

# Spin and a Running Radius in RS1

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## Abstract

We develop a renormalization group formalism for the compactified Randall-Sundrum scenario wherein the extra-dimensional radius serves as the scaling parameter. Couplings on the hidden brane scale as we move within local effective field theories with varying size of the warped extra dimension. We consider this RG approach applied to  $U(1)$  gauge theories and gravity. We use this method to derive a low energy effective theory.

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# 1 Introduction

The compactified Randall-Sundrum (RS) scenario provides a natural explanation for the separation of the weak/Planck scales observed in nature [1]. The warped space-time presents a difficulty, however, in evaluating Feynman diagrams and estimating their size. In particular, the exponentially small warp factor responsible for setting the large hierarchy can appear in ways that are difficult to predict, having dramatic consequences for considerations such as the stability of the RS hierarchy and gauge unification [2]. Recently, a novel approach was begun for studying this scenario wherein holographic renormalization group (RG) ideas [3,4] were used to identify scaling effects and simplify power counting in an effective field theory approach [5]. Here we continue this program, extending the analysis to RS theories with bulk photons and gravitons.

We will consider the geometry of the RS1 scenario where five dimensional AdS space is compactified on an orbifold  $S_1/\mathbb{Z}_2$  with two branes (the hidden and the visible brane) located at the fixed points. Following Ref. [5], we match to an effective theory with a smaller proper distance between the visible brane and an effective hidden brane. This matching is accomplished by requiring that the classical solutions of the field equations are identical in the overlapping region. This procedure requires an effective hidden brane action local in the fields and their derivatives and gauge invariant (or general covariant in the case of spin 2 fields). The couplings on the effective hidden brane flow at the classical level in the sense of a Wilsonian RG flow. In fact, our method closely resembles Wilson's picture of renormalization, integrating out degrees of freedom close to the hidden brane we flow in the space of local theories with various radii. In running the radius down to a negligible extra-dimensional size, the large warp factor is integrated into the hidden brane couplings in predictable ways simplifying power counting analysis. We may identify in the running of these couplings certain features of the AdS/CFT duality [6] applied to the RS scenario [7]. In fact, by moving the visible brane, the same method can be applied to holographic computations in the RS/CFT correspondence, as shown in [8] for the case of scalar fields.

The analysis of photons and gravitons is similar to the analysis of bulk scalar fields of Ref. [5] in many respects. Particular to the case of the massive scalar however, is the existence of an attractive local fixed point. We will see that such a fixed point is absent in gauge theories or any theory containing massless fields. From the CFT side the absence of a fixed point is understood as the dual of these RS theories contains a massless field breaking conformal invariance at any scale. In the brane running approach developed here, we find that it is precisely the necessity of the massless zero mode in gauge theories that precludes the existence of a fixed point.

We find an equivalent field theory description of these bulk gauge and gravity theories with

an attractive local fixed point. By adding a massless hidden brane field to the bulk photon or graviton theories we treat the zero mode explicitly in the running. This massless field decouples from the massive Kaluza-Klein (KK) tower at the fixed point. One can use the attractive nature of the fixed point to simplify the running to find sensible low energy effective theories.

The paper is organized as follows. In Section 2 we discuss the matching procedure for a free bulk photon propagating in the RS1 geometry. We consider spin 2 fields in the linearized approximation in Section 3. In Section 4 we introduce a method of improving the effective field theory description by adding a massless hidden brane field. We examine this method for the case of massless scalar fields and  $U(1)$  gauge theories. Section 5 contains a brief discussion of the results. In the Appendix we extend the procedure to the case of scalar QED with general hidden brane interactions.

## 2 The free photon

We consider a fixed AdS<sub>5</sub> background with metric

$$ds^2 = \frac{1}{(kz)^2}(\eta_{\mu\nu}dx^\mu dx^\nu - dz^2). \quad (2.1)$$

The  $z$  coordinate parameterizes the direction in the orbifolded space. The space is bounded by a hidden brane at  $z = z_h = 1/k \sim \mathcal{O}(1/M_{Plank})$  and a visible brane at  $z = z_v \sim \mathcal{O}(1/\text{TeV})$ .

We examine the case of a free  $U(1)$  gauge field with action

$$S = S_{bulk} + S_{hid} + S_{vis}, \quad (2.2)$$

where

$$S_{bulk} = -\frac{1}{4g_5^2} \int d^4x \int_{z_v}^{z_h} dz \sqrt{G} G^{MN} G^{SP} F_{MS} F_{NP}, \quad (2.3)$$

$$S_{hid} = -\frac{1}{4g_5^2 k} \int d^4x \sqrt{-g_h} g_h^{\mu\nu} g_h^{\sigma\delta} F_{\mu\sigma} \tau_h F_{\nu\delta}, \quad (2.4)$$

$$S_{vis} = -\frac{1}{4g_5^2 k} \int d^4x \sqrt{-g_v} g_v^{\mu\nu} g_v^{\sigma\delta} F_{\mu\sigma} \tau_v F_{\nu\delta}, \quad (2.5)$$

and

$$g_{\mu\nu}^h(x) = G_{\mu\nu}(x, z = z_h), \quad (2.6)$$

$$g_{\mu\nu}^v(x) = G_{\mu\nu}(x, z = z_v). \quad (2.7)$$

The brane actions are the most general quadratic actions compatible with gauge invariance and  $4D$  general covariance. The coefficient  $\tau_h$  is a series of derivative operators,

$$\tau_h = \tau_h^{(0)} - k^{-2} \tau_h^{(2)} g_h^{\mu\nu} \partial_\mu \partial_\nu + k^{-4} \tau_h^{(4)} g_h^{\mu\nu} g_h^{\rho\sigma} \partial_\mu \partial_\nu \partial_\rho \partial_\sigma + \dots \quad (2.8)$$

and similarly for  $\tau_v$ . We assume  $A_\mu$  to be even and  $A_z$  odd on the orbifold.

