Decomposing the total effects of one variable on another into direct and indirect effects has long been of interest to researchers who use path analysis. In this paper, I review the decomposition of effects in general structural equation models with latent and observed variables. I present the two approaches to defining total effects. One is based on sums of powers of coefficient matrices. The other defines total effects as reduced-form coefficients. I show the conditions under which these two definitions are equivalent. I also compare the different types of specific indirect effects. These are the influences that are transmitted through particular

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variables in a model. Finally, I propose a more general definition of specific effects that includes the effects transmitted by any path or combination of paths. I also include a section on computing standard errors for all types of effects.

Since its introduction into sociology, path analysis has been used to decompose the influences of one variable on another into total, direct, and indirect effects (Duncan 1966, 1975; Finney 1972; Alwin and Hauser 1975; Greene 1977; Fox 1980). Several researchers have generalized the decomposition procedures from observed variable models to structural equations with latent variables (Schmidt 1978; Graff and Schmidt 1982; Jöreskog and Sörbom 1981, 1982). This latter development opens up new applications. For instance, the total effects of a second-order factor on indicators can reveal which measures are most closely related to it. Or, if a measure has a factor complexity of two or more, an effect decomposition can show which latent variable has the greatest effect on the measure, not just the largest direct effect. The first purpose of this paper is to review the decomposition of effects for these general models.

The second purpose is to clarify the definitions of total effects. Some have defined total effects as sums of powers of the coefficient matrices; these effects are defined only if certain stability conditions are met (Bentler and Freeman 1983; Jöreskog and Sörbom 1981). Others have used the reduced-form coefficients to define total effects (Alwin and Hauser 1975). Most researchers have used some combination of these (e.g., Graff and Schmidt 1982). In this paper, I will explain the relationship between both definitions.

Decompositions typically provide the total indirect effects, which consist of all paths from one variable to another mediated by at least one additional variable. Alwin and Hauser (1975), Greene (1977), and Fox (1985) provide techniques to estimate specific indirect effects that are transmitted by selected variables rather than by all variables. Although they use the same term, their definitions of specific effects differ both conceptually and operationally. As these techniques become more widely used, this is sure to be a source of confusion, since estimates of specific indirect effects will differ depending on the definition. The third purpose of this paper is to describe the alternative meanings of specific indirect effects and the techniques for calculating them.
The final purpose of the paper is to propose a more general definition of specific effects, a definition that includes the effects transmitted by any path or combination of paths in a model. This definition encompasses all the previously proposed specific indirect effects plus a new set of specific effects. The approach is unlike the others in that it is path-oriented rather than variable-oriented. I develop and illustrate a simple technique for estimating all these effects. I also draw on the work of Folmer (1981, pp. 1440–42) and Sobel (1982, 1986) to demonstrate the use of the delta method to compute standard errors for these decompositions.

**DECOMPOSITION OF EFFECTS**

I discuss total, direct, and indirect effects in a structural equation model with latent variables, often referred to as the LISREL model (see Jöreskog and Sörbom 1981; Wiley 1973). This model consists of a latent variable equation and two measurement equations. The latent variable model is

\[ \eta = B \eta + \Gamma \xi + \zeta, \]

where \( \eta \) is an \( m \times 1 \) vector of latent endogenous variables, \( B \) is an \( m \times m \) matrix of the coefficients linking the \( \eta \) variables, \( \xi \) is an \( n \times 1 \) vector of latent exogenous variables, \( \Gamma \) is an \( m \times n \) matrix of coefficients relating \( \xi \) to \( \eta \), and \( \zeta \) is an \( m \times 1 \) vector of errors in the equation or disturbance terms. It is assumed that \( \zeta \) is uncorrelated with \( \xi \), that \( E(\zeta) \) is zero, that \( (I-B) \) is nonsingular,\(^1\) and that \( \eta \) and \( \xi \) are deviated from their means. The measurement equations are

\[ x = \Lambda_{x} \xi + \delta, \]
\[ y = \Lambda_{y} \eta + \varepsilon, \]

where \( x \) and \( y \) denote observed variables, \( \Lambda_{x} \) is a \( q \times n \) coefficient matrix relating \( x \) to \( \xi \), \( \Lambda_{y} \) is a \( p \times m \) coefficient matrix of the effects of \( \eta \) on \( y \). It is assumed that \( E(\delta) \) and \( E(\varepsilon) \) equal zero, that \( \delta \) and \( \varepsilon \) are uncorrelated with each other and with \( \xi, \eta, \) and \( \zeta \), and that \( x \) and \( y \) are written in deviation form.

\(^1\) Readers familiar with LISREL versions I–IV will note the different definitions of \( B \) in versions V and VI. The new \( B \) matrix equals the \( m \)-order identity matrix minus the old \( B \): \( B_{\text{new}} = (I - B_{\text{old}}) \).
In all decompositions, the total effects are equal to the direct effects plus the indirect effects. The direct effects are those influences unmediated by any other variable in the model. The coefficient matrices in the structural equations (1), (2), and (3) are the direct effects. For instance, (1) shows the direct effects of $\xi_i$ on $\eta_1$ as $\Gamma$. Equation (3) gives the direct effects of $\xi_i$ on $x$ as $\Lambda_x$. Indirect effects are mediated by at least one intervening variable. They are determined by subtracting the direct effects from the total effects.

Total effects are defined in two ways. Some researchers define them as the sum of powers of coefficient matrices. Others define them using reduced-form coefficients. I examine both approaches in this section, starting with the infinite-sum definition.

Fox (1980, p. 12) defines the total effects of $\eta_i$ on $\eta_1$ or $T_{\eta_1}$ as

$$T_{\eta_1} = \sum_{k=1}^{\infty} B^k. \quad (4)$$

$T_{\eta_1}$ is defined only if the infinite sum in (4) converges to a matrix with finite elements. To understand this definition, consider the relationship between three latent endogenous variables, as drawn in the path diagram in Figure 1. The $B$ matrix is

$$B = \begin{bmatrix} 0 & 0 & 0 \\ \beta_1 & 0 & 0 \\ \beta_2 & \beta_3 & 0 \end{bmatrix}. \quad (5)$$

$B$ provides the direct effects of the latent endogenous variables on one another. The first few terms of the infinite series defining the total

FIGURE 1. A simple recursive model for three endogenous variables.
effects of $\eta$ on $\eta$ are
\[
T_{\eta\eta} = B + B^2 + B^3 + \ldots
\]
\[
= \begin{bmatrix}
\beta_1 & 0 & 0 \\
0 & \beta_3 & 0 \\
\beta_2 & \beta_3 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
\beta_1 \beta_3 & 0 & 0
\end{bmatrix}
+ 0 + \ldots.
\]

(6)

Clearly, since all $B^k$ for $k \geq 3$ equal zero, the series converges and the total effects are defined. The first term in the series represents the direct effects of $\eta$ on $\eta$. The second and higher-power terms represent the indirect effects of varying lengths. In Figure 1, the only indirect effect of length 2 is $\beta_1 \beta_3$, the influence of $\eta_1$ on $\eta_3$ mediated by $\eta_2$. The zero values for $B^3$ and the higher-power terms indicate that all indirect effects of length 3 or greater equal zero. Summing the right-hand side of (6) gives

\[
T_{\eta\eta} = \begin{bmatrix}
0 & \beta_1 & 0 \\
\beta_2 + \beta_1 \beta_3 & \beta_3 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

(7)

In general, the indirect effects are the differences between the total effects and the direct effects. Subtracting $B$ from $T_{\eta\eta}$ yields $B^2$, the indirect effects ($I_{\eta\eta}$) for this model.

FIGURE 2. Path diagram of simple latent-variable model and measurement model.
This result generalizes. For recursive systems in which $B$ can be written as a lower triangular matrix, $B^k$ equals zero for $k \geq m$ (where $m$ is the number of $\eta$‘s). Thus, $B$ converges and the total effects are defined. These total effects equal $\sum_{k=1}^{m-1} B^k$. It then follows that the indirect effects are $T_{\eta \eta} - B$, or $\sum_{k=2}^{m-1} B^k$.

