddot{Q}_{mn} + H\dot{Q}_{mn} - 2H^2\dot{Q}_{mn} + H\ddot{Q}_{mn} + H^2\dot{Q}_{mn} - 2H^3\bar{Q}_{mn} \right] \quad (56)$$

$$\mathcal{L}_T g_{mn} = -2H \left[ \ddot{Q}_{mn} + H\dot{Q}_{mn} + H\ddot{Q}_{mn} + H^2\dot{Q}_{mn} \right] \quad \Rightarrow \quad (57)$$

$$\partial_\eta \chi_{mn}^{tt}(\eta_{ret}) = \frac{2}{r} \frac{\eta}{\eta - r} \mathcal{Q}_{mn}^{tt} \quad \text{with,} \quad (58)$$

$$\mathcal{Q}_{mn}^{tt} := \left[ \ddot{Q}_{mn} + 3H\dot{Q}_{mn} + 2H^2\dot{Q}_{mn} + H\ddot{Q}_{mn} + 3H^2\dot{Q}_{mn} + 2H^3\bar{Q}_{mn} \right]^{tt} (t_{ret}) \quad (59)$$

Here we have also used  $(1 - a(t_{ret})rH) = \frac{\eta}{\eta - r}$ . Collecting all expressions, we write the flux through a segment of  $r_{phy} = \rho$  hypersurface in a convenient form as,

$$\begin{aligned}\int_{\Sigma_\rho} d\Sigma_\alpha J_T^\alpha &= \int_{\tau_1}^{\tau_2} d\tau \int_{S^2} d^2s \left[ -\rho^2 H^2 \eta(\tau) r(\tau) \right] \left[ \frac{(1 + H\rho)^2}{32\pi H\rho} \right] \left\langle \left[ \frac{2}{r} \frac{\eta}{\eta - r} \right]^2 \mathcal{Q}_{ij}^{tt} \mathcal{Q}_{tt}^{ij} \right\rangle \\ &= \int_{\tau_1}^{\tau_2} d\tau \int_{S^2} d^2s \left[ \frac{1}{8\pi} \right] \langle \mathcal{Q}_{ij}^{tt} \mathcal{Q}_{tt}^{ij} \rangle (t_{ret})\end{aligned}\quad (60)$$

In the appendix, we show that for large  $\rho$ , the expression within the square brackets inside the angular brackets, can be taken outside. Then, using  $r = -H\rho\eta$  which is valid over the hypersurface  $\forall \rho \in \mathbb{R}^+$ , we see that the explicit dependence on  $\rho$  (for large enough  $\rho$ ) disappears from the integrand but there is an implicit dependence on  $\rho$  and  $\tau$  through  $t_{ret}$ . If however, the  $\tau$ -integration is extended over its full range,  $(-\infty, \infty)$ , then the integral is independent of  $\rho$  as well. Hence, *for sufficiently large  $\rho$ , all Killing observers infer the same energy flux in the limit  $(\tau_1, \tau_2) \rightarrow (-\infty, \infty)$ .*

The  $\rho$  independence of the full flux integral in particular means that the total flux across  $\mathcal{J}^+$  equals the total flux across the cosmological horizon,  $\mathcal{H}^+$ .

$$\lim_{\rho \rightarrow \infty} \int_{\Sigma_\rho} d\Sigma_\mu J_T^\mu = \int_{\Sigma_{(H\rho=1)}} d\Sigma_\mu J_T^\mu \Leftrightarrow \int_{\mathcal{J}^+} d\Sigma_\mu J_T^\mu = \int_{\mathcal{H}^+} d\Sigma_\mu J_T^\mu. \quad (61)$$

## 2. Flux through a constant $\eta$ slice:

The hypersurface  $\Sigma_{\eta_0}$  defined by  $\eta = \eta_0$  is a cosmological slice  $\sim \mathbb{R}^3$ . It is space-like, with a normal  $n_\mu = -|H\eta_0|^{-1}(1, \vec{0}) \leftrightarrow n^\mu = |H\eta_0|(1, \vec{0})$  which is future directed. We choose

a finite portion of it with  $r \in [r_1, r_2]$ . The hypersurface is topologically  $\Delta r \times s^2$ . Choosing the  $(r, \theta, \phi)$  coordinates on the hypersurface, the embedding is given by

$$\eta(r, \theta, \phi) = \eta_0, \quad x = r \sin\theta \cos\phi, \quad y = r \sin\theta \sin\phi, \quad z = r \cos\theta.$$

The induced metric is given by  $h_{ab} = (H\eta_0)^{-2} \text{diag}(1, r^2, r^2 \sin^2\theta)$  giving  $\sqrt{|\det h_{ab}|} = |H\eta_0|^{-3} r^2 \sin\theta$ . Denoting  $N^\mu := (1, \vec{0})$ , the hypersurface integral is given by,

$$\int_{\Sigma_{\eta_0}} d\Sigma_\mu J_T^\mu = \int_{r_1}^{r_2} dr \int_{S^2} d^2s \, r^2 a^2(\eta_0) (-N^\mu t_{\mu\nu} T^\nu) \quad \text{with} \quad (62)$$

$$\begin{aligned} N^\mu t_{\mu\nu} T^\nu &= (-H) (t_{\eta\eta}\eta + t_{\eta i}x^i) \\ &= \frac{-H}{32\pi} (\eta \langle \partial_\eta \chi_{mn} \partial_\eta \chi^{mn} \rangle + x^i \langle \partial_\eta \chi_{mn} \partial_i \chi^{mn} \rangle) \end{aligned} \quad (63)$$

$$= \frac{-H}{32\pi} \left( (\eta - r) \langle \partial_\eta \chi_{mn} \partial_\eta \chi^{mn} \rangle - \left\langle \frac{f_{mn}}{r} \partial_\eta \chi^{mn} \right\rangle \right) \quad (64)$$

$$\therefore \int_{\Sigma_{\eta_0}} d\Sigma_\mu J_T^\mu = -\frac{1}{32\pi H^2 \eta_0^2} \int_{r_1}^{r_2} dr \int_{S^2} d^2s \, r^2 \left\{ \frac{1}{a(\eta_{ret})} \langle \partial_\eta \chi_{mn} \partial_\eta \chi^{mn} \rangle \right\} \quad (65)$$

$$\approx \int_{r_1}^{r_2} \frac{dr}{H(r - \eta_0)} \int_{S^2} d^2s \left[ \frac{-1}{8\pi} \right] \langle \mathcal{Q}_{mn}^{tt} \mathcal{Q}_{tt}^{mn} \rangle \quad (66)$$

By the same reasoning as before, we have dropped the  $\frac{f_{mn}}{r}$  and also used equation (A12). In the limit  $\eta_0 \rightarrow 0$  with  $(r_1, r_2) \rightarrow (0, \infty)$ , the hypersurface becomes  $\mathcal{J}^+$  and the integration measure becomes  $\frac{dr}{Hr}$ . The limit  $\eta \rightarrow 0$  is thus finite.