The equations of motion are given by

$$\partial_M(\sqrt{G}G^{MN}G^{5P}F_{NP}) = 0 \quad (2.9)$$

$$\partial_M(\sqrt{G}G^{MN}G^{\delta P}F_{NP}) = -\frac{1}{k}(\tau_h\delta(z-z_h) + \tau_v\delta(z-z_v))\partial_\mu(\sqrt{-g_h}g^{\mu\nu}g^{\delta\sigma}F_{\nu\sigma}). \quad (2.10)$$

These equations greatly simplify in the standard gauge  $A_z(x, z) = 0$  and  $\partial^\mu A_\mu(x, z) = 0$ . This is a consistent gauge choice as there is no Aharonov-Bohm phase around the fifth dimension for  $A_z$ . It is the most convenient choice for our considerations here.

Fourier transforming the 4D Minkowskian directions, equation (2.10) becomes the boundary independent differential equation

$$(z^2\partial_z^2 - z\partial_z + q^2z^2)A_\mu(q, z) = 0, \quad (2.11)$$

with the following boundary conditions determined by the brane actions

$$z\partial_z A_\mu(z)\Big|_{z=z_h} = -\frac{(qz_h)^2}{2}\tau_h(q^2z_h^2)A_\mu(z_h) \quad (2.12)$$

$$z\partial_z A_\mu(z)\Big|_{z=z_v} = +\frac{(qz_v)^2}{2}\tau_v(q^2z_v^2)A_\mu(z_v). \quad (2.13)$$

where  $q$  is the 4D momentum.

We now consider an effective theory with a new hidden brane bounding the space at  $a > z_h$ . We require the new theory to reproduce the same physics from the point of view of an observer living at  $z > a$ . We take the same visible brane and bulk action as in (2.5) and introduce the following effective hidden brane action,

$$S_a = -\frac{1}{4g_5^2k}\int d^4x\sqrt{-g_a}g_a^{\mu\nu}g_a^{\sigma\delta}F_{\mu\sigma}\tau(a)F_{\nu\delta}, \quad (2.14)$$

where  $g_a$  is the induced metric at  $z = a$ . The coupling  $\tau(a)$  has a derivative expansion similar to (2.8),

$$\tau(a) = \sum_j (-1)^j \tau^{(2j)}(a) (k^{-2}g_a^{\mu\nu}\partial_\mu\partial_\nu)^j. \quad (2.15)$$

Notice that  $\tau(a)$  contains an explicit dependence on  $a$ , coming from the warped metric and an intrinsic one from the individual couplings  $\tau^{(2i)}(a)$ .

The effective brane action implies the boundary condition at  $a$  on the fields,

$$z\partial_z A_\mu(z)\Big|_{z=a} = -\frac{(qa)^2}{2}\tau(q^2a^2, a)A_\mu(a). \quad (2.16)$$

By assumption the solutions of the classical equations of motion in the restricted region  $z \in (a, z_v]$  are unchanged under the motion of the brane. This condition gives a *classical* flow equation for the coupling  $\tau$  similar to a  $4D$  RG flow.

Explicitly, we take the logarithmic derivative with respect to  $a$  of the boundary condition (2.16). Using the bulk equation of motion to eliminate the second derivatives of the fields one finds

$$a \frac{d}{da} \tau(q^2 a^2, a) = \frac{(qa)^2}{2} \tau^2 + 2. \quad (2.17)$$

This equation can be interpreted as a set of coupled flow equations for the individual couplings  $\tau^{(2j)}(a)$  in (2.15), obtained by expanding in powers of  $q$  (2.17),

$$a \partial_a \tau^{(2j)} = -2j \tau^{(2j)} + \frac{1}{2} \sum_i \tau^{(2i)} \tau^{(2j-2i-2)} + 2\delta_{j,0}. \quad (2.18)$$

The flow equations for the couplings  $\tau^{(2j)}$  have a lower diagonal form. Operators mix with operators of lesser or equal dimension. This feature will be preserved in the interacting theory considered in detail in the Appendix.

We may search for fixed points to this flow satisfying

$$\partial_a \tau^{(2j)}(a) = 0. \quad (2.19)$$

It is not hard to check that

$$\tau^* = -2 \frac{J_0(qa)}{qa J_1(qa)}. \quad (2.20)$$

is a solution to (2.19). There are also other solutions to (2.19) with different linear combinations of Bessel functions with the same index but these have an expansion non polynomial in momentum. These correspond to non-local effective brane operators and so, in the spirit of the local effective field theory, they are rejected. They are however important in the context of the AdS/CFT correspondence [8].

The first term in the expansion of  $\tau^*$  is proportional to  $1/q^2$  corresponding to a brane localized mass term. A gauge theory can only approach such a fixed point if the gauge symmetry is broken. In fact, if we add a mass term for the gauge field on the hidden brane, that is  $\tau^{(-2)}(z_h) \neq 0$ , then  $\tau(a)$  approaches  $\tau^*$  for large  $a$ . If the gauge symmetry is unbroken however there is no fixed point to the flow.