Nonrecursive models are more complicated. Consider the example in Figure 2:

$$B = \begin{bmatrix} 0 & \beta_2 \\ \beta_1 & 0 \end{bmatrix},$$

(8)

where $\beta_1$ represents the direct effect of $\eta_1$ on $\eta_2$ and $\beta_2$ represents the direct effect of $\eta_2$ on $\eta_1$. The indirect effects of $\eta$ on $\eta$ of particular lengths can be determined by raising $B$ to the appropriate powers. The indirect effects for lengths 2, 3, and 4 are

$$B^2 = \begin{bmatrix} \beta_1 \beta_2 & 0 \\ 0 & \beta_1 \beta_2 \end{bmatrix}, \quad B^3 = \begin{bmatrix} 0 & \beta_1 \beta_2^2 \\ \beta_1^2 \beta_2 & 0 \end{bmatrix},$$

$$B^4 = \begin{bmatrix} \beta_1^2 \beta_2^2 & 0 \\ 0 & \beta_1^2 \beta_2^2 \end{bmatrix}.$$

(9)

Unlike $B^k$ in recursive models, $B^k$ in nonrecursive models is not necessarily zero for $k \geq m$. Also, as $B^2$ and $B^4$ illustrate, an endogenous variable can have an indirect effect on itself. For instance, $\eta_1$ changes $\eta_2$ by $\beta_1$ units, but the $\beta_1$ change in $\eta_2$ leads to a $\beta_1 \beta_2$ shift in $\eta_1$. To determine the total effects, $T_{\eta \eta}$, we let $k \to \infty$ and sum all $B^k$:

$$T_{\eta \eta} = \sum_{k=1}^{\infty} B^k = \begin{bmatrix} \sum_{i=1}^{\infty} (\beta_1 \beta_2)^i & \sum_{j=1}^{\infty} \beta_1^{-1} \beta_2^j \\ \sum_{j=1}^{\infty} \beta_1 \beta_2^{-1} & \sum_{i=1}^{\infty} (\beta_1 \beta_2)^i \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=1}^{\infty} (\beta_1 \beta_2)^i & \sum_{j=1}^{\infty} (\beta_1 \beta_2)^j \beta_1^{-1} \\ \sum_{j=1}^{\infty} (\beta_1 \beta_2)^j \beta_2^{-1} & \sum_{i=1}^{\infty} (\beta_1 \beta_2)^i \end{bmatrix}.$$

(10)

Each element of $T_{\eta \eta}$ involves an infinite sum. For instance, the $(1,1)$
element is

$$\sum_{i=1}^{\infty} \left( \beta_1 \beta_2 \right)^i = \beta_1 \beta_2 + \left( \beta_1 \beta_2 \right)^2 + \left( \beta_1 \beta_2 \right)^3 + \cdots .$$  \hspace{1cm} (11)

The total effect of $\eta_1$ on itself is defined if (11) converges. To determine the conditions for convergence and the value to which it converges, first add $(\beta_1 \beta_2)^0$ to both sides of (11):

$$\sum_{i=0}^{\infty} \left( \beta_1 \beta_2 \right)^i = \left( \beta_1 \beta_2 \right)^0 + \left( \beta_1 \beta_2 \right)^1 + \left( \beta_1 \beta_2 \right)^2 + \cdots .$$  \hspace{1cm} (12)

Equation (12) is a geometric series. For this series to converge, $(\beta_1 \beta_2)^i$ must go to zero as $i$ approaches infinity. This occurs only if $|\beta_1 \beta_2|$ is less than one.

To find the convergent value, multiply the right-hand side of (12) by $(1 - \beta_1 \beta_2)$:

$$(1 - \beta_1 \beta_2) \left( 1 + \beta_1 \beta_2 + \left( \beta_1 \beta_2 \right)^2 + \cdots + \left( \beta_1 \beta_2 \right)^i \right) = 1 - (\beta_1 \beta_2)^{i+1} .$$  \hspace{1cm} (13)

Since $|\beta_1 \beta_2|$ is less than one, $(\beta_1 \beta_2)^{i+1}$ goes to zero as $i$ goes to infinity. This leaves one on the right-hand side of (13). For the product on the left-hand side to equal one, $(1 + \beta_1 \beta_2 + \left( \beta_1 \beta_2 \right)^2 + \cdots + \left( \beta_1 \beta_2 \right)^i) \rightarrow (1 - \beta_1 \beta_2)^{-1}$ as $i \rightarrow \infty$. Thus,

$$\sum_{i=0}^{\infty} \left( \beta_1 \beta_2 \right)^i = (1 - \beta_1 \beta_2)^{-1} \text{ for } |\beta_1 \beta_2| < 1 \text{ and } i \rightarrow \infty .$$  \hspace{1cm} (14)

To obtain the original series in equation (11), subtract $1 (= (\beta_1 \beta_2)^0)$ from both sides of (14):

$$\sum_{i=1}^{\infty} \left( \beta_1 \beta_2 \right)^i = \frac{\beta_1 \beta_2}{1 - \beta_1 \beta_2} .$$  \hspace{1cm} (15)

This is the convergent value for equation (11) and the total effect of $\eta_1$ on itself. To sum up, if $|\beta_1 \beta_2| < 1$, the $(1,1)$ element of (10) converges to $\beta_1 \beta_2/(1 - \beta_1 \beta_2)$. Since the $(1,1)$ and $(2,2)$ elements are identical, the $(2,2)$ element also converges to this value. When we apply a similar argument to the other elements of (10), we see that under the same condition of $|\beta_1 \beta_2| < 1$, the $(1,2)$ element converges to $\beta_2/(1 - \beta_1 \beta_2)$ and the $(2,1)$ element converges to $\beta_1/(1 - \beta_1 \beta_2)$. Thus, the total
effects of $\eta$ on $\eta$ are
\[ T_{\eta\eta} = \sum_{k=1}^{\infty} B^k = (1 - \beta_1 \beta_2)^{-1} \begin{bmatrix} \beta_1 \beta_2 \\ \beta_1 \beta_2 \end{bmatrix}, \]
and the indirect effects are
\[ I_{\eta\eta} = T_{\eta\eta} - B = (1 - \beta_1 \beta_2)^{-1} \begin{bmatrix} \beta_1 \beta_2 \\ \beta_1 \beta_2 \end{bmatrix}. \]

If $|\beta_1 \beta_2|$ equals or exceeds unity, then the total and indirect effects among the latent endogenous variables in Figure 2 are not defined.

In general, for the total effects $T_{\eta\eta} (= \sum_{k=1}^{\infty} B^k)$ to be defined, $B^k$ must converge to zero as $k \to \infty$ (Ben-Israel and Greville 1974). The matrix is convergent if and only if the modulus or absolute value of the largest eigenvalue of $B$ is less than one (Bentler and Freeman 1983, p. 144). To continue the generalization of the previous results, we next search for the value to which $T_{\eta\eta}$ converges. First, we add $I$ ($= B^0$) to $T_{\eta\eta}$ and then premultiply the sum $I + B + B^2 + \cdots + B^k$ by $(I - B)$:
\[ (I - B)(I + B + B^2 + \cdots + B^k) = I - B^{k+1}. \]

Since $B^{k+1} \to 0$ as $k \to \infty$, the last term in (18) approaches zero, leaving $I$. For the product of the two left-hand terms in (18) to equal $I$, $(I + B + B^2 + \cdots + B^k)$ must converge to $(I - B)^{-1}$ as $k \to \infty$. If $I$ is subtracted from this value, $T_{\eta\eta}$ results:
\[ T_{\eta\eta} = (I - B)^{-1} - I. \]

The indirect effects are
\[ I_{\eta\eta} = (I - B)^{-1} - I - B. \]

The above shows the decomposition of effects for the latent endogenous variables on one another using the infinite-sum definition. Decompositions of the latent exogenous variables on the latent endogenous variables are closely related. Following Bentler and Freeman (1983, p. 143), we can determine the total effects of $\xi$ on $\eta$ by

The similar canonical form of $B$ is useful in studying $B^k$ and in understanding the justification for placing this condition on the eigenvalues of $B$. See Searle (1982, pp. 282–89) and Luenberger (1979) for further discussion.
repeatedly substituting equation (1) for $\eta$ into the right-hand side of (1):

$$
\eta = B\eta + \Gamma \xi + \zeta \\
= B(B\eta + \Gamma \xi + \zeta) + \Gamma \xi + \zeta \\
= B^2\eta + (I + B)(\Gamma \xi + \zeta) \\
= B^2(B\eta + \Gamma \xi + \zeta) + (I + B)(\Gamma \xi + \zeta) \\
= B^3\eta + (I + B + B^2)(\Gamma \xi + \zeta) \\
= B^3\eta + (I + B + B^2 + \cdots + B^{k-1})(\Gamma \xi + \zeta). 
$$

(21)

The total effects of $\xi$ on $\eta$ are in the coefficient matrix for $\xi$ in the last line of (21): $(I + B + B^2 + \cdots + B^{k-1})\Gamma$. The $(I + B + B^2 + \cdots + B^{k-1})$ term is an infinite sum that converges to $(I - B)^{-1}$ under the same conditions stated above. That is, the absolute value or modulus of the largest eigenvalue of $B$ must be less than one. The $T_{\eta \xi}$ matrix under this condition is

$$
T_{\eta \xi} = (I - B)^{-1}\Gamma. 
$$

(22)

Since the direct effects of $\xi$ on $\eta$ are in $\Gamma$, the indirect effects of $\xi$ on $\eta$ are

$$
I_{\eta \xi} = (I - B)^{-1}\Gamma - \Gamma = ((I - B)^{-1} - I)\Gamma. 
$$

(23)

Equation (23) shows that the indirect effects of $\xi$ on $\eta$ equal the product of the total effects of $\eta$ on $\eta$ and the direct effects of $\xi$ on $\eta$.