As noted earlier, the hypersurface integral when expressed in terms of the Killing parameter, has a minus sign due to the reversal of the induced orientation. The measures (positive) themselves are related as  $\frac{dr}{Hr} = d\tau$ , leading to  $\int_0^\infty dr/Hr = -\int_{-\infty}^\infty d\tau$  and we get,

$$\lim_{\eta_0 \rightarrow 0} \int_{\Sigma_{\eta_0}} d\Sigma_\mu J_T^\mu = \lim_{\rho \rightarrow \infty} \int_{\Sigma_\rho} d\Sigma_\mu J_T^\mu. \quad (67)$$

### 3. Flux through null hypersurfaces:

There are two families of *future directed* null hypersurfaces given by  $\eta + \epsilon r + \sigma = 0$ , see figure (2). For  $\epsilon = +1$ , these 45 degree lines in the Penrose diagram are parallel to the cosmological horizon while for  $\epsilon = -1$ , the lines are parallel to the null boundary of the Poincare patch. We refer to these as the *in-coming* ( $\epsilon = 1$ ) and *out-going* ( $\epsilon = -1$ ) null hypersurfaces. The parameter  $\sigma$  labels members of these families.

The null normals of these families are of the form  $n_\mu = \gamma(1, \epsilon \hat{x}_i) \leftrightarrow n^\mu = (H\eta)^2 \gamma(-1, \epsilon \hat{x}^i)$ , where  $\gamma$  is to be chosen suitably and should be *negative* for future directed hypersur-

faces. Choosing coordinates  $(\lambda, \theta, \phi)$  on a null hypersurface, its embedding may be taken as  $\eta(\lambda), r(\lambda)$  with identity mapping of the angles. Here  $\lambda$  is an affine parameter of the null geodesics generating the null hypersurfaces. The induced metric is obtained as  $h_{ab} = (H\eta)^{-2} \text{diag}(0, r^2, r^2 \sin^2\theta)$ . Note that the orientation of the hypersurfaces, relative to that defined by  $(r, \theta, \phi)$  is the same for the out-going hypersurfaces and opposite for the in-coming hypersurfaces. The hypersurface integral is then given by  $(N^\mu := (-1, \epsilon \hat{x}^i))$ ,

$$\int_{\Sigma_{(\epsilon, \sigma)}} d\Sigma_\mu J_T^\mu = -\epsilon \int_{\lambda_1}^{\lambda_2} d\lambda \int_S d^2s \left[ \frac{r^2}{H^2 \eta^2} \right] (H^2 \eta^2) \gamma (-N^\mu t_{\mu\nu} T^\nu) \quad , \quad (68)$$

$$\begin{aligned} N^\mu t_{\mu\nu} T^\nu &= -H (-t_{\eta\eta} \eta - t_{\eta j} r \hat{x}^j + \epsilon \hat{x}^i t_{i\eta} \eta + \epsilon r \hat{x}^i t_{ij} \hat{x}^j) \\ &= -\frac{H}{32\pi} (-\eta \langle \partial_\eta \chi_{mn} \partial_\eta \chi^{mn} \rangle + (\epsilon \eta - r) \langle \partial_\eta \chi_{mn} \partial_r \chi^{mn} \rangle + \epsilon r \langle \partial_r \chi_{mn} \partial_r \chi^{mn} \rangle) \\ &= -\frac{H}{32\pi} (1 + \epsilon)(r - \eta) \langle \partial_\eta \chi_{mn}^{tt} \partial_\eta \chi_{tt}^{mn} \rangle + o(r^{-2}) \end{aligned} \quad (69)$$

$$= -\frac{H}{32\pi} (1 + \epsilon)(r - \eta) \left[ \frac{4}{r^2} \frac{\eta^2}{(\eta - r)^2} \right] \langle \mathcal{Q}_{mn}^{tt} \mathcal{Q}_{tt}^{mn} \rangle + o(r^{-2}) \quad (70)$$

$$\therefore \int_{\Sigma_{(\epsilon, \sigma)}} d\Sigma_\mu J_T^\mu = \epsilon \int_{\lambda_1}^{\lambda_2} d\lambda \int_{S^2} d^2s [\gamma H] \left[ \frac{(1 + \epsilon)}{8\pi} \frac{\eta^2}{\eta - r} \right] \langle \mathcal{Q}_{mn}^{tt} \mathcal{Q}_{tt}^{mn} \rangle \quad (71)$$

As before, we have dropped the  $f_{mn}/r^2$  terms from  $\partial_r \chi_{mn}$  and used the equation (A12).

It is immediately clear that the flux through the out-going null hypersurfaces ( $\rho$  or  $r$  increase along these) vanishes. In the  $\epsilon = 1$  family, only the cosmological horizon is of interest. For this we have  $(\eta = -r)$  and we *choose* the factor  $\gamma = -(Hr)^{-1}$  so the null normal matches with the Killing vector ( $\gamma$  is negative as desired for future orientation) and the affine parameter  $\lambda$  matches with the Killing parameter  $\tau$ . With this choice, the flux in eqn. (71) matches with that given in eq. (60) for  $H\rho = 1$ . Thus, once again, the full flux through cosmological horizon is exactly same as that of  $r_{\text{physical}} = \text{const}$  hypersurfaces.

*Remarks:* All three calculations consistently have the same  $[+1/8\pi]$  factor, with integrals oriented along the stationary Killing vector.

It is surprising at first that the flux through  $\eta - r = \text{constant}$  hypersurfaces is zero, which indicates sharp propagation of the energy, even though the retarded solution has a tail contribution. This can be seen more directly as follows. Let us recast eqn. (54) as

$$\int_{\Sigma_\rho} d\Sigma_\alpha J_T^\alpha = \frac{1}{32\pi} \int_{\tau_1}^{\tau_2} d\tau \int_{S^2} d^2s \left\langle r^2 (\eta - r)^2 \left( \frac{1}{\eta} \partial_\eta \chi_{ij} \right) \left( \frac{1}{\eta} \partial_\eta \chi_{ij} \right) \right\rangle \quad (72)$$

where we have neglected the  $1/r^2$  terms and have used  $r = -H\rho\eta$ .