We can integrate analytically equation (2.17),

$$\tau(a) = -2 \frac{Y_0(qa) + B J_0(qa)}{qa(Y_1(qa) + B J_1(qa))}. \quad (2.21)$$

The boundary condition at the hidden brane

$$\tau(q^2 a^2, a)|_{a=z_h} = \tau_h, \quad (2.22)$$

determines the integration constant  $B$ ,

$$B = -\frac{Y_0(qz_h) + qz_h\tau_h Y_1(qz_h)}{J_0(qz_h) + qz_h\tau_h J_1(qz_h)}. \quad (2.23)$$

Running the hidden brane to the visible brane the entire contribution to the action is given by the brane actions since the bulk contribution becomes negligible. Notice that the zeros in  $q^2$  of  $\tau(a)|_{a=z_v} + \tau_{z_v}$  are the  $4D$  masses of the KK tower of the gauge field. This follows from the fact that the exact propagator in the effective field theory coincides with the brane-to-brane propagator of the full theory.

The leading term in the momentum expansion of (2.21) is given by

$$\tau^{(0)}(a) = 2 \log\left(\frac{a}{z_h}\right) + \tau_h^{(0)}. \quad (2.24)$$

This can be easily obtained from (2.18). Running to  $a = z_v$  we find that the  $4D$  effective gauge coupling is:

$$\frac{1}{g_4^2} = \frac{2 \log\left(\frac{z_v}{z_h}\right) + \tau_h^{(0)} + \tau_v^{(0)}}{g_5^2 k}. \quad (2.25)$$

which reproduces the zero mode calculation. The kinetic terms on the hidden or on the visible brane modify the effective  $4D$  gauge coupling by additive corrections. As expected the coupling  $\tau(a)$  has a local expansion in momentum.

The solution of equation (2.21) gives the exact flow for any  $\tau_h$ . It is useful to examine the flow equations in the form of equation (2.18) to estimate the size of the coefficients  $\tau^{(2j)}(a)$ . We see first that  $\tau^{(0)} \sim \log a$ . Iterating in equation (2.18) we deduce that  $\tau^{(2j)} \sim (\log a)^{j+1}$  and no larger. In terms of external momentum  $q$  the operator with coefficient  $\tau^{(2j)}$  will power count as  $(q^2 a^2 \log(a/z_h))^{j+1}$ . The coefficients are enhanced by large logs. However, the effective theory is valid in the regime  $q^2 a^2 \log(a/z_h) < 1$ . In the Appendix we show that this conclusion holds in the interacting theory as well.

It is important to understand the physical origin of the logarithmic enhancement of the  $q^2 a^2$  expansion. This is a result of the fact that the massless zero mode of a gauge field is not localized near the visible brane but flat throughout the extra-dimensional space. The KK modes of a gauge field couple to the visible brane parametrically stronger than the zero mode [9]. As a consequence, they become important at energies below the mass of the first KK mode. The existence of a non-localized zero mode increases the sensitivity to the hidden brane position and decreases the region of validity of the effective theory lowering the cutoff to  $1/(a \log(a/z_h))$ . A similar story occurs whenever we consider theories with massless modes. In Section 4 we will show how to find improved effective theories with massless zero modes having a cutoff at the scale  $1/a$  (no log suppression).

### 3 Gravity

The same RG formalism can also be applied to gravity. Here we consider the linearized fluctuations of the gravitational field.

We start by briefly reviewing the Randall-Sundrum solution of Einstein's equations. Following the conventions of [1] we take the gravitational action to be

$$S = \int d^4x dz \sqrt{G} (2M_5^3 R - \Lambda) - T_h \int d^4x \sqrt{-g_h} - T_v \int d^4x \sqrt{-g_v}, \quad (3.1)$$

where  $T_h$  and  $T_v$  are the tensions of the hidden and visible brane, necessary to reproduce the singularities of the metric at the fixed points of the orbifold. The metric

$$ds^2 = \frac{1}{(kz)^2} (\eta_{\mu\nu} dx^\mu dx^\nu - dz^2), \quad (3.2)$$

defined on the orbifold is a solution of Einstein's equations when the brane tensions are tuned as

$$T_h = -T_v = 24M_5^3 k \quad (3.3)$$

and the bulk cosmological constant is

$$\Lambda = -24M_5^3 k^2. \quad (3.4)$$

The zero mode fluctuations of the metric (obtained by replacing  $\eta_{\mu\nu}$  with  $g_{\mu\nu}(x)$  in (3.2)) reproduce ordinary four dimensional gravity with  $4D$  Planck mass given by

$$M_4^2 = \frac{M_5^3}{k^3} \left( \frac{1}{z_h^2} - \frac{1}{z_v^2} \right). \quad (3.5)$$

We now consider the linearized tensor fluctuations of the metric  $G_{\mu\nu}(x, z) = 1/(kz)^2 \eta_{\mu\nu} + h_{\mu\nu}(x, z)$ . Choosing the gauge  $\partial^\mu h_{\mu\nu} = 0$  and  $h^\mu_\mu = 0$ , the equations of motion assume a particularly simple form,

$$\left( \frac{1}{2} (z^2 \partial_z^2 + z \partial_z - 4 - z^2 \square) + 2z\delta(z - z_h) - 2z\delta(z - z_v) \right) h_{\mu\nu}(x, z) = 0 \quad (3.6)$$

where  $q$  is the four dimensional momentum. The delta-Dirac terms include the contributions coming from the singularities of the derivatives of the background metric as well as the brane terms. This is different from the case of scalar and gauge fields where delta-Dirac terms in the equations of motion arise solely from brane contributions.