The derivation of the total and indirect effects of $\zeta$ on $y$ follows a similar strategy. Repeated substitution of equation (1) for $\eta$ in the measurement equation for $y$ yields

$$
y = \Lambda_j B^k \eta + \Lambda_j (I + B + B^2 + \cdots + B^{k-1})\Gamma \xi \\
+ \Lambda_j (I + B + B^2 + \cdots + B^{k-1})\xi + \epsilon. 
$$

(24)

For convergent $B$, the total effects $T_{y \zeta}$ are

$$
T_{y \zeta} = \Lambda_j (I - B)^{-1}\Gamma. 
$$

(25)
The indirect effects also equal (25), since \( \xi \) has no direct effects on \( y \).\(^3\)

By the same logic, the total effects of \( \eta \) on \( y \) are

\[
T_{y\eta} = \Lambda_y(I + B + B^2 + \cdots + B^k) = \Lambda_y(I - B)^{-1},
\]

and the indirect effects are

\[
I_{y\eta} = \Lambda_y(I - B)^{-1} - \Lambda_y = \Lambda_y((I - B)^{-1} - I).
\]

Total and indirect effects are defined only under certain conditions, i.e., when the absolute value or modulus of the eigenvalues of \( B \) is less than one.\(^4\) The eigenvalues are not always readily available. However, two shortcuts can identify convergent \( B \). I have already discussed the first: If \( B \) is a lower triangular matrix, then \( B^k \) is zero for \( k \geq m \). The second applies to nonrecursive models: If the elements of \( B \) are positive and the sum of the elements in each column is less than one, the absolute values of the eigenvalues are less than one (Goldberg 1958, pp. 237-38). This is a sufficient condition, but it is not necessary for stability. Given its ease of calculation, it should prove useful in many situations.

The preceding approach defines the total effects as infinite sums of functions of the coefficient matrices, and it requires that stability conditions be met for these effects to exist. A second strategy uses reduced-form equations to identify total effects (Alwin and Hauser 1975; Fox 1980, p. 25; Graff and Schmidt 1982). In classical econometric models, reduced form refers to solutions of structural equations in which all endogenous variables are written as functions of only exogenous (or predetermined) variables, structural coefficients, and disturbances. My use of the concept of reduced forms is similar to Graff and Schmidt's (1982). It differs from the classical econometric concept in that \( \xi \), the vector of exogenous variables, is a vector of latent variables rather than directly observed variables.

\(^3\) Jöreskog and Sörbom (1981, p. 39; 1982, p. 409) incorrectly list the direct effect of \( \xi \) on \( y \) as \( \Lambda_y \Gamma \) and the indirect effects as \( \Lambda_y(I - B)^{-1} \Gamma - \Lambda_y \Gamma \). Freeman (1982), in his dissertation, also notes this error.

\(^4\) This condition is also necessary for the existence of \( T_{\eta \eta} \), but it is not necessary for the existence of \( T_{\xi \xi} \), \( T_{y \eta} \), and \( T_{\eta \xi} \) (see Sobel 1986, p. 166). In practice, we would rarely, if ever, be interested in these latter total effects if \( T_{\eta \eta} \) did not exist.
The relationship between the reduced-form coefficients for $\xi$ and the total effects of $\xi$ on $\eta$ is easily shown by writing equation (1) in reduced form:

$$\eta = B\eta + \Gamma \xi + \zeta$$

$$(I - B)\eta = \Gamma \xi + \zeta$$

$$\eta = (I - B)^{-1} \Gamma \xi + (I - B)^{-1} \zeta.$$  \hspace{1cm} (28)

The reduced-form coefficients of $\xi$ on $\eta$ are $(I - B)^{-1}$. They equal $T_{\eta \xi}$, derived above.

More steps are required to show the relationship between the reduced-form coefficients and the total effects of $\eta$ on $\eta$. First, the effects of $\xi$ on $\eta$ are decomposed. To determine the total effects of $\xi$ on $\eta$, we return to equation (28). A largely unrecognized aspect of this reduced-form equation is that $(I - B)^{-1}$, the coefficient matrix of $\xi$, contains the total effects of $\xi$ on $\eta$. When equation (1), the structural equation for $\eta$, is slightly revised, the direct effects of $\xi$ on $\eta$ are immediately apparent:

$$\eta = B\eta + \Gamma \xi + I\xi,$$  \hspace{1cm} (29)

where $I$ is an $m \times m$ identity matrix. In (29), the direct effects of $\xi$ equal $I$. Since $\xi$ directly affects only variables comprising $\eta$, all its indirect effects must be mediated by $\eta$. The indirect effects of $\xi$ include all effects from $\eta$ (the mediating variable) to $\eta$. But all influences from $\eta$ to $\eta$ are the total effects of $\eta$ on $\eta$. Thus, the indirect effects of $\xi$ on $\eta$ equal the total effects of $\eta$ on $\eta$.

Equation (28) shows that the total effects of $\xi$ on $\eta$ are $(I - B)^{-1}$. Equation (29) shows that the direct effects of $\xi$ on $\eta$ equal $I$. The indirect effects of $\xi$ on $\eta$ and the total effects of $\eta$ on $\eta$ both equal $(I - B)^{-1} - I$.

Similarly, the total effects of $\xi$ on $y$ are equal to the reduced-form coefficients for $\xi$. The reduced form of the measurement model for $y$ is

$$y = \Lambda_y(I - B)^{-1} \Gamma \xi + \Lambda_y(I - B)^{-1} \zeta + \varepsilon.$$  \hspace{1cm} (30)

The reduced-form coefficients for $\xi$ in (30) correspond to the total effects derived earlier for $T_{\eta \xi}$. Finally, the total effects of $\eta$ on $y$ also equal reduced-form coefficients. In equation (30), the reduced-form coefficients for $\xi$ equal the total effects of $\xi$ on $y$, or $\Lambda_y(I - B)^{-1}$. Since $\xi$ has no influence on $y$ unmediated by $\eta$, the direct effects of $\xi$ on $y$
are zero, and the indirect and total effects are equal. The indirect effects of $\xi$ on $y$ are all the influences that $\eta$ exerts on $y$; therefore, $T_{y\eta}$ is $\Lambda_y(I - B)^{-1}$. Thus, when the reduced-form and structural equations are used, all the total and indirect effects can be identified.

Defining total effects as the sum of an infinite series and defining them in terms of reduced-form coefficients appear to yield the same results. However, there is one important difference. To illustrate, consider $T_{\eta\eta}$. For the total effects $\sum_{k=1}^{\infty} B^k$ to be defined, the convergence condition for $B$ must be met. To write the structural equation for $\eta$ in reduced form, $(I - B)^{-1}$ must exist. The existence of $(I - B)^{-1}$ need not imply the convergence condition. For example, specify $B$ as

$$B = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}. \quad (31)$$

The matrix $(I - B)$ is nonsingular with an inverse of

$$ (I - B)^{-1} = -\frac{1}{3} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}. \quad (32)$$

Nevertheless, $|\beta_1 \beta_2| (= 4)$ exceeds one, so the convergence condition is not met. In this example, the total effects would be defined by the reduced-form coefficients, although they would be nonexistent because of the convergence criterion. In such a situation, it is questionable whether the reduced-form coefficients can be interpreted as total effects. Since the convergence condition is sufficient for $(I - B)^{-1}$ to exist (Goldberg 1958, p. 238), it makes sense to require convergence even if the decomposition of effects is approached via reduced-form equations.

To illustrate the decomposition, I again refer to Figure 2. Here, $\xi_1$ and $\xi_2$ are latent exogenous variables, $\eta_1$ and $\eta_2$ are latent endogenous variables, and there are two indicators each for $\xi_1$, $\xi_2$, $\eta_1$, and $\eta_2$. The matrix representation for the latent variable model is

$$ \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 0 & \beta_2 \\ \beta_1 & 0 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} + \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}. \quad (33)$$

The equations for the measurement models are

$$ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ \lambda_2 & 0 \\ 0 & \lambda_3 \\ 0 & \lambda_4 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix}. \quad (34)$$
TOTAL, DIRECT, AND INDIRECT EFFECTS

and

\[
\begin{bmatrix}
  y_1 \\
  y_2 \\
  y_3 \\
  y_4
\end{bmatrix} = \begin{bmatrix}
  \lambda_5 & 0 \\
  \lambda_6 & 0 \\
  0 & \lambda_7 \\
  0 & \lambda_8
\end{bmatrix} \begin{bmatrix}
  \eta_1 \\
  \eta_2
\end{bmatrix} + \begin{bmatrix}
  \epsilon_1 \\
  \epsilon_2 \\
  \epsilon_3 \\
  \epsilon_4
\end{bmatrix}. \tag{35}
\]