Now in taking the  $\eta$ -derivative, contribution of the tail term in (5) cancels out, leaving only the contribution from the sharp term:

$$\frac{1}{\eta} \partial_\eta \chi_{ij} = \frac{4}{r \eta_{ret}} \partial_\eta \int d^3 x' T_{ij}(\eta - r, x'). \quad (73)$$

## B. Momentum and angular momentum fluxes

For the same three classes of hypersurfaces, we present the momentum and angular momentum fluxes. We already have the measures for these hypersurfaces as well as the currents given in (49, 50). The full fluxes, only to the leading order in  $r^{-1}$ , are given by,

$$\Sigma_\rho : - \int_{-\infty}^{\infty} d\tau \int_{S^2} d^2 s \rho^2 \frac{1}{H\eta} (H\rho, \hat{x}_i) J^\mu \quad (74)$$

$$\Sigma_{\eta_0} : \int_0^\infty dr \int_{S^2} d^2 s \left[ \frac{r^2}{|H\eta_0|^3} \right] \left[ \frac{1}{H\eta_0} \right] (1, \vec{0}) J^\mu \quad (75)$$

$$\Sigma_{(\epsilon, \sigma)} : -\epsilon \int_{\lambda_1}^{\lambda_2} d\lambda \int_{S^2} d^2 s \left[ \frac{r^2 \gamma}{|H\eta_0|^2} \right] (1, \epsilon \hat{x}_i) J^\mu \quad (76)$$

*Momentum fluxes:* The momentum current is given by,

$$J_{\xi_j}^\mu = - \frac{a^{-2}}{32\pi} \left[ \frac{2}{r} \frac{\eta}{\eta - r} \right]^2 \langle \mathcal{Q}_{tt}^{mn} \mathcal{Q}_{mn}^{tt} \rangle (\hat{x}_j) (1, -\hat{x}^i) \quad (77)$$

Dotting with the  $n_\mu$  produces a rotational scalar and the average is a rotational scalar too. Then the angular integration with  $\hat{x}_j$  vanishes, in all three cases. Hence, the momentum flux is zero across the three classes of hypersurfaces.

*Angular Momentum fluxes:* Replacing  $\partial_r \chi_{tt}^{mn} \approx -\partial_\eta \chi_{tt}^{mn}$ , we can write the angular momentum current as,

$$J_{L_j}^\mu = - \frac{a^{-2}}{16\pi} \left[ \frac{2}{r} \frac{\eta}{\eta - r} \right] [\epsilon_{jmn} \hat{x}^m \hat{x}^k \langle \mathcal{Q}_{tt}^{nl} \chi_{kl} \rangle] (1, \hat{x}^i)$$

The fluxes then take the form,

$$\Sigma_\rho : - \frac{\rho}{8\pi} \int_{-\infty}^{\infty} d\tau \int_{S^2} d^2 s [\epsilon_{jmn} \hat{x}^m \hat{x}^k \langle \mathcal{Q}_{tt}^{nl} \chi_{kl} \rangle] \quad (78)$$

$$\Sigma_{\eta_0} : \frac{1}{8\pi H^2 |\eta_0|} \int_0^\infty dr \frac{r}{r - \eta_0} \int_{S^2} d^2 s [\epsilon_{jmn} \hat{x}^m \hat{x}^k \langle \mathcal{Q}_{tt}^{nl} \chi_{kl} \rangle] \quad (79)$$

$$\Sigma_{(\epsilon, \sigma)} : -\epsilon \frac{1 + \epsilon}{8\pi} \int_{\lambda_1}^{\lambda_2} d\lambda (-\gamma) \frac{r \eta}{\eta - r} \int_{S^2} d^2 s [\epsilon_{jmn} \hat{x}^m \hat{x}^k \langle \mathcal{Q}_{tt}^{nl} \chi_{kl} \rangle] \quad (80)$$

Consider the average. The function enclosed in averaging is product of the  $\Lambda$ -projector containing angular dependence and a function having dependence on  $(\eta, r)$ . The averaging

can then be split into averaging over a cell  $\Delta\omega$  in the angular coordinates around the direction  $\hat{r}$  and averaging over a cell in the  $(\eta, r)$  plane, see equation (A9). Thus, we write,

$$\langle \mathcal{Q}_{tt}^{nl} \chi_{kl} \rangle(\eta, r, \hat{r}) = \left[ \frac{1}{\Delta\omega} \int_{\Delta\omega} d^2 s' \Lambda_{rs}^{nl}(\hat{r}') \right] [\langle \mathcal{Q}^{rs} \chi_{kl} \rangle(\eta, r)] \quad (81)$$

$$= \Lambda_{rs}^{nl}(\hat{r}) \langle \mathcal{Q}^{rs} \chi_{kl} \rangle(\eta, r) \quad (82)$$

The angular integration over the sphere can be done explicitly:

$$\int_{S^2} d^2 s \epsilon_{jmn} \hat{x}^m \hat{x}^k \Lambda_{rs}^{nl}(\hat{r}) \langle \mathcal{Q}^{rs} \chi_{kl}(\eta, r) \rangle = \frac{8\pi}{15} \epsilon_{jmn} \langle \mathcal{Q}^{nl} \chi_l^m \rangle(\eta, r). \quad (83)$$

This is to be integrated over the Killing parameter  $\tau$  or  $r$  or  $\lambda$  for the three classes of hypersurfaces. The average is now over an  $(\eta, r)$  cell.

This integration in the flux expressions above, can be expressed in terms of the Killing parameter  $\tau$  and then they all take the same form *provided* for  $\Sigma_{\eta_0}$  we consider the  $\eta_0 \approx 0 \rightarrow \mathcal{J}^+$  and for the null hypersurface we choose the cosmological horizon,  $\mathcal{H}^+$  ( $\epsilon = +1, \eta = -r$ ):

$$(\text{Flux of the of (Angular Momentum)}_j) = -\frac{1}{15} \int_{-\infty}^{\infty} d\tau a(\eta(\tau)) r(\tau) \epsilon_{jmn} \langle \mathcal{Q}^{nl} \chi_l^m \rangle \quad (84)$$

The radiation field  $\mathcal{Q}^{nl}$  is given in equation (34) but without the  $tt$  label and,

$$\begin{aligned} \chi_{lm} = & \frac{2}{ra(\eta)} \left[ \ddot{Q}_{lm} + 2H\dot{Q}_{lm} + H\dot{Q}_{lm} + 2H^2\bar{Q}_{lm} \right] (\eta_{ret}) \\ & + 2H^2 \left[ \dot{Q}_{lm} + H\bar{Q}_{lm} \right] (\eta_{ret}) - 2H^2 \left[ \dot{Q}_{lm} + H\bar{Q}_{lm} \right] (-\infty). \end{aligned} \quad (85)$$

This flux does *not* have a finite limit to  $\mathcal{J}^+$  due to the *tail term* in  $\chi_l^m$  and does *not* match with the flux given by [2]. It is finite along the  $\mathcal{H}^+$  though. It does *not* match with the correct angular momentum flux in the flat space limit as well and it is well known [9, 13] that the Isaacson effective stress tensor does not suffice to capture the flux of angular momentum. The sharp propagation property still holds in the sense that the flux across out-going null hypersurface is zero.