The effective theory with a smaller radius has an effective hidden brane located at  $a$  where  $z_h < a < z_v$ . The quadratic hidden brane action at  $a$  is given by

$$S_a = 2M_5^3 k \int \sqrt{-g_a} h_{\mu\nu} (3\alpha - 4) \mathcal{O}^{\mu\nu\rho\sigma} h_{\rho\sigma}. \quad (3.7)$$

Here we have introduced the operator  $\mathcal{O}^{\mu\nu\rho\sigma}$  in order to preserve the invariance of the action under infinitesimal diffeomorphisms. In the gauge  $\partial^\mu h_{\mu\nu} = 0$  and  $h_\mu^\mu = 0$  considered above,  $\mathcal{O}^{\mu\nu\rho\sigma}$  simply becomes  $g_a^{\mu\rho} g_a^{\nu\sigma}$ . This brane action is derived generalizing the hidden brane tension to a running coupling  $T(a) = 24M^3 k \alpha(a)$ , where  $\alpha(a)$  represents the series of derivative couplings,

$$\alpha(a) = \sum_j (-1)^j \alpha^{(2j)}(a) (k^{-2} g_a^{\mu\nu} \partial_\mu \partial_\nu)^j. \quad (3.8)$$

These couplings are the coefficients of some linear combination of higher derivative general covariant operators.

The gauge fixed linearized equations of motion derived from the effective theory are

$$\left( \frac{1}{2} (z^2 \partial_z^2 + z \partial_z - 4 - z^2 \square) - (6\alpha - 8) z \delta(z - a) - 2z \delta(z - z_h) \right) h_{\mu\nu}(x, z) = 0. \quad (3.9)$$

We obtain the boundary condition at  $a$

$$z \partial_z h_{\mu\nu}(z) \Big|_{z=a} = (6\alpha - 8) h_{\mu\nu}(a). \quad (3.10)$$

Taking the logarithmic derivative with respect to  $a$  of this equation and using the bulk equations of motion we find the flow for  $\alpha$

$$a \frac{d}{da} \alpha(q^2 a^2, a) = -6\alpha^2 + 16\alpha - 10 - \frac{1}{6} (qa)^2 \quad (3.11)$$

This flow has a local fixed point given by

$$\alpha^* = 1 + \frac{1}{6} qa \frac{J_1(qa)}{J_2(qa)} = \frac{5}{3} - \frac{(qa)^2}{36} + \dots \quad (3.12)$$

The value of the brane tension at the fixed point  $\alpha^{(0)*}$  is inconsistent with the Poincaré ansatz which requires  $\alpha^{(0)}(z_h) = 1$ . In fact, there is no local fixed point consistent with 4D Poincaré invariance.

The solution of equation (3.11) is

$$\alpha(a) = \frac{qaY_1(qa) + 6Y_2(qa) + C(6J_2(qa) + qaJ_1(qa))}{6Y_2(qa) + 6CJ_2(qa)} \quad (3.13)$$

where  $C$  is an integration constant determined by the initial condition for the flow at  $z = z_h$ . For the original theory with action (3.1), we impose  $\alpha(z_h) = 1$  (in general one can include brane localized curvature terms). Expanding (3.13) in powers of momentum we find

$$\alpha(a) = 1 + \frac{1}{12} \left(1 - \frac{a^2}{z_h^2}\right) (qa)^2 - \frac{a^4 - 4a^2 z_h^2 + 3z_h^4 + 4z_h^4 \log(a/z_h)}{96z_h^4} (qa)^4 + \mathcal{O}((qa)^6). \quad (3.14)$$



This equation is particularly interesting. It tells us that the hidden brane tension is at a fixed point since it does not scale with  $a$ . This is a consequence of the fact that the distance between the two branes is undetermined in the original theory.

Running the hidden brane to  $a \sim z_v$  the bulk contribution to the action goes to zero. The quadratic action is given by the sum of the visible and effective brane actions. Up to two derivatives, the linearized analysis and general covariance completely determine the effective action. In terms of the field  $\hat{g}_{\mu\nu}(x) = k^2 z_v^2 G_{\mu\nu}(x, z_v)$  we have,

$$S_4 = 24 \frac{M_5^3}{k^3 z_v^4} \int d^4x \sqrt{-\hat{g}} (-z_v^2 \alpha^{(2)} R(\hat{g}) - \alpha^{(0)} + 1) + \mathcal{O}(R(\hat{g})^2) \quad (3.15)$$

Notice that, with the tensions tuned, the visible and effective hidden brane tensions cancel leaving, as expected, zero cosmological constant in the four dimensional effective theory. The value of the 4D Planck mass (3.5) can be readily obtained from the coupling  $\alpha^{(2)}(a)$ . The coefficients of the higher derivative operators are not fully determined by the linearized analysis.

For a general local hidden brane action at  $z_h$  equation (3.11) dictates a flow for the couplings  $\alpha^{(2j)}(a)$ . These couplings scale as  $(a/z_h)^{2j}$ . The  $q^2$  expansion appears to break down well below the scale of the lowest lying KK mode  $\sim 1/z_v$ . As in the case of gauge fields this effect is related to the shape of the modes in the extra-dimension: the massless graviton mode is localized near the hidden brane while the wave function of the KK modes is peaked on the visible brane.

The graviton zero mode couples to the visible brane a factor of  $z_h/z_v \sim \mathcal{O}(10^{-15})$  more weakly than the KK modes. Because of this, at least in principle, in a scattering experiment with gravitons, deviations from ordinary gravity could be detected at energies exponentially smaller than the masses of the KK modes. On the other hand, these KK modes give corrections to the Newtonian potential only at distances which are parametrically smaller than the typical mass so they are in practice unobservable.