In equations (9) through (16), I derived the total, direct, and indirect effects of \( q_1 \) on \( r \) for this model. I now do the same for the effects of \( q \) on \( q^1 \). According to the formula above, the total effects of \( q \) on \( q^1 \) equal \((I - \mathbf{B})^{-1}\Gamma\). When this result is applied to the model shown in Figure 2, we get

\[
\mathbf{T}_{q^1q} = (I - \mathbf{B})^{-1}\Gamma = (1 - \beta_1\beta_2)^{-1} \begin{bmatrix}
  1 & \beta_2 \\
  \beta_1 & 1
\end{bmatrix} \begin{bmatrix}
  \gamma_1 & 0 \\
  0 & \gamma_2
\end{bmatrix}
\]

\[
= (1 - \beta_1\beta_2)^{-1} \begin{bmatrix}
  \gamma_1 & \beta_2\gamma_2 \\
  \beta_1\gamma_1 & \gamma_2
\end{bmatrix}. \tag{36}
\]

The first term on the right-hand side of this expression, \((1 - \beta_1\beta_2)^{-1}\), results from the reciprocal relationship between \( \eta_1 \) and \( \eta_2 \), which may be activated by the effect of \( \xi_1 \) (or \( \xi_2 \)) on \( \eta_1 \) (or \( \eta_2 \)). For this feedback loop to converge, \(|\beta_1\beta_2|\) must be less than one. The direct effects of \( q \) on \( q^1 \) appear as coefficients in \( \Gamma \). The indirect effects of \( q \) on \( q^1 \) equal \((I - \mathbf{B})^{-1} - I)\Gamma\), which for Figure 2 are

\[
\mathbf{I}_{q^1q} = (1 - \beta_1\beta_2)^{-1} \begin{bmatrix}
  \beta_1\beta_2\gamma_1 & \beta_2\gamma_2 \\
  \beta_1\gamma_1 & \beta_1\beta_2\gamma_2
\end{bmatrix}. \tag{37}
\]

Similarly, the total and indirect effects for the remaining variables can be determined by substituting the \( \mathbf{B}, \Gamma, \) and \( \Lambda \), in Figure 2 into the appropriate decomposition formula derived above.

Table 1 summarizes the decomposition of effects for the general structural equation model with latent variables. It is well known that the classical econometric simultaneous equation model, confirmatory factor analysis, and the MIMIC model are special cases of the LISREL model (Jöreskog and Sörbom 1982; Jöreskog and Goldberger 1975). Thus, Table 1 can be applied to these models given the appropriate substitutions. For example, if \( x \) were substituted for \( \xi \), and \( y \) for \( q \), the decomposition of effects in Table 1 would match those in Fox (1980), except for a slight difference in notation.
### TABLE 1
Direct, Indirect, and Total Effects of $\xi$ and $\eta$ on $\eta$, $y$, and $x$

<table>
<thead>
<tr>
<th>Effects on:</th>
<th>$\eta$</th>
<th>$y$</th>
<th>$x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Effects of $\xi$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Direct</td>
<td>$\Gamma$</td>
<td>0</td>
<td>$\Lambda_x$</td>
</tr>
<tr>
<td>Indirect</td>
<td>$(I - B)^{-1} \Gamma - \Gamma$</td>
<td>$\Lambda_y(I - B)^{-1} \Gamma$</td>
<td>0</td>
</tr>
<tr>
<td>Total</td>
<td>$(I - B)^{-1} \Gamma$</td>
<td>$\Lambda_y(I - B)^{-1} \Gamma$</td>
<td>$\Lambda_x$</td>
</tr>
<tr>
<td>Effects of $\eta$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Direct</td>
<td>$B$</td>
<td>$\Delta_y$</td>
<td>0</td>
</tr>
<tr>
<td>Indirect</td>
<td>$(I - B)^{-1} - I - B$</td>
<td>$\Lambda_y(I - B)^{-1} - \Lambda_y$</td>
<td>0</td>
</tr>
<tr>
<td>Total</td>
<td>$(I - B)^{-1} - I$</td>
<td>$\Lambda_y(I - B)^{-1}$</td>
<td>0</td>
</tr>
</tbody>
</table>

**SPECIFIC EFFECTS**

The indirect effects comprise all the indirect paths from one variable to another. Consequently, the contribution of particular mediating variables can be obscured. Alwin and Hauser (1975), Greene (1977), and Fox (1985) have proposed strategies to analyze *specific indirect effects*, i.e., those effects transmitted by a particular variable or group of variables. Their conceptions of specific indirect effects differ. In this section, I contrast their definitions of these effects and their procedures for calculating them.

Alwin and Hauser (1975, p. 42) define specific indirect effects as those influences mediated by the variable in question and *those after it*. They exclude from the definition those paths running through endogenous variables that precede the variable of interest. Consider Figure 3 (also used by Alwin and Hauser), which depicts an exactly identified recursive model with three exogenous variables ($x_1, x_2, x_3$) and three endogenous variables ($y_1, y_2, y_3$). According to Alwin and Hauser's definition, the specific indirect effect of $x_1$ on $y_3$ mediated by $y_2$ in this model is the compound path $x_1 \rightarrow y_2 \rightarrow y_3$. They exclude the indirect path operating through $y_1$ and $y_2$ (i.e., $x_1 \rightarrow y_1 \rightarrow y_2 \rightarrow y_3$). The specific indirect effects of $x_1$ on $y_3$ through $y_1$ appear in two compound paths: $x_1 \rightarrow y_1 \rightarrow y_3$ and $x_1 \rightarrow y_1 \rightarrow y_2 \rightarrow y_3$. The second path through $y_2$ is included as part of the specific indirect effect of $y_1$ because in the causal sequence, $y_2$ comes after $y_1$. Alwin and Hauser
conceptualize specific indirect effects as increments in the causal sequence of variables in a model. I call these indirect effects *incremental specific effects*.

Incremental specific effects are restrictive, i.e., they do not include compound paths operating through prior endogenous variables. Greene's (1977) idea of specific effects is even more restrictive. It includes as specific effects only those influences mediated by the intervening variable or variables of interest. For instance, according to his definition, the specific effect of $x_1$ on $y_3$ operating through $y_1$ is the compound path $x_1 \rightarrow y_1 \rightarrow y_3$ (see Figure 3). This differs from the incremental specific effects for $y_1$ because it excludes the path going through $y_2$ (i.e., $x_1 \rightarrow y_1 \rightarrow y_2 \rightarrow y_3$). Greene's notion of specific indirect effects focuses on those effects that operate through $y_1$ but no other endogenous variable. Alwin and Hauser's concept of incremental specific effects includes all compound paths subsequent to the endogenous variable of interest. I call Greene's specific indirect effect *exclusive specific effects*.

---

Greene's (pp. 377–78) emphasis is on presenting a computational method for computing specific effects for recursive systems. He does not give a detailed definition of specific effects nor does he explain how his use of the term differs from Alwin and Hauser's. Greene's computation method suggests a particular definition of specific effects, which is what I discuss.
Fox's (1985) definition of specific effects is more inclusive than Greene’s or Alwin and Hauser’s. Fox defines a specific indirect effect as all compound paths that traverse a particular intervening variable. For instance, Fox’s definition implies that the specific effect of $x_1$ on $y_3$ through $y_2$ consists of two compound paths: $x_1 \rightarrow y_1 \rightarrow y_2 \rightarrow y_3$ and $x_1 \rightarrow y_2 \rightarrow y_3$ (see Figure 3). The incremental and the exclusive specific effects in this case include only a single path: $x_1 \rightarrow y_2 \rightarrow y_3$. According to Fox’s definition, the specific effect of $x_1$ on $y_3$ by way of $y_1$ incorporates $x_1 \rightarrow y_1 \rightarrow y_3$ and $x_1 \rightarrow y_1 \rightarrow y_2 \rightarrow y_3$. The incremental specific effect is identical to this, but the exclusive specific effect is not. The latter is composed of a single path: $x_1 \rightarrow y_1 \rightarrow y_3$. Fox includes as specific indirect effects all paths coming into or going out of a particular variable. I call these inclusive specific effects.

Each definition of specific indirect effects is tied closely to a particular method for decomposing effects. Alwin and Hauser (1975, p. 42) summarize their procedure as follows:6

For each endogenous (dependent) variable in the model, obtain the successive reduced-form equations, beginning with that containing only exogenous (predetermined) variables, then adding intervening variables in sequence from cause to effect. The total effect of a variable is its coefficient in the first reduced-form equation in which it appears as a regressor. . . . Indirect components of effects are given by differences between coefficients of a causal variable in two equations in the sequence, where the mediating variable (or variables) is that which appears as regressor in one equation and not in the other. However, it must be understood that indirect effects so computed will include those mediated by the intervening variable(s) in question and later variables.

Greene’s procedure, unlike Alwin and Hauser’s, requires matrix manipulations. The first step is to form a square matrix with a row to represent each variable and a column to do the same. If one variable

6 Their procedure is designed for exactly identified, recursive models.
affects another, its coefficient is listed in its row and in the column of the variable it affects. To determine an exclusive specific effect, the rows and columns corresponding to endogenous variables other than those of interest are first removed, creating a new and smaller square matrix. This new matrix is premultiplied by itself again and again until the resulting matrix is null. When each matrix product, beginning with the first, is summed, the end result is a matrix of the exclusive specific effects.