### C. Extending from ‘tt’ to ‘TT’

We have used the algebraic ‘tt’ projection on the approximated, retarded solution. How would the results change if we were to use the ‘TT’ decomposition of the exact solution prior to the  $|\vec{x}'|/|\vec{x}| \ll 1$  approximation? For this we note a few points.



It is easy to see that the TT part of the retarded solution is given by [2],

$$\begin{aligned} \mathcal{X}_{ij}^{TT}(\eta, x) = & 4 \int d^3x' \frac{\eta}{|x-x'|(\eta-|x-x'|)} T_{ij}^{TT'}(\eta', x') \Big|_{\eta'=\eta-|x-x'|} \\ & + 4 \int d^3x' \int_{-\infty}^{\eta-|x-x'|} d\eta' \frac{T_{ij}^{TT'}(\eta', x')}{\eta'^2} \end{aligned} \quad (86)$$

where the  $TT'$  refers to the *second* argument of the stress tensor. This follows by checking that the divergence,  $\partial_x^i$ , of the right hand side converts into the divergence,  $\partial_{x'}^i$ , on the second argument of the stress tensor. For this relation, it is important to have the exact  $|x-x'|$  dependence and that the source has compact support. The TT part of the approximated solution cannot be similarly expressed in terms of TT part of the source stress tensor.

We can now consider the solution (86) for  $|x| \gg |x'|$ , and replace  $|x-x'| \approx r$  which simplifies the source integral. We denote this approximated expression as  $\chi_{ij}^{TT}$ . This satisfies the transversality condition to  $o(r^{-1})$  only<sup>3</sup>. Furthermore, since the transverse, traceless part of the stress tensor *drops out of its conservation equation*, we cannot directly express  $\int_{source} T_{ij}^{TT'}$  in terms of correspondingly defined moments. Nevertheless, we do get,

$$\partial_\eta \chi_{ij}^{TT}(\eta, x) = 4 \frac{\eta}{r(\eta-r)} \partial_\eta \mathcal{M}_{ij}^{TT} \quad , \quad \mathcal{M}_{ij}^{TT}(\eta-r) := \int d^3x' T_{ij}^{TT'}(\eta-r, x') \quad (87)$$

$$\partial_m \chi_{ij}^{TT}(\eta, x) = 4 \frac{\hat{x}_m}{r} \left( \frac{\eta}{\eta-r} \partial_r \mathcal{M}_{ij}^{TT} - \frac{\mathcal{M}_{ij}^{TT}}{r} \right) = -\hat{x}_m \partial_\eta \chi_{ij}^{TT} - 4 \frac{\hat{x}_m}{r^2} \mathcal{M}_{ij}^{TT} \quad (88)$$

$$\therefore \partial_r \chi_{ij}^{TT} = -\partial_\eta \chi_{ij}^{TT} - 4 \frac{\mathcal{M}_{ij}^{TT}}{r^2} \quad (89)$$

The equation (89) has the same form as eq.(13). The equation (87) has the same form as eq.(58) which introduced the radiation field  $\mathcal{Q}_{ij}^{tt}$ . We can thus introduce a new ‘radiation field’,  $\mathcal{Q}_{ij}^{TT} := 2\partial_\eta \mathcal{M}_{ij}^{TT}$ . With this, the form of the expressions for fluxes will remain the same with  $\mathcal{Q}_{ij}^{tt} \rightarrow \mathcal{Q}_{ij}^{TT}$ . Note that unlike  $\mathcal{Q}_{ij}^{tt}$ , the  $\mathcal{Q}_{ij}^{TT}$  does *not* have a simple relation to the source moments defined earlier. Nevertheless, it shares the important property with  $\mathcal{Q}_{ij}^{tt}$ , namely, it too is a function of  $\eta-r$  alone. This enables the space-time averaging to be reduced to averaging over  $\rho = \text{constant}$  hypersurfaces, as shown in eqn. (A13).

$$\langle \partial_\eta \chi_{mn}^{TT} \partial_\eta \chi_{TT}^{mn} \rangle(t, r, \hat{r}) = 4 \frac{a^2(\bar{t}_0)}{\rho_0^2} \langle \mathcal{Q}_{ij}^{TT} \mathcal{Q}_{TT}^{ij} \rangle(\bar{t}_0, \hat{r}) \quad (90)$$

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<sup>3</sup> Extracting the TT part and making the approximation for  $|\vec{x}| \gg |\vec{x}'|$ , do not commute i.e.  $[(\mathcal{X}_{ij})_{approx}]^{TT} \neq [(\mathcal{X}_{ij})^{TT}]_{approx}$ . This is so because the  $\partial^j$  of the l.h.s. is always zero by definition while that of the r.h.s. is non-zero in general. We are using the r.h.s.

In the next section we restrict to the energy fluxes and see two applications of the conservation equation and the sharp propagation property.

#### IV. IMPLICATIONS OF CONSERVATION EQUATION AND SHARP PROPAGATION

In the previous subsection, we assembled fluxes through various hypersurfaces, all having the topology  $\Delta \times S^2$ . We considered  $\Delta$  to be a finite interval and also the cases with  $\Delta = \mathbb{R}$ . The relevant hypersurfaces have  $\rho = \text{constant}$ . In all cases, the energy flux integral had the form,

$$\mathcal{F}(a, b) := \int_a^b d\tau \int_{S^2} d^2s \left[ \frac{1}{8\pi} \right] \langle \mathcal{Q}_{mn}^{tt} \mathcal{Q}_{tt}^{mn} \rangle =: \int_a^b d\tau \langle F \rangle(\tau) . \quad (91)$$

As shown in the appendix, equations (A12), the angular brackets denote averaging over  $\tau$ -intervals and a trivial averaging over the angular intervals. Since the angular average is trivial, we have taken the angular integration across the averaging and denoted the integration over the sphere by  $\langle F \rangle(\tau)$ . Using the mean value theorem, we write,

$$\int_a^b d\tau \langle F \rangle(\tau) = \langle F \rangle(c)(b - a) = \int_{c-\delta}^{c+\delta} d\tau F \frac{(b - a)}{2\delta} , \quad c \in (a, b) . \quad (92)$$

Let us choose  $(a, b)$  to be an averaging interval i.e.  $(b - a) = 2\delta$ . Recall that the averaged quantities are slowly varying i.e.  $\langle F \rangle$  is varying only over the scale  $L \gg 2\delta$  and thus essentially constant over the averaging interval. Therefore, we can choose  $c = (a + b)/2$  possibly making a small error. But then the right hand side of the last equality in the above equation becomes  $\int_a^b d\tau F(\tau)$ . In effect, for integral over an averaging interval, we can drop the angular brackets in equation (91).