To illustrate this explicitly, we consider the exact propagator of the effective theory. This can be directly obtained from effective brane coupling evaluated at  $z_v$ . Suppressing the tensor structure of the propagator we have:

$$\Delta(q, z_v) = \frac{1}{\alpha - 1} = \frac{J_1(qz_h)Y_2(qz_v) - J_2(qz_v)Y_1(qz_h)}{qz_v(J_1(qz_h)Y_1(qz_v) - J_1(qz_v)Y_1(qz_h))} \quad (3.16)$$

This propagator has an isolated pole at zero momenta. The mass scale of the KK states is set by the visible brane. The propagator can be rewritten as a sum of contributions from each mode:

$$\Delta(q, z_v) = \sum_n \frac{R(m_n^2)}{q^2 - m_n^2} \quad (3.17)$$

where  $R(m_n^2)$  is the residue of the propagator at the pole  $m_n^2$ . In general, the residue is just (up to normalization) the square of the coupling of the mode to matter. It is possible to check

that:

$$R(m_n^2) \sim \frac{z_v^2}{z_h^2} R(0) \quad n \geq 1 \quad (3.18)$$

In the language of modes the interaction lagrangian takes the form

$$\mathcal{L} = -\frac{1}{M_4} h_{\alpha\beta}^{(0)}(x) T^{\alpha\beta}(x) - \frac{1}{M_4} \left(\frac{z_v}{z_h}\right) \sum_{n=1}^{\infty} h_{\alpha\beta}^{(n)}(x) T^{\alpha\beta}(x) \quad (3.19)$$

reproducing the result in Ref. [10]. In this equation  $T^{\alpha\beta}(x)$  is the energy momentum tensor obtained by running down the hidden brane so it automatically includes the effects of bulk matter.

## 4 An Improved EFT for Massless Fields

We have seen that the effective field theories derived above are not valid at energies just below  $1/a$ , the cutoff expected from holographic arguments. In the case of gravity the cutoff scale is exponentially lower. The cutoff is lower by a large log in the case of the photon. In this section we introduce a method of brane running where the massless zero mode is treated explicitly as a massless field on the effective hidden brane. This theory is equivalent to the theory without such a field. However, it has an attractive fixed point where the massless field decouples from the bulk field representing the massive KK tower. The cutoff of the improved theory is  $\mathcal{O}(1/a)$ .

### 4.1 Introducing the massless brane field

For simplicity, we introduce this method in the case of a massless bulk scalar field. Consider a free scalar field in AdS space.

$$\begin{aligned} S &= S_{bulk} + S_{vis} + S_{hid} \\ S_{bulk} &= \frac{1}{2} \int d^4x dz \sqrt{G} (G^{MN} \partial_M \chi \partial_N \chi) \\ S_{hid} &= -\frac{1}{2} \int d^4x \sqrt{-g_h} (\chi^k \lambda_{h2} \chi) \\ S_{vis} &= -\frac{1}{2} \int d^4x \sqrt{-g_v} (\chi^k \lambda_{v2} \chi) \end{aligned} \quad (4.1)$$

In the set of effective theories of different radii it was found in Ref. [5] that the quadratic coupling on the effective hidden brane at  $a$  obeys the RG equation

$$a \frac{d\lambda_2}{da} = 4\lambda_2 - \frac{1}{2}\lambda_2^2 - 2q^2 a^2. \quad (4.2)$$

If there is no hidden brane mass term ( $\lambda_{h2}^{(0)} = 0$ ) one will not be generated in the flow. The theory does not flow to a fixed point because the fixed point contains a mass term. The couplings scale as  $\lambda_2^{(j)} = (a/z_h)^j$ , becoming exponentially large as we approach the visible brane. This effective theory suffers from the same problem as the photon and graviton theories discussed earlier. The cutoff of the effective field theory is far below the scale  $1/a$ .

We can find an equivalent description of the theory presented above by adding a massless hidden brane field. We take for the hidden brane action

$$S_{hid} = -\frac{1}{2} \int d^4x \sqrt{-g_h} \left( Z_b g_h^{\mu\nu} \partial_\mu b \partial_\nu b + k(\chi - \sqrt{k}b) \gamma_h (\chi - \sqrt{k}b) \right), \quad (4.3)$$

where  $b$  is a four dimensional field confined to the hidden brane,  $Z_b$  its field strength and

$$\gamma_h = \gamma_h^{(0)} - k^{-2} \gamma_h^{(2)} g_h^{\mu\nu} \partial_\mu \partial_\nu + \dots \quad (4.4)$$

From the equation of motion for  $\chi$  one derives the boundary condition in momentum space

$$z \partial_z \chi_q(z) \Big|_{z=z_h} = \frac{\gamma_h}{2} (\chi_q(z_h) - \sqrt{k} b_q). \quad (4.5)$$

The field  $b$  satisfies the equation

$$Z_b q^2 z_h^2 b_q - \gamma \left( \frac{\chi_q}{\sqrt{k}} - b_q \right) = 0. \quad (4.6)$$

Integrating out  $b$  according to its equation of motion and replacing in (4.5) we obtain

$$z \partial_z \chi_q(z) \Big|_{z=z_h} = \frac{1}{2} \frac{\gamma_h Z_b q^2 a^2}{\gamma_h + Z_b q^2 a^2} \chi_q(z_h). \quad (4.7)$$

This is the same boundary condition as the one implied by the original action (4.1) when

$$\lambda_{h2} = \frac{\gamma_h Z_b q^2 a^2}{\gamma_h + Z_b q^2 a^2}. \quad (4.8)$$

Now we require in the running of the radius that the effective theory with the brane field included be equivalent to the effective theory without that field. This requirement is clearly

$$\lambda_2(a) = \frac{\gamma(a) Z_b(a) q^2 a^2}{\gamma(a) + Z_b(a) q^2 a^2}. \quad (4.9)$$