To see how this procedure works, consider the matrix representation of Figure 3:

\[
\begin{bmatrix}
Y_1 \\
Y_2 \\
Y_3
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 \\
\beta_1 & 0 & 0 \\
\beta_2 & \beta_3 & 0
\end{bmatrix} \begin{bmatrix}
Y_1 \\
Y_2 \\
Y_3
\end{bmatrix} + \begin{bmatrix}
\gamma_1 & \gamma_2 & \gamma_3 \\
\gamma_4 & \gamma_5 & \gamma_6 \\
\gamma_7 & \gamma_8 & \gamma_9
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} + \begin{bmatrix}
\delta_1 \\
\delta_2 \\
\delta_3
\end{bmatrix}. \tag{38}
\]

Following Greene's (1977) procedure, we form the matrix \( M \), which contains all the structural coefficients:

\[
M = \begin{bmatrix}
0 & 0 & 0 & \gamma_1 & \gamma_2 & \gamma_3 \\
0 & 0 & 0 & \gamma_4 & \gamma_5 & \gamma_6 \\
0 & 0 & 0 & \gamma_7 & \gamma_8 & \gamma_9 \\
0 & 0 & 0 & \beta_1 & \beta_2 & 0 \\
0 & 0 & 0 & 0 & \beta_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
0 & \Gamma' \\
0 & \mathbf{B}'
\end{bmatrix}. \tag{39}
\]

Several characteristics of \( M \) should be noted. First, the ordering of the row variables from first to last is \( x_1, x_2, x_3, y_1, y_2, \) and \( y_3 \). The columns are arranged in the same order from left to right. The influences of one variable on other variables appear as the coefficients in that variable's row. For instance, the effect of \( x_2 \) on \( y_2 \) is \( \gamma_5 \). Second, the matrix \( M \) will always have a \( (p + q) \times q \) zero submatrix, where \( p \) is the number of \( y \) variables and \( q \) is the number of \( x \) variables. In (39), the zero submatrix is \( 6 \times 3 \). The zero submatrix is created by the assumption that the exogenous variables do not directly affect one another and that the endogenous variables do not affect the exogenous variables. Third, the upper right-hand corner of (39) is \( \Gamma' \), and the lower right-hand corner is \( \mathbf{B}' \).

To find the specific indirect effects, Greene (1977) deletes row(s) and column(s) in \( M \) that correspond to all endogenous variables except
those whose exclusive specific effects he wants to estimate. He also keeps the rows and columns of the exogenous variables. To determine the exclusive specific effect of $x_1$ on $y_3$ mediated by $y_1$, he removes the row and column corresponding to $y_2$, the only other intervening variable in the model. Thus, he forms $M^*$ by removing the fifth row and fifth column from $M$:

$$M^* = \begin{bmatrix}
0 & 0 & 0 & \gamma_1 & \gamma_7 \\
0 & 0 & 0 & \gamma_2 & \gamma_8 \\
0 & 0 & 0 & \gamma_3 & \gamma_9 \\
0 & 0 & 0 & 0 & \beta_2 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}. \quad (40)$$

Next, he premultiplies $M^*$ by itself, yielding

$$M^{*2} = \begin{bmatrix}
0 & 0 & 0 & 0 & \gamma_1\beta_2 \\
0 & 0 & 0 & 0 & \gamma_2\beta_2 \\
0 & 0 & 0 & 0 & \gamma_3\beta_2 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}. \quad (41)$$

As is easily verified, further premultiplication will result in a matrix of zeros. Therefore, the only nonzero matrix in the sum of product matrices is (41). Thus, the element in the first row and last column of $M^{*2}$ gives the exclusive specific effect of $x_1$ on $y_3$ through $y_1$. In this case, this effect equals $\gamma_1\beta_2$.

In recursive systems with even a moderate number of endogenous variables, Greene's procedure may generate a number of terms to multiply and sum. Furthermore, his discussion and examples of exclusive specific indirect effects do not treat nonrecursive systems. Recently, Fox (1985) proposed a procedure for deriving inclusive specific indirect effects that is computationally similar to Greene's method but avoids the summing of a power series of matrices. It is also more general than either Greene's or Alwin and Hauser's methods. Fox assumes a classical econometric model in which $x = \xi$ and $y = \eta$. Computations start with $T_{yx}$, the total effects of $x$ on $y$, formed as $(I - B)^{-1}\Gamma$. From $T_{yx}$ he deletes the row(s) of the variable(s) whose specific effects he wishes to analyze. Next, he removes the row(s) and column(s) of $B$ and the row(s) of $\Gamma$ corresponding to the variable(s) of interest, leading to $\Gamma^*$.
and $\mathbf{B}^*$, where the "*" indicates the matrices with the proper row(s) or column(s) absent. Then, he forms a new total effects matrix, $(\mathbf{I}^* - \mathbf{B}^*)^{-1} \Gamma^*$. He subtracts the $(\mathbf{I}^* - \mathbf{B}^*)^{-1} \Gamma^*$ matrix from the original $\mathbf{T}_{yx}$ matrix with the appropriate row(s) removed to determine inclusive specific effects through the variable(s) being analyzed. He follows a similar series of steps to find the inclusive specific effects of $y$ on $y$.

A generalization of specific effects. The procedures reviewed above share one limitation: They are oriented toward variables rather than paths. Suppose we want to know that part of the indirect or total effects transmitted by a path or combination of individual paths. The preceding methods would be helpful only if these paths happened to correspond to incremental, exclusive, or inclusive specific indirect effects. This need not be the case, of course. In this subsection, I explain a technique that is coefficient (or path) oriented. Since the coefficient or path is a more elementary unit than the variable, this procedure is more general than the existing procedures. By choosing appropriate combinations of paths, we can determine the specific indirect effects of variables. In the next few pages, I use this idea to show that the specific indirect effects proposed by Alwin and Hauser, Greene, and Fox are special cases of effects estimated with this alternative method. Then, I apply this method to a type of specific effect that falls outside the scope of these other procedures. Unlike Alwin and Hauser's procedures for estimating specific indirect effects, this method is not restricted to exactly identified recursive models nor to incremental specific effects. Like Greene's and Fox's procedures, it uses matrix operations; but unlike their procedures, it does not require row and column removal.

The method has several steps. The last is optional.

1. Identify the changes needed in the coefficient matrices.
2. Modify the coefficient matrices.
3. If $\mathbf{B}$ is changed, check the modulus or absolute value of the largest eigenvalue of the new $\mathbf{B}$ to insure that it is less than one.
4. Calculate the direct, indirect, or total effects with the modified matrices.
5. Subtract the new decompositions from the old.

Fox also shows how Alwin and Hauser's incremental specific effects can be computed with his procedure.
I demonstrate this procedure first for calculating inclusive specific indirect effects as defined by Fox (1985). Suppose that we want to know the inclusive specific indirect effects of \( x \) on \( y \) through \( y_1 \) (see Figure 3). These include all paths that traverse \( y_1 \). The standard indirect effects with the original coefficient matrices provide the effects through \( y_1 \) and through the other variables in the model. If we can find the decomposition resulting if paths through \( y_1 \) are eliminated, we will know the decomposition of effects due not to \( y_1 \) but only to the remaining variables. Subtracting the second from the first gives only those specific indirect effects through \( y_1 \), the quantity desired. In Figure 3, the original indirect effects of \( x \) on \( y \) were \((I - B)^{-1} \Gamma - \Gamma\). To remove the influence of \( y_1 \), all paths coming into or leaving \( y_1 \) are set to zero. Thus, the \((2,1)\) and \((3,1)\) elements of \( B \) and the \((1,1), (1,2), \) and \((1,3)\) elements of \( \Gamma \) are set to zero (see equation [38] for the original \( B \) and \( \Gamma \)). The altered \( B \) and \( \Gamma \) matrices are represented as \( B_{(l_1)} \) and \( \Gamma_{(l_2)} \):

\[
B_{(l_1)} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & \beta_3 & 0
\end{bmatrix}
\]

\[
\Gamma_{(l_2)} = \begin{bmatrix}
0 & 0 & 0 \\
\gamma_4 & \gamma_5 & \gamma_6 \\
\gamma_7 & \gamma_8 & \gamma_9
\end{bmatrix},
\]

where \( l_1 = \{(., 1) = 0\} \) and \( l_2 = \{(1,.) = 0\} \). The \((l_1)\) subscript indicates a set of modifications that are given in the statement defining \( l_1 \). In (42), the first column of \( B \) is set to zero, which is represented by \((.,1) = 0\). Similarly, the first row of \( \Gamma \) is set to zero, and this is symbolized as \( \Gamma_{(l_2)} \), with \( l_2 \) representing \((1,.) = 0\). In this notation, the “.” in the row position indicates that all rows in the stated column are zero. The “.” in the column position defines all columns in the stated row as zero.