For  $a \ll 0, b \gg 0$ , the  $\tau$  integral can be replaced by a sum with each sub-interval,  $[a_k, b_k]$  being an averaging interval. Using the above argument, we can write,

$$\mathcal{F}(a, b) \approx \sum_k \int_{a_k}^{b_k} d\tau \int_{S^2} d^2s \left[ \frac{1}{8\pi} \right] \mathcal{Q}_{mn}^{tt} \mathcal{Q}_{tt}^{mn} \quad (93)$$

However, the averaging  $\tau$ -intervals cannot be made arbitrarily finer and the Riemann sum cannot be taken to the integral. Hence, flux integral over an averaged integrand matches with the flux integral over an *un-averaged* integrand only at a *coarse grained level*. The same arguments also hold for  $\mathcal{Q}_{mn}^{tt} \rightarrow \mathcal{Q}_{mn}^{TT}$  and then the fluxes defined using the averaged stress tensor match with the expressions (36) at a *coarse grained level*.

By judicious choices of hypersurfaces comprising the boundary  $\partial\mathcal{V}$  of a space-time region  $\mathcal{V}$ , we can relate different fluxes using the conservation equation (51). The sharp propagation of energy comes in very useful. We note two of its implications.

(1) The flux across two hypersurfaces  $\Sigma_{\eta_1}$  and  $\Sigma_{\eta_2}$  *cannot* be equal, see the right side figure of (2).

Let  $\eta_2 > \eta_1$ . Let  $\Sigma_{\eta_1}$  meet the  $r = 0$  line at  $A_1$ . Let the out-going null hypersurface through  $A_1$  intersect the  $\Sigma_{\eta_2}$  in a  $S^2$  at  $B_1$  with the radial coordinate being  $r_1$ . The three hypersurfaces  $\Sigma_{\eta_1}$ , the out-going null hypersurface and the hypersurface  $\Sigma_{\eta_2}$  bounded by the sphere at  $B_1$  enclose a space-time region,  $A_1BB_1$ . By the conservation equation (51), the sum of the fluxes through these bounding hypersurfaces must vanish. But the flux through the out-going null hypersurface vanishes as shown before. Hence the fluxes through  $\Sigma_{\eta_1}$  and the partial hypersurface  $\Sigma_{\eta_2}$  between  $B_1$  and  $B$ , must be equal. However, this leaves the contribution of the flux through the ‘remaining’ portion of the  $\Sigma_{\eta_2}$  hypersurface between  $B_2$  and  $B_1$ . Hence the result. Alternatively, one can also see this explicitly by writing the full flux through the two hypersurfaces using the expression given in equation (66) and matching the integrands along the out-going null hypersurface. Evidently, the full flux through  $\Sigma_{\eta \neq 0}$  is also not equal to that through  $\mathcal{J}^+$ . Physically this is understandable since the hypersurface at a later value of  $\eta$  receives energy emitted *after* the earlier value of  $\eta$ . The null infinity of course records *all* the energy emitted by the source and so does the cosmological horizon. We also conclude that the total flux at  $\mathcal{J}^+$  computed by Ashtekar et al, as given in eq.(36), matches (at coarse grained level) with that given in equation (66) (with  $\mathcal{Q} \rightarrow \mathcal{R}$ ) *only* for  $\eta = 0$ . Note that unlike the spatial slices  $\Sigma_\eta$ , all hypersurfaces  $\Sigma_{\rho > 0}$  intercept all the emitted energy.

(2) The sharp propagation of energy can also be used to infer the *instantaneous emitted power*. Consider two out-going null hypersurfaces intersecting the cosmological horizon in spheres with radii  $r(\tau)$  and  $r'(\tau')$ . The same hypersurfaces intersect the null infinity at corresponding spheres at  $R(\tau)$  and  $R'(\tau')$ , see the left side figure in (2). For  $\tau' > \tau$ , we have  $r'(\tau') < r(\tau)$  and  $R'(\tau') < R(\tau)$ . By the conservation equation and sharp propagation, the flux integral over the portion bounded by the spheres  $R, R'$  on  $\mathcal{J}^+$  and the flux integral over the portion bounded by the spheres  $r(\tau), r'(\tau')$  on the  $\mathcal{H}^+$ , are equal. Taking  $\tau' = \tau + \delta\tau$ , the integral becomes  $\delta\tau \times$  the integral over the sphere at  $r(\tau)$ . The *emitted power* is then defined by dividing the flux integral by  $\delta\tau$  and taking the limit. Thus we get the instantaneous power

as:

$$\mathcal{P}(\tau) := \lim_{\delta\tau \rightarrow 0} \frac{\mathcal{F}(\tau + \delta\tau, \tau)}{\delta\tau} = \frac{1}{8\pi} \int_{S^2} d^2s \langle \mathcal{Q}_{ij}^{TT} \mathcal{Q}_{TT}^{ij} \rangle. \quad (94)$$

This is manifestly positive.

This is very similar to the definition given by Ashtekar et al [2] in the form of equation (38) *except that* the integrand is an average over  $\tau$  and angular windows. The power is usually averaged over a few periods. If this is done to the power expression in [2], it will match with the above expression, again at a coarse grained level.

The upshot is that the quadrupole power defined above is gauge invariant and can be computed at the cosmological horizon.

## V. DISCUSSION AND SUMMARY

We have dealt with two aspects namely the role of the cosmological horizon and the use of ripple stress tensor in the limited context of rapidly changing, distant sources.