We would like the field  $b$  to represent the massless zero mode. We therefore require that field strength flows according to

$$a \frac{dZ_b}{da} = 2Z_b - 2. \quad (4.10)$$

i.e.  $Z_b$  is identified with the coupling  $\lambda_2^{(2)}$  whose flow can be extracted from (4.2). Obeying this equation, the field strength grows large as  $a \rightarrow z_v$  and the field  $b$  decouples from  $\chi$ . The flow for  $\gamma$  is determined by the flow equation for  $\lambda_2$  and by equation (4.9),

$$a \frac{d\gamma}{da} = 4\gamma - \frac{1}{2}\gamma^2 - 2q^2 a^2 - \frac{4\gamma}{Z_b}. \quad (4.11)$$

The important point in all of this is that the flow equations have an attractive fixed point given by

$$1/Z_b^* = 0, \quad (4.12)$$

$$\gamma^* = 2qa \frac{J_1(qa)}{J_2(qa)} \approx 8 - \frac{(qa)^2}{3} - \frac{(qa)^4}{144} + \dots \quad (4.13)$$

Flowing towards this fixed point the massless field  $b$  decouples.

The improved effective theory is valid up to the naive scale  $1/a$ . In the limit where  $a \rightarrow z_v$  the field  $\tilde{\chi}(x) = \chi(x, z_v) - \sqrt{k}b(x)$  incorporates the contribution of the KK modes and  $b$  is the massless zero mode.

## 4.2 EFT improvement for gauge fields

We can improve the effective theory running for the  $U(1)$  gauge theory and gravity as we have done in the case of the scalar field. The basic structure wherein the massless field is added is the same with additional gauge and index structure. In the case of the bulk photon we add a four-dimensional  $U(1)$  gauge field localized on the hidden brane. The quadratic hidden brane action is

$$S_{hid} = -\frac{1}{4g_5^2 k} \int d^4x \sqrt{g_h} (Z_B g_h^{\mu\rho} g_h^{\nu\sigma} B_{\mu\nu} B_{\rho\sigma} + 2k^2 g_h^{\mu\nu} (A_\mu - B_\mu - \partial_\mu \alpha) \rho_h (A_\nu - B_\nu - \partial_\nu \alpha)), \quad (4.14)$$

where  $B_\mu$  is the massless gauge field on the hidden brane and  $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$ . We have introduced a  $4D$  goldstone mode  $\alpha$  in order to break spontaneously one combination of the  $U(1)$  gauge symmetries. In the previous equation

$$\rho_h = \rho_h^{(0)} - k^{-2} \rho_h^{(2)} g_h^{\mu\nu} \partial_\mu \partial_\nu + \dots \quad (4.15)$$

By making an appropriate identification of the couplings of this theory with the theory of Section 2, namely

$$\tau_h = \frac{\rho_h Z_B}{\rho_h + Z_B q^2 z_h^2}, \quad (4.16)$$

we can insure that the two are equivalent. Running the radius down with the requirement

$$\tau(a) = \frac{\rho(a) Z_B(a)}{\rho(a) + Z_B(a) q^2 a^2}, \quad (4.17)$$

the couplings flow according to

$$a \frac{dZ_B}{da} = 2 \tag{4.18}$$

$$a \frac{d\rho}{da} = 2\rho + \frac{\rho^2}{2} + 2q^2 a^2 + 4 \frac{\rho}{Z_B}, \tag{4.19}$$

where the flow of  $Z_B$  is chosen to reproduce the zero mode of the bulk field. The flow possess an attractive fixed point,

$$1/Z_B^* = 0 \tag{4.20}$$

$$\rho^* = -2qa \frac{J_0(qa)}{J_1(qa)} \approx -4 + \frac{(qa)^2}{2} + \frac{(qa)^4}{48} + \dots \tag{4.21}$$

where the field  $B_\mu$  decouples.

We can extend the analysis above to include interactions of the gauge field on the hidden brane (see the Appendix for a detailed analysis of interacting gauge theories). We write a hidden brane action in terms of the fields  $\tilde{A}_\mu = A_\mu - B_\mu - \partial_\mu \alpha$  and  $B_{\mu\nu}$ . We choose the coefficients of the operators so that this theory is equivalent to a particular interacting theory upon removing the field  $B_\mu$ . We know that a fixed point for this theory exists, the quadratic field fixed point where all coefficients of higher field operators are zero. Consider now a small perturbation from the fixed point by a gauge invariant operator schematically of the form  $\epsilon \tilde{A}^n B^m$  where indices and derivatives are suppressed. Working to first order in  $\epsilon$ , we remove the field  $B_\mu$  and determine the flow for  $\epsilon$  from the flow equation (A.8) presented in the Appendix. The operator is irrelevant for all  $m$  and  $n$ . Therefore, in the case of the  $U(1)$  gauge theory it is in general possible to find a sensible low energy effective theory with a cutoff  $1/a$ .

We can also examine the gravitational theory at the linearized order by adding a massless brane graviton. As before the effective theory has an attractive fixed point and the cutoff is  $1/a$ .

## 5 Discussion

We have extended the RG formalism proposed in [5] to gauge fields and gravity in the RS1 scenario. Moving within the space of effective field theories with varying radius of the extra dimension, the hidden brane couplings flow in a way that resembles a  $4D$  RG flow. The flows for the theories we consider do not have an attractive fixed point due to the presence of a massless zero mode (photon or graviton). The cutoff of the effective theory with the hidden brane located at  $a$  is lower than the expected scale  $1/a$ . We have found an improved procedure where we introduce a massless hidden brane field. This equivalent theory has an attractive fixed

point. The cutoff is  $1/a$ . Running the radius down, the massless field becomes the massless zero mode of the gauge/gravity theory.