Before calculating indirect effects with \( B_{(l_1)} \), we need to check whether it satisfies the convergence condition presented earlier. That is, the modulus or absolute value of the largest eigenvalue must be less than one. In general this is done because \( B_{(l_1)} \) can fail this condition even if \( B \) does not (Fisher 1970; Sobel 1986). In this example, \( B \) and \( B_{(l_1)} \) are lower triangular and the condition is satisfied.
The indirect effects $I_{yx}$ based on the original $B$ and $\Gamma$ (see [38]) are

$$(I - B)^{-1} \Gamma - \Gamma = \begin{bmatrix}
0 & 0 & 0 \\
\beta_1 \gamma_1 & \beta_1 \gamma_2 & \beta_1 \gamma_3 \\
(\beta_2 + \beta_1 \beta_3) \gamma_1 + \beta_3 \gamma_4 & (\beta_2 + \beta_1 \beta_3) \gamma_2 + \beta_3 \gamma_5 & (\beta_2 + \beta_1 \beta_3) \gamma_3 + \beta_3 \gamma_6
\end{bmatrix}.$$  \hspace{1cm} (43)

The indirect effects based on the $B_{(i)}$ and $\Gamma_{(i)}$ of (42), in which those paths into or out of $y_i$ have been eliminated, are

$$(I - B_{(i)})^{-1} \Gamma_{(i)} - \Gamma_{(i)} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
\beta_3 \gamma_4 & \beta_3 \gamma_5 & \beta_3 \gamma_6
\end{bmatrix}. \hspace{1cm} (44)$$

Equation (44) reveals that the indirect effects of $x$ on $y$ that remain after those passing through $y_i$ have been eliminated are limited to indirect effects of $x$ on $y_3$ operating through $y_2$. Subtracting (44) from (43) produces the inclusive specific effects of $y_1$:

$$\begin{bmatrix}
0 & 0 & 0 \\
\beta_1 \gamma_1 & \beta_1 \gamma_2 & \beta_1 \gamma_3 \\
(\beta_2 + \beta_1 \beta_3) \gamma_1 & (\beta_2 + \beta_1 \beta_3) \gamma_2 & (\beta_2 + \beta_1 \beta_3) \gamma_3
\end{bmatrix}. \hspace{1cm} (45)$$

These results match those obtained using Fox’s (1985) procedure. In general, to calculate inclusive specific effects, we first form the indirect effects with the original matrices. Second, we alter the $B$ and $\Gamma$ matrices so that all paths leading to or from the variable or variables of interest are set to zero. Third, assuming that the new $B$ meets the convergence condition, we recalculate the indirect effects with these modified matrices. Finally, we subtract the modified indirect effects from the original effects to obtain the inclusive specific effects.

Greene (1977) limited his discussion of exclusive specific effects to recursive models with observed variables. The alternative procedure I have proposed can handle exclusive specific effects for these types of models and for nonrecursive or latent variable models. The first step is to determine the changes needed in the coefficient matrices for exclusive specific effects. Greene’s specific effects include only those paths through a particular endogenous variable. Excluded are paths to the
selected endogenous variable from the other endogenous variables. Also excluded are the paths from the exogenous variables to the endogenous variables not of interest. All columns of B are set to zero except those representing the influence of the endogenous variable(s) whose exclusive specific effects are to be calculated. This modification forces to zero the direct effects of the excluded endogenous variables. For Γ, all rows are set to zero except those corresponding to the effects of the exogenous variables on the endogenous variable(s) selected. The indirect effects are formed using these modified matrices. The equation for indirect effects is computed by substituting the modified matrices for the original matrices. This creates exclusive specific effects. An example will help clarify this procedure.

In Figure 3, suppose that we want to know the exclusive specific indirect effects of \( y_1 \). The modified B and Γ are

\[
B(\text{i}_1) = \begin{bmatrix}
0 & 0 & 0 \\
\beta_1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad \Gamma(\text{i}_2) = \begin{bmatrix}
y_1 & y_2 & y_3 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad (46)
\]

where \( \text{i}_1 = \{(.), 2) = 0, (.), 3) = 0 \} \) and \( \text{i}_2 = \{(2,.), 0, (3,.), 0 \} \). The only nonzero elements in \( B(\text{i}_1) \) appear in the first column and represent the direct effects of \( y_1 \) on \( y_2 \) and \( y_3 \). The nonzero elements in \( \Gamma(\text{i}_2) \) denote effects of the exogenous variables on \( y_1 \). The modified versions of the coefficient matrices replace the original matrices in the indirect effects formula, yielding

\[
(\mathbf{I} - B(\text{i}_1))^{-1}\Gamma(\text{i}_2) - \Gamma(\text{i}_2) = \begin{bmatrix}
0 & 0 & 0 \\
\beta_1 y_1 & \beta_1 y_2 & \beta_1 y_3 \\
\beta_2 y_1 & \beta_2 y_2 & \beta_2 y_3
\end{bmatrix}. \quad (47)
\]

As the first row of (47) shows, \( x \) has no exclusive specific indirect effects on \( y_1 \); all its effects are direct. The second row contains the specific effects of \( x \) on \( y_2 \) exclusively through \( y_1 \), and the third row contains the exclusive specific effects of \( x \) on \( y_3 \). The procedure illustrated above can be applied to other types of models and can be used to find the exclusive specific effects of more than one endogenous variable at a time.\(^8\)

\(^8\)Alwin and Hauser’s (1975) incremental specific effects can also be calculated with this method. However, these effects are not clearly defined when a nonrecursive model is analyzed because with a feedback or reciprocal relationship, it is unclear which variable precedes or follows in the causal hierarchy.
In addition to computing exclusive, inclusive, and incremental specific effects, this procedure allows us to compute new types of effects. For instance, we determine the specific effects that result from any individual path, or group of paths, by first setting these paths to zero in the coefficient matrices. Then, we compute the decompositions with the modified matrices. Finally, we subtract the results from those obtained with the original matrices. The difference is attributable to the specific path(s) identified. I return to Figure 2 to illustrate this procedure.

Suppose that we want to estimate that part of the total effects of \( \xi \) on \( \eta \) that is transmitted through the path from \( \eta_2 \) to \( \eta_1 \). All effects operating through this link would be eliminated if \( \beta_2 \) in \( B \) were set to zero. Since we are interested only in the direct effect of \( \eta_2 \) on \( \eta_1 \), only the \( B \) matrix requires modification:

\[
B_{(l)} = \begin{bmatrix} 0 & 0 \\ \beta_1 & 0 \end{bmatrix},
\]

where \( l = \{ \beta_2 = 0 \} \). Next, we check the convergence condition for \( B_{(l)} \). Since \( B_{(l)} \) is a lower triangular matrix, the condition is satisfied. The next step is to substitute (48) for \( B \) in the total effects formula for \( \xi \) on \( \eta \):

\[
(I - B)^{-1} \Gamma = \begin{bmatrix} 1 & 0 \\ \beta_1 & 1 \end{bmatrix} \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix} = \begin{bmatrix} \gamma_1 & 0 \\ \beta_1 \gamma_1 & \gamma_2 \end{bmatrix}.
\]

The total effects of \( \xi \) on \( \eta \) are

\[
(I - B)^{-1} \Gamma = (1 - \beta_1 \beta_2)^{-1} \begin{bmatrix} \gamma_1 & \beta_2 \gamma_2 \\ \beta_1 \gamma_1 & \gamma_2 \end{bmatrix}.
\]

When equation (49) is subtracted from (50), we get that part of the total effects that is attributable to the direct influence of \( \eta_2 \) on \( \eta_1 \):

\[
(I - B)^{-1} \Gamma - (I - B_{(l)})^{-1} \Gamma = (1 - \beta_1 \beta_2)^{-1} \begin{bmatrix} \beta_1 \beta_2 \gamma_1 & \beta_2 \gamma_2 \\ \beta_1 \beta_2 \gamma_1 & \beta_1 \beta_2 \gamma_2 \end{bmatrix}.
\]

Equation (51) shows, for instance, that \( \beta_1 \beta_2 \gamma_1/(1 - \beta_1 \beta_2) \) of the total effects of \( \xi_1 \) on \( \eta_1 \) results from the effect of \( \eta_2 \) on \( \eta_1 \).
The decompositions I have discussed can be calculated in several ways. The formulas in Table 1 can be programmed using software that has matrix capabilities. For instance, APL, GAUSS, the PROC MATRIX procedure in SAS, or SAS/IML can perform the required matrix algebra. The alternative is to use LISREL V or VI. Each of these versions calculates the estimated total effects of $\xi$ and $\eta$. The estimated direct effects are the structural coefficients in $\hat{\Lambda}_x$, $\hat{\Lambda}_y$, $\hat{\Gamma}$, and $\hat{\mathbf{B}}$. The indirect effects can be obtained by subtracting the direct effects from the total effects. Finally, for simple models, the decomposition of effects can be calculated with a hand calculator when the structural coefficient estimates are available. The more complicated the model, the less practical is this option.

**ASYMPTOTIC VARIANCES OF EFFECTS**

So far, I have limited my discussion to methods of decomposing various types of effects. Once these effects are in hand, the question of their statistical significance arises. This section briefly describes the means of obtaining estimated asymptotic variances and tests of significance for all types of effects.