A question regarding the validity of the ‘short wavelength approximation’ near  $\mathcal{J}^+$  arises due to the understanding that the physical wavelength will diverge near the future null infinity thanks to the scale factor  $a(t)$ . Let us recall that background plus ripple decomposition is premised over the expectation:  $\partial_\alpha \bar{g}_{\mu\nu} \sim \bar{g}_{\mu\nu}/L$  and  $\partial_\alpha h_{\mu\nu} \sim h_{\mu\nu}/\lambda$ . In the cosmological chart, the non-zero coordinate derivatives of the background are:  $\partial_t \bar{g}_{ij} = 2H \bar{g}_{ij} \sim \bar{g}_{ij}/L$ . For the retarded solution we have,

$$\frac{\partial_t h_{ij}}{h_{ij}} = \partial_t [\ln(a^2(t) \chi_{ij})] = 2H + \partial_t \eta \partial_\eta \ln(\chi_{ij}) = 2H + \frac{\partial_\eta \ln(\chi_{ij})}{a(t)} \sim \frac{1}{L} + \frac{1}{a(t)\lambda}, \quad (95)$$

$$\frac{\partial_k h_{ij}}{h_{ij}} = \partial_k \ln(\chi_{ij}) = \hat{r}_k \partial_r \ln(\chi_{ij}) \approx -\hat{r}_k \partial_\eta \ln(\chi_{ij}) \sim \frac{\hat{r}_k}{\lambda} \quad (96)$$

The first equation shows that the  $t$ -derivative of the perturbation does *not* satisfy the premise, near  $\mathcal{J}^+$  thanks to the presence of the scale factor. The second equation however does *not* have the scale factor and the ripple indeed has short scale of spatial variation. Interestingly, in the calculation of the fluxes, spatial components of the ripple stress tensor (and hence the spatial derivatives of the perturbation) do contribute since all Killing vectors are space-like near  $\mathcal{J}^+$  and the ‘short wavelength approximation’ can justifiably be used.

As noted in the introduction, the cosmological horizon is unambiguously defined for a spatially compact source. This follows because worldlines with finite physical separation at

every  $\eta$  must converge to  $i^+$ , the point  $A$  of figure (1). If  $\Delta$  denotes the physical radial distance corresponding to the radial coordinate difference  $\delta$ , then  $\Delta^2 = \frac{\delta^2}{H^2\eta^2}$ . To maintain  $\Delta^2$  to be finite as  $\eta \rightarrow 0_-$ , we must have  $\delta^2 \sim \alpha^2\eta^2 + O(|\eta|^3)$  near  $i^+$ . This identifies  $\delta$  with  $-\alpha\eta$  or  $\alpha = H\rho$ . Thus, the worldlines approach  $i^+$  along the  $\rho = \text{constant}$  hypersurfaces. The cosmological horizon is then the past lightcone of  $i^+$ . The same argument also shows that any observer, who remains at finite physical distance away from the source must remain confined within the cosmological horizon. Furthermore, neither any such observer, nor the source has any access to energy/momentum which has crossed the horizon. Hence cosmological horizon does share physically relevant properties with the future infinity. Incidentally, any future directed causal curve reaching  $\mathcal{J}^+$  also registers on  $\mathcal{H}^+$ .

Further support for the role of cosmological horizon as future null infinity comes from the computations of the energy momentum fluxes. For these, we employed the effective ripple stress tensor and showed that the fluxes defined at  $\mathcal{J}^+$  match with those defined at  $\mathcal{H}^+$ . Furthermore, these fluxes also matched (at a coarse grained level) with the energy momentum fluxes defined by the more geometric methods of the covariant phase space framework. This provides a further support to the utility of the ripple stress tensor. The quadrupole power too matches likewise. The ripple stress tensor, although limited to short wavelength regimes (which covers most common sources), provides a convenient picture of energy momentum flows much like the flows for matter. There is a shortcoming of the ripple stress tensor - it does not capture the angular momentum flux correctly. A clearer understanding of this failure is lacking at present.

It should be noted that definition of fluxes is not necessarily *unique*. Apart from a definition being well defined, its ‘correctness’ should be tested in conjunction with the definition of the Bondi-type quantities having a loss formula relating to flux. Recent work within a Bondi-type framework may be seen in [14–16]. The observation that the cosmological horizon is a Killing horizon and hence an isolated horizon should be helpful in this regard.

## Acknowledgments

We would like to thank Béatrice Bonga for discussions and clarifications regarding [2].

## Appendix A: An averaging procedure

In the main body we specified an averaging procedure by stipulating its properties namely, (i) average of odd powers of  $h$  vanishes and (ii) average of space-time divergence is sub-leading. This was then used to simplify the expression for the ripple stress tensor. An averaging procedure satisfying these properties is indeed given by Isaacson [8]. We will use the same one and give more explicit details in the present context.

Isaacson defines the *space-time average* of a tensor by using the *parallel propagator bi-tensor*,  $g_{\mu}^{\mu'}(x, x')$  as:

$$\langle X_{\mu\nu} \rangle(x) := \frac{\int_{\text{cell}} d^4x' \sqrt{|g(x')|} g_{\mu}^{\mu'}(x, x') g_{\nu}^{\nu'}(x, x') X_{\mu'\nu'}(x')}{\int_{\text{cell}} d^4x' \sqrt{|g(x')|}}. \quad (\text{A1})$$

In the present context, we need average of the stress tensor for ripples due to an retarded solution which has certain explicit form. We will use this information to choose suitable integration variables and corresponding ‘cell’ denoting the averaging region. Because of this, we have not used any weighting function as given by Isaacson [8].

To keep track of the powers of  $H$ , we begin by going from the conformal chart  $(\eta, x^i)$  to the cosmological chart  $(t, x^i)$ ,  $\eta := -H^{-1}e^{-Ht}$  with the spatial coordinates unchanged. In the cosmological chart:

$$\text{Metric} : ds^2 = -dt^2 + a^2(t)(\delta_{ij}dx^i dx^j) \quad , \quad a(t) := e^{Ht} \quad (\text{A2})$$

$$\begin{aligned} \text{Connection} : \Gamma^t_{tt} = 0 \quad , \quad \Gamma^t_{tj} = 0 \quad , \quad \Gamma^t_{ij} = Ha^2(t)\delta_{ij} \\ \Gamma^i_{tt} = 0 \quad , \quad \Gamma^i_{tj} = H\delta^i_j \quad , \quad \Gamma^i_{jk} = 0. \end{aligned} \quad (\text{A3})$$