Our method allows one to compute classical effective field theories for a low energy observer living near the visible brane in a simple and systematic way. We believe that these techniques can be used to simplify  $5D$  computations. It would be interesting to extend this formalism to the quantum level. This, for example, could shed some light on the quantum running of the  $4D$  zero mode gauge coupling, relevant for studies of unification in the RS1 scenario. More general gravitational backgrounds could also be analyzed using the techniques presented in this paper.

Consideration of the Goldberger-Wise scenario including the gravitational back reaction and the extension of this work into the quantum regime has the potential to provide a comprehensive demonstration of the stability of the hierarchy between the weak/Planck scales in the RS1 model.

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## A Scalar QED

In this Appendix we apply the method developed in this paper to the case of scalar QED. The action is

$$S = S_{bulk} + S_{hid} + S_{vis}, \tag{A.1}$$

where

$$S_{bulk} = \int d^4x \int_{z_v}^{z_h} dz \sqrt{G} \left( -\frac{1}{4g_5^2} G^{MN} G^{PQ} F_{MP} F_{NQ} + \frac{1}{2} G^{MN} (D_M \chi)(D_N \chi)^\dagger - \frac{m^2}{2} \chi \chi^\dagger \right). \tag{A.2}$$

Here  $\chi$  is a massive scalar field charged under the  $U(1)$  gauge symmetry.  $S_v$  and  $S_h$  are the most general brane actions local in  $\chi$ ,  $A_\mu$  and their  $x$ -derivatives and consistent with gauge invariance. We look for a flow of the hidden brane couplings that preserves this form. Although the calculation of this flow is technically more complicated, the procedure is conceptually the same as in the free photon theory.

The equations of motion

$$\frac{\delta S}{\delta A_\mu} = 0 \quad \text{and} \quad \frac{\delta S}{\delta \chi} = 0 \quad (\text{A.3})$$

require boundary conditions for  $A_\mu$  and  $\chi$  at the hidden and visible branes. At the hidden brane the boundary conditions are

$$z\partial_z A_\mu \Big|_{z=z_h} = -\frac{1}{2\sqrt{-g_h}} \frac{\delta S_h}{\delta A_\nu(x)} g_{\mu\nu}^h \Big|_{A_\nu(x)=A_\nu(x,z_h)} \quad (\text{A.4})$$

$$z\partial\chi \Big|_{z=z_h} = -\frac{1}{\sqrt{-g_h}} \frac{\delta S_h}{\delta \chi^*(x)} \Big|_{\chi^*(x)=\chi^*(x,z_h)}. \quad (\text{A.5})$$

Notice that the functional derivatives are with respect to  $4D$  fields.

We look for a new local action  $S_a$  such that the boundary conditions

$$z\partial_z A_\mu \Big|_{z=a} = -\frac{1}{2\sqrt{-g}} \frac{\delta S_a}{\delta A_\nu(x)} g_{\mu\nu} \Big|_{A_\nu(x)=A_\nu(x,a)} \quad (\text{A.6})$$

$$z\partial\chi \Big|_{z=a} = -\frac{1}{\sqrt{-g}} \frac{\delta S_a}{\delta \chi^*(x)} \Big|_{\chi^*(x)=\chi^*(x,a)} \quad (\text{A.7})$$

are satisfied, where  $z_v > a \geq z_h$  and  $S_a = S_h$  when  $a = z_h$ . Here  $g_{\mu\nu}(x) = G_{\mu\nu}(x, a)$ . The fields that satisfy these boundary conditions must be solutions to the original equations of motion in the restricted region  $z_v \geq z \geq a$ .

We are interested in the differential flow of  $S_a$  as we vary  $a$ . Taking the logarithmic derivative of these equations and using the equations of motion we write the flow equations in terms of the brane action. The first boundary condition leads to the flow equation

$$\begin{aligned} a \frac{\partial}{\partial a} \left( \frac{\delta S_a}{\delta A_\mu(x)} \right) &= \int d^4 x' \left( \frac{\delta^2 S_a}{\delta A_\mu(x) \delta \chi(x')} \left( \frac{1}{\sqrt{-g}} \frac{\delta S_a}{\delta \chi^*(x')} \right) + \frac{\delta^2 S_a}{\delta A_\mu(x) \delta \psi^*(x')} \left( \frac{1}{\sqrt{-g}} \frac{\delta S_a}{\delta \chi(x')} \right) \right. \\ &\quad \left. - 2 \frac{\delta^2 S_a}{\delta A_\mu(x) \delta g^{\rho\sigma}(x')} g^{\rho\sigma} + \frac{\delta^2 S_a}{\delta A_\mu(x) \delta A_\rho(x')} \left( \frac{1}{2\sqrt{-g}} \frac{\delta S_a}{\delta A_\sigma(x')} \right) g_{\rho\sigma} \right) \\ &\quad - 2\sqrt{-g} g^{\mu\nu} g^{\rho\sigma} \partial_\rho F_{\sigma\nu} + i\sqrt{-g} g^{\mu\nu} (\chi^* D_\nu \chi - \chi D_\nu \chi^*). \end{aligned} \quad (\text{A.8})$$

where as before  $A_\mu(x)$  and  $\chi(x)$  are  $4D$  fields evaluated such that  $A_\mu(x) = A_\mu(x, a)$  and  $\chi(x) = \chi(x, a)$ . From the second boundary condition we obtain