The asymptotic variances of the direct effects are readily available for maximum likelihood (ML) estimators when $y$ and $x$ have multinormal distribution (Jöreskog 1978, p. 447) and for generalized least squares (GLS) estimators under suitable conditions (Browne 1984). Significance testing of the indirect, total, and specific effects is more complicated, since it typically involves obtaining the variances of products of coefficient estimates. For instance, we can estimate the asymptotic variance of an ML estimator of a direct effect (e.g., $\hat{\beta}_1$), but the asymptotic variance of an indirect effect (e.g., $\hat{\beta}_1 \hat{\gamma}_1$) is less straightforward. The multivariate delta method (Bishop, Fienberg, and Holland 1975, pp. 486–500; Rao 1973, pp. 385–89) proves helpful in this situation. The delta method begins with the assumption that a parameter estimator has an asymptotically normal distribution with a mean of the parameter and an asymptotic covariance matrix. It then provides a method of estimating the asymptotic covariance matrix of functions of the parameter.

Folmer (1981, pp. 1440–42) and Sobel (1982, 1986) suggested applying the delta method to estimate the asymptotic variances of
total, indirect, and other types of effects. The procedure is as follows. First, define $\theta$ as an $s$-dimensional vector of the unknown elements in $B$, $\Gamma$, and $\Lambda_j$, and define $\hat{\theta}_N$ as the corresponding sample estimator of $\theta$ for a sample of size $N$. Choose an estimator so that $\hat{\theta}_N$ has an asymptotically normal distribution with a mean of $\theta$ and an asymptotic covariance matrix, $N^{-1}V(\theta)$. Under appropriate assumptions, the ML and GLS estimators of $\theta$ satisfy these conditions (Jöreskog 1978, p. 447; Browne 1984).

Next, define an $r$-dimensional vector $f(\theta)$ that is a differentiable function of $\theta$. In this case, $f(\theta)$ contains the indirect (or total) effects that are functions of the direct effects. Under these conditions, the multivariate delta method states that the asymptotic distribution of $f(\hat{\theta}_N)$ is normal, with a mean of $f(\theta)$ and an asymptotic covariance matrix of $N^{-1} \left( \frac{\partial f}{\partial \theta} \right)' V(\theta) \left( \frac{\partial f}{\partial \theta} \right)$. The first row of $\frac{\partial f}{\partial \theta}$ is $\frac{\partial f_1}{\partial \theta_1}$, $\frac{\partial f_2}{\partial \theta_1}$, ..., $\frac{\partial f_r}{\partial \theta_1}$, where $f_i$ is the $i$th element of $f(\theta)$. The second row is $\frac{\partial f_1}{\partial \theta_2}$, $\frac{\partial f_2}{\partial \theta_2}$, ..., $\frac{\partial f_r}{\partial \theta_2}$, and so on, so that $\frac{\partial f}{\partial \theta}$ is an $s \times r$ matrix. For large samples, $\hat{\theta}_N$ is substituted for $\theta$ to obtain an estimate of the asymptotic covariance matrix for $f(\hat{\theta}_N)$:

$$N^{-1} \left[ \left( \frac{\partial f}{\partial \hat{\theta}_N} \right)' V(\hat{\theta}_N) \left( \frac{\partial f}{\partial \hat{\theta}_N} \right) \right]. \quad (52)$$

To illustrate this procedure, consider the simple casual chain model:

$$\eta_1 = \gamma_1 \xi_1 + \zeta_1,$$

$$\eta_2 = \beta_1 \eta_1 + \xi_2. \quad (53)$$

In (53), assume that $\zeta_1$ and $\xi_2$ are uncorrelated with each other and with $\xi_1$, that $E(\xi_i)$ is zero, and that $\eta_1$, $\eta_2$, and $\xi_1$ are deviated from their means. Define $\gamma_1$, $\beta_1$, the indirect effect of $\xi_1$ on $\eta_2$, as the single element of $f(\theta)$, with $\theta$ containing $\gamma_1$ and $\beta_1$. The $\frac{\partial f}{\partial \theta}$ is $[ \beta_1 \gamma_1 ]'$.

---

9 In addition to requiring a consistent and asymptotically normal estimator of $\theta$, $f(\theta)$ must be continuously differentiable with respect to $\theta$ in the neighborhood of the true parameter, and I assume the continuity of $N^{-1}V(\theta)$ with respect to $\theta$ in the neighborhood of the true parameter. See Bishop et al. (1975) for additional discussion of these conditions.
and the asymptotic covariance matrix of the estimator of $\theta$ is

$$N^{-1} \begin{bmatrix} V(\gamma_1) & 0 \\ 0 & V(\beta_1) \end{bmatrix}. \quad (54)$$

The main diagonal of (54) contains the asymptotic variances of $\hat{\gamma}_{1N}$ and $\hat{\beta}_{1N}$. The off-diagonal elements are zero, since these two coefficients are uncorrelated in a recursive system. When these matrices are combined using the multivariate delta method, the asymptotic variance of $\hat{\gamma}_1 \hat{\beta}_1$ is the scalar:

$$N^{-1} \left[ \beta_1^2 V(\gamma_1) + \gamma_1^2 V(\beta_1) \right]. \quad (55)$$

If $\beta_1$ and $\gamma_1$ are zero, the delta method cannot be applied. Otherwise, by substituting the sample estimates into (55), we can obtain an estimate of the asymptotic variance of $\hat{\gamma}_{1N} \hat{\beta}_{1N}$ for large samples.

Considering each element of the indirect (or total) effects in the above fashion for more complicated models is extremely tedious. Sobel (1986) proposes a matrix formulation that is a far more efficient means of finding $\partial f(\theta)/\partial \theta$, which is required for the covariance matrix of $f(\theta)$. In the appendix, I list the $\partial f(\theta)/\partial \theta$ for the indirect effects and use them to estimate the standard errors of effects reported in the next section.

**EMPIRICAL ILLUSTRATION**

To illustrate the preceding methods, I utilize a model relating objective and subjective components of stratification for individuals. Figure 4 depicts this model. Define $x_1$ as income, $x_2$ as occupational prestige, $\eta_1$ as subjective income, $\eta_2$ as subjective occupational prestige, and $\eta_3$ as subjective overall class ranking. Let $y_1$, $y_2$, and $y_3$ be measures of the $\eta_1$, $\eta_2$, and $\eta_3$ latent variables. In the model, subjective income ($\eta_1$) and subjective occupational prestige ($\eta_2$) are dependent upon their objective counterparts and are reciprocally related to each other. In turn, the subjective income and occupation variables ($\eta_1$ and $\eta_2$) have direct effects on overall subjective class ($\eta_3$). The data are taken from a white subsample of 432 individuals from Indianapolis, Indiana (Kluegal, Singleton, and Starnes 1977).
The ML estimates based on the covariance matrix of these five variables give

\[
\hat{B} = \begin{bmatrix}
0 & .288 & 0 \\
.330 & 0 & 0 \\
.399 & .535 & 0 \\
.139 & .201 & 0
\end{bmatrix}
\]  
\[
\hat{\Gamma} = \begin{bmatrix}
.101 & 0 \\
0 & .007 \\
0 & 0
\end{bmatrix}
\]  

(56)

The standard errors are in parentheses. All coefficient estimates carry the expected positive signs, and all are statistically significant (p < .05, one-tailed test), though \(\hat{\beta}_1\) is barely significant. The goodness of fit of the overall model is excellent (\(\chi^2 = 0.27, df = 1, \text{GFI} = 1.00, \text{AFGI} = .996\)).

The estimated indirect effects of \(\eta\) on \(\eta\) and \(x\) on \(\eta\) are

\[
\hat{I}_{\eta \eta} = \begin{bmatrix}
.105 & .030 & 0 \\
.035 & .105 & 0 \\
.237 & .183 & 0
\end{bmatrix}
\]  
\[
\hat{I}_{\xi \eta} = \begin{bmatrix}
.011 & .0022 \\
.006 & .0014 \\
.011 & .0004
\end{bmatrix}
\]  

(57)

\(\hat{\xi}_i\) is an identity matrix. Also, the estimate of the eigenvalues for \(\hat{B}\) are considerably less than one. Therefore, the convergence condition is met.
In the $\hat{I}_{\eta x}$ matrix, all the indirect effects of $x$ on $\eta$ except the indirect effect of occupational prestige ($x_2$) on subjective income ($\eta_1$) are statistically significant ($p < .05$, one-tailed test). In the $\hat{I}_{\eta \eta}$ matrix, in contrast, none of the indirect effects of $\eta$ on $\eta$ except the effect of subjective income ($\eta_1$) on subjective overall class ($\eta_3$) are statistically significant. This suggests that with one exception, the most significant effects of the subjective variables on one another are direct, not indirect. Statistically significant direct effects need not be accompanied by significant indirect effects.