The parallel propagator is computed in terms of the parallel transport of an arbitrary co-tetrad (or tetrad):  $g_{\mu}^{\mu'}(x, x') := e_{\mu}^a(x)e_a^{\mu'}(x')$ . The averaging region is small enough that for a cell around a point  $P$  with coordinates  $x^\alpha$ , there is unique geodesic to points  $P'$  with coordinates  $x'^\alpha$ . The parallel transported co-tetrad is obtained using Taylor expansions of the co-tetrad, the affine connection and the coordinates along the geodesic, in terms of its affine parameter and eliminating the affine parameter afterwards in favour of the coordinate differences  $\Delta x^\alpha := x'^\alpha - x^\alpha$ . Details may be seen in the appendix B of [3]. There is a slight difference from [3] since that calculation was given in the context of Fermi normal coordinates where the connection is already of order  $H^2$  while in the cosmological chart, the

connection is of order  $H$ . The final expressions are:

$$e_\mu^a(x') = \hat{e}_\lambda^a \left[ \delta_\mu^\lambda + \hat{\Gamma}_{\mu\alpha}^\lambda \Delta x^\alpha + \frac{1}{2} \left( \widehat{\partial}_\rho \hat{\Gamma}_{\mu\sigma}^\lambda + \hat{\Gamma}_{\mu\sigma}^\alpha \hat{\Gamma}_{\alpha\rho}^\lambda \right) \Delta x^\rho \Delta x^\sigma \right] \quad (\text{A4})$$

$$g_\mu^{\mu'}(x, x') = \delta_\mu^{\mu'} - \hat{\Gamma}_{\mu\alpha}^{\mu'} \Delta x^\alpha - \frac{1}{2} \left( \widehat{\partial}_\rho \hat{\Gamma}_{\mu\sigma}^{\mu'} - \hat{\Gamma}_{\alpha\rho}^{\mu'} \hat{\Gamma}_{\sigma\mu}^\alpha \right) \Delta x^\rho \Delta x^\sigma \quad (\text{A5})$$

In the above, the hatted quantities are evaluated at  $x$ .

The connection dependent terms are linear and quadratic in  $H\Delta x$ . Although the coordinate differences are much larger than the length scale  $\lambda$  they are much smaller than  $H^{-1}$ . Hence, these terms can be neglected and *effectively the parallel propagator reduces to just the Kronecker delta*. For purposes of illustration of averaging, this suffices. It remains to integrate the  $X_{\mu'\nu'}$  over the cell and as noted in the main text in the paragraph below equation (60), the components of the ripple stress tensor are essentially determined in terms of  $\partial_\eta \chi_{ij}^{tt} = 2\frac{\eta}{r} \frac{\mathcal{Q}_{ij}^{tt}(\eta-r)}{\eta-r}$  or alternatively in terms of  $\partial_\eta \chi_{ij}^{TT} = 2\frac{\eta}{r} \frac{\mathcal{R}_{ij}^{TT}(\eta-r)}{\eta-r}$ .

The angular dependence is introduced due to the ‘tt’ part, eg as is explicit in the  $\Lambda_{ij}^{kl}(\hat{r})$  projector. The  $(\eta, r)$  dependence has a convenient factorised form. It is thus natural to change the integration variables from  $(\eta, r)$  to  $(\bar{t}, \rho)$ , where  $\bar{t}$  is the retarded synchronous time defined through,  $\eta - r := -H^{-1}e^{H\bar{t}}$  and  $H\rho := -\frac{r}{\eta}$  defines  $\rho$ . For definiteness, consider the average,

$$\langle \partial_\eta \chi_{mn}^{tt} \partial_\eta \chi_{tt}^{mn} \rangle(t, r, \hat{r}) := \frac{\int_{\text{cell}} dt dr r^2 d^2s a^3(t) \partial_\eta \chi_{mn}^{tt}(\bar{t}) \partial_\eta \chi_{tt}^{mn}(\bar{t})}{\int_{\text{cell}} dt dr r^2 d^2s a^3(t)} \quad (\text{A6})$$

Here  $\hat{r}$  denotes a point on  $S^2$  (a spatial direction). We will specify the cell after changing over to  $(\bar{t}, \rho, \hat{r})$ .

From the definitions, we arrive at the coordinate transformations,

$$\begin{aligned} \eta(\bar{t}, \rho) &= -\frac{1}{H(1+H\rho)} e^{-H\bar{t}} \quad , \quad r(\bar{t}, \rho) = \frac{\rho}{1+H\rho} e^{-H\bar{t}} \quad \Rightarrow \\ a(t) &= a(\bar{t})(1+H\rho) \quad , \quad \text{with } a(t) := e^{Ht} \end{aligned} \quad (\text{A7})$$

The Jacobian of transformation is  $\frac{\partial(t,r)}{\partial(\bar{t},\rho)} = \{a(\bar{t})(1+H\rho)\}^{-1}$ . We choose the cell so that  $\bar{t} \in [\bar{t}_0 - \delta, \bar{t}_0 + \delta]$  and  $\rho \in [\rho_0 - \Delta, \rho_0 + \Delta]$  and  $\hat{r} \in \Delta\omega$ . The coordinate windows  $\delta, \Delta$  and  $\sqrt{r^2\Delta\omega}$  are several times the ripple scale while  $(\bar{t}_0, \rho_0)$  are the transforms of  $(t, r)$ . In terms of these choices, the average becomes,

$$\langle \partial_\eta \chi_{mn}^{tt} \partial_\eta \chi_{tt}^{mn} \rangle(t, r, \hat{r}) := \frac{\int_{\bar{t}_0-\delta}^{\bar{t}_0+\delta} d\bar{t} \int_{\rho_0-\Delta}^{\rho_0+\Delta} d\rho \rho^2 \int_{\Delta\omega} d^2s \left[ 4 \frac{a^2(\bar{t})}{\rho^2} \mathcal{Q}_{mn}^{tt}(\bar{t}) \mathcal{Q}_{tt}^{mn}(\bar{t}) \right]}{\int_{\bar{t}_0-\delta}^{\bar{t}_0+\delta} d\bar{t} \int_{\rho_0-\Delta}^{\rho_0+\Delta} d\rho \rho^2 \int_{\Delta\omega} d^2s} \quad (\text{A8})$$

Consider the angular integration. The angular dependence arises in taking the ‘tt’ part of the solution  $\chi_{ij}(\eta, r)$ . For illustration purpose, consider  $r$  to be sufficiently large so that we can use the  $\Lambda_{ij}{}^{kl}(\hat{r})$  projector, giving  $\partial_\eta \chi_{ij}^{tt} \partial_\eta \chi_{tt}^{ij} \sim \Lambda_{ij}{}^{kl} \partial_\eta \chi^{ij} \partial_\eta \chi_{kl}$ . For large  $r$ , the angular coordinate windows are  $\sim \lambda/r \ll 1$ . Using the mean value theorem in the angular integration in the numerator, we get

$$\frac{\int_{\Delta\omega} d^2 s(\hat{r}') \Lambda_{ij}{}^{kl}(\hat{r}')}{\int_{\Delta\omega} d^2 s(\hat{r}')} \approx \Lambda_{ij}{}^{kl}(\hat{r}). \quad (\text{A9})$$