$$\begin{aligned} a \frac{\partial}{\partial a} \left( \frac{\delta S_a}{\delta \chi(x)} \right) &= \int d^4 x' \left( \frac{\delta^2 S_a}{\delta \chi(x) \delta \chi(x')} \left( \frac{1}{\sqrt{-g}} \frac{\delta S_a}{\delta \chi^*(x')} \right) + \frac{\delta^2 S_a}{\delta \chi(x) \delta \chi^*(x')} \left( \frac{1}{\sqrt{-g}} \frac{\delta S_a}{\delta \chi(x')} \right) \right. \\ &\quad \left. - 2 \frac{\delta^2 S_a}{\delta \chi(x) \delta g^{\rho\sigma}(x')} g^{\rho\sigma} + \frac{\delta^2 S_a}{\delta A_\rho(x') \delta \chi(x)} \left( \frac{1}{2\sqrt{-g}} \frac{\delta S_a}{\delta A_\sigma(x')} \right) g_{\rho\sigma} \right) \\ &\quad - \sqrt{-g} g^{\mu\nu} D_\mu D_\nu \chi^* + \sqrt{-g} m^2 \chi^*. \end{aligned} \quad (\text{A.9})$$

These equations represent the  $\beta$ -functions for the couplings of local operators on the effective hidden brane.

Notice that these flow equations are local. As a consequence, we will always flow within actions that contain only local operators. These equations also respect  $4D$  gauge invariance,

$$A_\mu \rightarrow A_\mu + \partial_\mu \Lambda \quad (\text{A.10})$$

$$\chi \rightarrow e^{i\Lambda} \chi \quad (\text{A.11})$$

This implies that if we start with a  $4D$  gauge invariant action at  $z_h$  any boundary action consistent with the flow equations will also be  $4D$  gauge invariant. The full action is still  $5D$  gauge invariant.

We now show that the effective theory generated is sensible in the sense that the effects of the higher dimension operators generated on the effective hidden brane are suppressed at energies below a cutoff scale of the order  $1/(a \log(a/z_h))$ . We will do this by showing that the dimensionless coefficient of no operator flows to an excessively large number. Therefore one can truncate to the leading operators as in standard effective field theory.

In the case of the massive free scalar with no photon present, studied in [5], it was found that there is an attractive fixed point to which a general local brane action flows. The coefficient of an operator with  $i$  fields and  $j$  derivatives scales as  $(a/z_h)^{\gamma_i^{(2j)}}$  where

$$\gamma_i^{(2j)} = 4 - 2j - \frac{i}{2}(4 + 2\nu) < 0 \quad (\text{A.12})$$

and  $\nu = \sqrt{4 + m^2/k^2}$ . Higher dimension operators are suppressed for momentum  $q$  in the region  $qa < 1$ .

In the case of a free photon without a scalar field we have found by explicit calculation in Section 2 that the coefficient of an operator with  $2k$  derivatives scales as  $(\log(a/z_h))^k$ . If we restrict momentum such that  $q^2 a^2 \log(a/z_h) < 1$ , higher derivative operators will be unimportant. The cutoff here is parametrically smaller than  $1/a$ .

Now consider operators with  $j$  derivatives and  $i$  powers of the  $A_\mu$  field. Let the coefficient of such an operator be denoted  $\tau_{i,j}$ . The flow of the coefficients of these operators will be dictated by equation (A.8). In the absence of scalars this gives a flow of the form

$$a\partial_a \tau_{i,j} = -j \tau_{i,j} + \sum_{klmn} c_{klmn} \tau_{k,l} \tau_{m,n} \quad (\text{A.13})$$

where  $c_{klmn}$  are order one numbers. We must have  $k+m = i+2$ ,  $n+l = j$ ,  $i \leq j$ ,  $k \leq l$  and  $m \leq n$ . These conditions are sufficient to inductively show that  $\tau_{i,j}$  scales as  $(a/z_h)^{-i} (\log(a/z_h))^{(j-i)/2}$  for  $i \geq 4$ . These operators are suppressed as we move the effective hidden brane closer to the visible brane.



We now consider those brane operators with interaction terms between the photon and scalar. Some of these operators will be contained in the gauge invariant operators with covariant derivatives of  $\chi$  only. Coefficients of these operators must flow as in the case of the self-interacting scalar without the photon field by gauge invariance. Operators not of this form (for example  $|\chi|^2 F^2$ ) must be examined by studying equation (A.8) to determine the flows of their coefficients. We denote the coefficient of an operator with  $i$  derivatives,  $j$  powers of  $A^\mu$  and  $k$  powers of  $\chi$  as  $\alpha_{i,j,k}$ . The flow equation for these coefficients takes the form

$$a\partial_a\alpha_{i,j,k} = \gamma_{ijk}\alpha_{i,j,k} + \delta\lambda_2^{(0)}\alpha_{i,j,k} + \sum c_{lmnpqr}\alpha_{l,m,n}\alpha_{p,q,r} \quad (\text{A.14})$$

where

$$\gamma_{ijk} = 4 - (i + j) - k(2 + \nu) < 0 \quad (\text{A.15})$$

Here the indices are constrained to obey  $l + p = i$ ,  $m + q = j + 2$ ,  $n + r = k$ ,  $i \geq j$ ,  $l \geq m$ ,  $p \geq q$ . By induction the terms with the strongest  $a$  dependence scale as  $\alpha_{ijk} \sim (a/z_h)^{4-k(2+\nu)-j}(\log(a/z_h))^{(i-j)/2}$  for  $k \geq 2$ .

The important thing to take away from this discussion is that the effective theory with hidden brane located at  $a$  will be valid up to momenta satisfying  $q^2 a^2 \log(a/z_h) < 1$ . The higher derivative operators will be suppressed at low energy allowing for simplification in effective field theory computations.

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