To explain the largely nonsignificant indirect effects of $\eta$ on $\eta$, we can begin with $\hat{\beta}_1$, the coefficient of the path from subjective occupational prestige ($\eta_2$) to subjective income ($\eta_1$). Relative to $\hat{\beta}_2$, the influence of subjective income on prestige, $\hat{\beta}_1$ is somewhat weak. In substantive terms, people who perceive their income as high are likely to perceive their occupational prestige as higher than those who perceive their income as low. Yet, individuals with positions they perceive as prestigious are less likely to believe that their incomes are high. Since the $\beta_1$ path channels a portion of the indirect effects, this may partially explain the relatively small magnitude of $\hat{I}_{\eta \eta}$'s compared to its estimated asymptotic variance. I investigate this possibility with the generalization of specific-effects methodology that I proposed earlier. First, I estimate the indirect effects transmitted through the path from $\eta_2$ to $\eta_1$ (i.e., $\beta_1$). If the above ideas are correct, these new specific indirect effects from $\eta$ to $\eta$ should be nonsignificant. Second, I estimate the indirect effects that do not cross the $\beta_1$ path. If these are significant, the idea that the largely nonsignificant $\hat{I}_{\eta \eta}$ are due at least partially to the weak link from $\eta_2$ to $\eta_1$ would receive further support.

With $\hat{B}(\beta)$ defined so that $l = \{ \beta_1 = 0 \}$, the estimated indirect effects transmitted by the $\beta_1$ path are

$$\hat{I}_{\eta \eta}^* = \begin{bmatrix} .105 & .030 & 0 \\ (.067) & (.036) & \\ .035 & .105 & 0 \\ (.026) & (.067) & \\ .061 & .183 & 0 \\ (.044) & (.127) & \end{bmatrix} \quad \hat{I}_{\eta x}^* = \begin{bmatrix} .011 & .0022 \\ (.006) & (.0014) & \\ .003 & .0007 \\ (.002) & (.0004) & \\ .006 & .0012 \\ (.004) & (.0008) & \end{bmatrix}. \quad (58)$$

In (58), the "*" indicates that these are indirect effects sent over particular paths—in this case, for $\beta_1$. When we compare $\hat{I}_{\eta \eta}^*$ of (58) to
\( \hat{I}_{\eta \eta} \) of (57), we see that all the indirect effects of \( \eta \) on \( \eta \) except the indirect effects of subjective income (\( \eta_1 \)) on subjective overall class (\( \eta_3 \)) are transmitted through the \( \beta_1 \) path. The indirect effect of \( \eta_1 \) on \( \eta_3 \) passed through the \( \beta_1 \) path is not statistically significant, although as shown in (57), the total indirect effect of \( \eta_1 \) on \( \eta_3 \) is statistically significant.

The indirect effects of \( x \) on \( \eta \) through the \( \beta_1 \) path are even more revealing. As \( \hat{I}_{\eta x}^* \) shows, the only indirect effects through the \( \beta_1 \) path that are statistically significant are the effect of income (\( x_1 \)) on subjective income (\( \eta_1 \)) and the effect of occupational prestige (\( x_2 \)) on its subjective counterpart (\( \eta_2 \)). The remaining indirect effects of income and occupational prestige via the \( \beta_1 \) path are not significant. When \( \hat{I}_{\eta x}^* \) is contrasted with \( \hat{I}_{\eta x} \) in (57), the secondary role that the influence of subjective occupational prestige on subjective income plays in generating total indirect effects of \( x \) on \( \eta \) is highlighted.

Finally the indirect effects from \( \eta \) to \( \eta \) and \( x \) to \( \eta \) that do not cross the path from \( \eta_2 \) to \( \eta_1 \) (i.e., \( \beta_1 \)) are

\[
\hat{I}_{\eta \eta}^{**} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
.177 & 0 & 0 \\
(.089)
\end{bmatrix} \quad \hat{I}_{\eta x}^{**} = \begin{bmatrix}
0 & 0 \\
.033 & 0 \\
(.010) & (.0013) \\
.058 & .0036 \\
(.012) & (.0013)
\end{bmatrix}. \quad (59)
\]

The only indirect effect of \( \eta \) on \( \eta \) is the one from \( \eta_1 \) to \( \eta_3 \) through \( \eta_2 \). This effect is statistically significant. In \( \hat{I}_{\eta x}^{**} \), all the indirect effects of \( x \) on \( \eta \) not mediated by the path associated with \( \beta_1 \) are statistically significant. Thus, the indirect effects that cross the \( \beta_1 \) path tend to be nonsignificant, and the indirect effects that do not cross the path are significant.

The example illustrates that not all specific indirect effects that comprise the total indirect effects have the same statistical significance. Examining the statistical significance of only the total indirect effects can give a misleading portrayal of the significance of the individual indirect paths. In other cases, if all the components of an effect are not significantly different from zero, then the delta method is inappropriate. For instance, if \( \hat{\gamma}_1 \) and \( \hat{\beta}_1 \) from equation (53) are essentially zero, then the delta method should not be applied to find the standard error of \( \hat{\gamma}_1 \hat{\beta}_1 \).
A final point bears repeating. The delta method is designed for large-sample problems. The accuracy of the method for small samples is not known.

**SUMMARY**

Path analysis decomposition techniques have developed on a number of fronts, and the results are dispersed throughout the social science literature. In this paper, I have attempted to consolidate some of these findings. Drawing on earlier literature, I derived the total, indirect, and direct effects for the general LISREL model. I showed that to estimate total effects, whether defined as an infinite sum or as reduced-form coefficients, the convergence condition must be met. I reviewed the existing techniques for determining specific indirect effects, and I proposed a new procedure to estimate all types of specific effects. More generally, this procedure allows the researcher to trace the influence of any path or combination of paths in a model. Finally, using results from Folmer (1981) and Sobel (1982, 1986), I presented the standard errors for the various decompositions.

**APPENDIX**

To utilize the delta method for computing the asymptotic variance of the indirect effects, we must use the $\frac{\partial f(\theta)}{\partial \theta}$. To simplify the results for the indirect effects, I define $\theta$ to include only those unrestricted elements in $B$, $\Gamma$, and $\Lambda_j$. Each indirect effect is treated separately. Sobel (1986, pp. 170–72) lists the partial derivatives for the indirect effects as

$$
\frac{\partial \text{vec} I_{\eta}}{\partial \theta} = V_B' \left( (I - B)^{-1} \otimes \left( (I - B)^{-1} \right)' - I_m \otimes I_m \right)
$$  \hspace{1cm} (A1)

$$
\frac{\partial \text{vec} I_{\xi}}{\partial \theta} = V_B' \left( (I - B)^{-1} \Gamma \otimes \left( (I - B)^{-1} \right)' \right) + V_{\Gamma} \left( I_n \otimes \left( (I - B)^{-1} - I \right)' \right)
$$  \hspace{1cm} (A2)

$$
\frac{\partial \text{vec} I_{\gamma}}{\partial \theta} = V_{\Lambda_j} \left( \left( (I - B)^{-1} - I \right) \otimes I_p \right) + V_B' \left( (I - B)^{-1} \otimes \left( \Lambda_j (I - B)^{-1} \right)' \right)
$$  \hspace{1cm} (A3)
\[
\frac{\partial \text{vec} I_{Y_i}}{\partial \theta} = V_B' \left( (I - B)^{-1} \Gamma \otimes (\Lambda_j (I - B)^{-1})' \right)
+ V_{\Gamma}' \left( I_n \otimes (\Lambda_j (I - B)^{-1})' \right)
+ V_{\Lambda_j}' \left( (I - B)^{-1} \Gamma \otimes I_p \right)
\]

where

\[
V_B = \begin{bmatrix}
\text{vec} \frac{\partial B}{\partial \theta_1}, \text{vec} \frac{\partial B}{\partial \theta_2}, \ldots, \text{vec} \frac{\partial B}{\partial \theta_3}
\end{bmatrix},
\]

\[
V_{\Gamma} = \begin{bmatrix}
\text{vec} \frac{\partial \Gamma}{\partial \theta_1}, \text{vec} \frac{\partial \Gamma}{\partial \theta_2}, \ldots, \text{vec} \frac{\partial \Gamma}{\partial \theta_3}
\end{bmatrix},
\]

\[
V_{\Lambda_j} = \begin{bmatrix}
\text{vec} \frac{\partial \Lambda_j}{\partial \theta_1}, \text{vec} \frac{\partial \Lambda_j}{\partial \theta_2}, \ldots, \text{vec} \frac{\partial \Lambda_j}{\partial \theta_3}
\end{bmatrix},
\]

\[
\text{vec} = \text{vec} \text{ operator},
\]

\[\otimes = \text{Kronecker's product}.\]

For the empirical illustration in Figure 4, \(\theta = [\beta_1 \beta_2 \beta_3 \beta_4 \gamma_1 \gamma_2]'\). The \(V_B\) and \(V_{\Gamma}\) matrices for \(I_{\eta}\) and \(I_{\eta*}\) are

\[
V_B = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

(A5)

and

\[
V_{\Gamma} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

(A6)

These are used to estimate the standard errors for \(\hat{I}_{\eta}\) and \(\hat{I}_{\eta*}\). The standard errors of the specific effects were estimated with appropriate modifications.
REFERENCES