In effect, the  $\Lambda$ -projector comes out of the averaging and the *angular average trivializes*. Of the remaining integrations, the  $\rho$  integration can be done explicitly and is independent of  $\Delta$  to the leading order in  $\Delta/\rho_0$ . Thus, in the numerator of (A8) we get,

$$\begin{aligned} \int_{\bar{t}_0-\delta}^{\bar{t}_0+\delta} d\bar{t} \int_{\rho_0-\Delta}^{\rho_0+\Delta} d\rho \ 4 a^2(\bar{t}) \mathcal{Q}^{ij}(\bar{t}) \mathcal{Q}_{kl}(\bar{t}) &\approx 8\Delta \int_{\bar{t}_0-\delta}^{\bar{t}_0+\delta} d\bar{t} \ a^2(\bar{t}) \mathcal{Q}^{ij}(\bar{t}) \mathcal{Q}_{kl}(\bar{t}) \\ &= 8\Delta \int_{-\delta}^{\delta} dy \ a^2(\bar{t}_0 + y) \mathcal{Q}^{ij} \mathcal{Q}_{kl}(\bar{t}_0 + y) \\ &= 8\Delta a^2(\bar{t}_0) \int_{-\delta}^{\delta} dy \ a^2(y) \mathcal{Q}^{ij} \mathcal{Q}_{kl}(\bar{t}_0 + y) \\ &:= (8\Delta) a^2(\bar{t}_0) (2\delta) \langle \mathcal{Q}^{ij} \mathcal{Q}_{kl} \rangle_{\bar{t}}(\bar{t}_0) \end{aligned} \quad (\text{A10})$$

In the third line, we have used  $a^2$  being an exponential function and in the last line we have *defined* the average over the retarded time around  $\bar{t}_0$  and put the suffix on the angular bracket as a reminder.

The  $\langle \rangle_{\bar{t}}$  averaging has the extra factor of  $a^2(y)$ . However, over the integration domain  $(-\delta, \delta)$ , we can approximate  $a^2(y) \approx 1 + 2Hy + \dots$  and neglect  $o(Hy)$  terms since  $H\delta \sim k\lambda/L \sim k\epsilon \ll 1$ . The extra factor thus introduces a small deviation from the usual averaging without the extra factor and *we neglect it henceforth and the reminder suffix,  $\bar{t}$  is also suppressed*.

In the denominator we get,

$$\int_{\bar{t}_0-\delta}^{\bar{t}_0+\delta} d\bar{t} \int_{\rho_0-\Delta}^{\rho_0+\Delta} d\rho \ \rho^2 \approx 2\rho_0^2 \Delta \int_{\bar{t}_0-\delta}^{\bar{t}_0+\delta} d\bar{t} = (2\Delta)(2\delta)\rho_0^2 \quad (\text{A11})$$

Combining equations (A8, A9, A10, A11), we get

$$\langle \partial_\eta \chi_{mn}^{tt} \partial_\eta \chi_{tt}^{mn} \rangle(t, r, \hat{r}) = 4 \frac{a^2(\bar{t}_0)}{\rho_0^2} \langle \mathcal{Q}_{ij}^{tt} \mathcal{Q}_{tt}^{ij} \rangle(\bar{t}_0, \hat{r}) \quad (\text{A12})$$

In the last equation, we have combined the averaging over retarded time and the (trivial) angular average. We have also inserted the  $\Lambda$ -projector. The averaging over *a space-time*



cell has been reduced to averaging over a 3-dimensional cell on a  $\rho = \text{constant}$  hypersurface. The pre-factor on the right hand side of the above equation exactly equals the last square bracket in the first line of the equation (60). In effect, the  $(\frac{2\eta}{r(\eta-r)})^2$  factor has come out of the averaging.

We can also reduce the space-time average to a hypersurface average for  $\partial_\eta \chi_{ij}^{TT} \partial_\eta \chi_{TT}^{ij}$ . Following the same steps as from eqn.(A6) onwards, we will arrive at eqn.(A8) with  $\mathcal{Q}_{mn}^{tt} \rightarrow \mathcal{Q}_{mn}^{TT}$ . We cannot do the angular averaging as before, but we don't need to. Crucially, the  $\rho$  dependence has factored out exactly as before and the average over  $\rho$  gives  $\rho_0^{-2}$  as before. The  $\bar{t}$  averaging too gives  $a^2(\bar{t}_0)$  and we get the desired result,

$$\langle \partial_\eta \chi_{mn}^{TT} \partial_\eta \chi_{TT}^{mn} \rangle(t, r, \hat{r}) = 4 \frac{a^2(\bar{t}_0)}{\rho_0^2} \langle \mathcal{Q}_{ij}^{TT} \mathcal{Q}_{TT}^{ij} \rangle(\bar{t}_0, \hat{r}) \quad (\text{A13})$$

We can relate the averaging over the retarded time,  $\bar{t}$ , to the averaging over the Killing time,  $\tau$  along the  $\rho = \rho_0$  curve. From the coordinate transformation, we have  $\eta - r = -H^{-1}e^{-H\bar{t}}$  while along  $\rho = \rho_0$  Killing trajectory,  $\eta - r = (\eta_* - r_*)e^{-H\tau} = -(H^{-1}e^{-H\bar{t}_*})e^{-H\tau}$ . Hence,  $\bar{t} = \tau + \bar{t}_*$  and the temporal averaging is related to averaging over a Killing time. Note that the averaging cell being bounded by two hypersurfaces of constant retarded times, the temporal averaging may be evaluated along the source worldline,  $r = 0$  or along the Killing trajectory on  $\mathcal{J}^+$ .

We also have mixed and spatial components of the ripple stress tensor. These involve  $\partial_i \chi_{mn}^{tt} \approx \hat{x}_i \partial_r \chi_{mn}^{tt} \approx -\hat{x}_i \partial_\eta \chi_{mn}^{tt}$ . While taking the average, the  $\hat{x}_i$  can be taken out of the average since the angular coordinate windows are of very small size  $\sim \lambda/r$ . This allows us to take  $\hat{x}^i$  across the angular averages and replace all components of the ripple stress tensor by  $t_{\eta\eta}$  in the conformal chart or by  $t_{00}$  in the cosmological chart.

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