Taming Multirelations

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Abstract

Binary multirelations generalise binary relations by associating elements of a set to its subsets. We study the structure and algebra of multirelations under the operations of union, intersection, sequential and parallel composition, as well as finite and infinite iteration. Starting from a set-theoretic investigation, we propose axiom systems for multirelations in contexts ranging from bi-monoids to bi-quantales.

1 Introduction

Multirelations generalise binary relations in that elements of a given set are not related to single elements of that set, but to its subsets. Hence a multirelation over a set $X$ is a relation of type $X \times 2^X$ and elements of a multirelation are pairs $(a, B)$ with $a \in X$ and $B \subseteq X$. Multirelations have found applications in program semantics and models or logics for games that require alternation [5], that is, two dual kinds of nondeterministic choice. The first one occurs within individual pairs $(a, B)$ when selecting an output $b \in B$ to an input $a$; the second one occurs between pairs $(a, B)$ and $(a, C)$ when selecting an output $B$ or $C$ to an input $a$.

Multirelations occur, in slightly generalised form, as transition relations of alternating automata [3, 5] with universal choices $(a, \sigma, B)$ forcing simultaneous moves from state $a$ along $\sigma$-labelled edges into all states in $B$, whereas existential choices between $(a, \sigma, B)$ and $(a, \sigma, C)$ for instance, allow the selection of a transition from $a$ into either $B$ or $C$ labelled with $\sigma$. Similarly, in two-player games, moves $(a, B)$ are made by an antagonist, and a protagonist must be prepared to play from each state in $B$; whereas a protagonist can select between moves $(a, B)$ and $(a, C)$ [32, 5, 23, 24, 26]. Otherwise, in multirelational semantics of computing systems [1, 4, 18, 29, 30] choices $(a, B)$ are controlled by the environment (the system must be correct under all of them), whereas choices between $(a, B)$ and $(a, C)$ are controlled by the system (the system must be correct under at least one of them). Moreover, in the semantics of concurrent dynamic logic [14, 20, 28, 27, 33], an element $(a, B)$ indicates an execution of a parallel program or component starting in state $a$ and terminating in the states within $B$, whereas $(a, B)$ and $(a, C)$ correspond to different executions. In these contexts, the first kind of choice is often called external, universal or demonic; the second one is known as internal, existential or angelic.

To reason about systems modelled by multirelations, modal logics such as game logics [23, 24] and concurrent dynamics logics [14, 20, 28, 27, 33] have been developed. Despite its obvious significance, however, the algebra of multirelations itself has so far received rather limited attention—except for the special case of up-closed multirelations [9, 18, 22, 29, 30], which are isomorphic to monotone predicate transformers [23]. The general algebra of multirelations, which forms the semantics of concurrent dynamics logics and captures alternation in its purest form, remains to be investigated in depth.

A first step towards taming multirelations has been an algebraic reconstruction of Peleg’s concurrent dynamic logic from a minimalist axiomatic basis [10, 12]. This article takes the next steps towards axiom systems for multirelations in the spirit of Tarski’s relation algebra (cf. [17]). Though there are many similarities, this is not straightforward: the sequential composition of multirelations, for instance, is more complex than its relational counterpart and not associative; the parallel composition of multirelations has no
relational counterpart; the relational converse makes no sense for multirelations. Thus alternative axioms, definitions and proofs for multirelational concepts are required.

Our main contributions are as follows. To obtain a basis for our axiom systems, we extend our previous investigation of multirelational properties by new interaction laws between sequential and parallel composition and an explicit definition of a multirelational domain operation. We also identify several sub-classes of multirelations—sequential and parallel subidentities, vectors or terminal multirelations, nonterminal multirelations—and mappings between these classes (cf. Figure 1 and 2). Based on these set-theoretic preliminaries we axiomatise multirelations at four different layers.

First we expand weak bi-monoids with operations for sequential and parallel composition by sequentiality-parallelism interaction axioms. We define a domain operation explicitly on these c-monoids and show that the domain elements form a sub-semilattice in which sequential and parallel composition coincide. We show that previous domain axioms for sequential monoids [6] and concurrent monoids [10] are derivable in this setting.

Second, we expand c-monoids to c-trioids by adding an operation of internal choice and providing another interaction axiom. The domain elements now form a distributive sublattice. All concurrent dynamic algebra axioms [25, 10] become derivable in the presence of a Kleene star and a few more simple multirelational properties.

Third, we study bounded distributive lattices with operations of sequential and parallel composition, to which we add different sequentiality-parallelism interaction axioms. The resulting c-lattices are also c-trioids, and a large number of laws can be derived in this setting. In particular, we prove that the algebras of subidentities and vectors are isomorphic (only sequential composition is usually not preserved) and characterise the subalgebras of these elements as well as that of nonterminal elements. In the latter, in fact, we find greater similarity to binary relations.

Finally, we consider notions of finite and infinite iteration over multirelations in an expansion of c-lattices to c-quantales. Due to the lack of distributivity and associativity laws in algebras of multirelations, our results are weaker than those for relations.

Because of the complexity of reasoning with multirelations and the task of minimising algebraic axiom systems, we have formalised all structures and proofs with the Isabelle/HOL theorem prover [21]. Our investigations are therefore also a study in formalised mathematics. Our Isabelle theories and a detailed proof document can be found online in the Archive of Formal Proofs [11] and consulted together with our article. Many proofs in this article are syntactic manipulations, which may be tedious, but carry little insight—consequently they are not displayed. Instead we provide human-readable Isabelle proofs whenever suitable and we have added pointers to the facts in this article to the Isabelle proof document. However we show some interesting proofs and counterexamples, as they are given, but not internally verified, by Isabelle.

The remainder of this article is organised as follows. Section 2 provides the basic definitions, operations and properties of multirelations. Section 3 introduces the sets of sequential subidentities, parallel subidentities and vectors and the isomorphisms between them. These are summarised in the three diagrams of Figure 1 and 2, on which much of the algebraic investigation is based. It also introduces some basic properties of nonterminal multirelations, which do not allow any pairs \((a, \emptyset)\) and provides the multirelational laws needed for algebraic soundness proofs in later sections. Section 4 studies c-monoids, our first axiom system for multirelations, Section 5 and 7 introduce c-trioids and c-lattices. Section 6 explains the relationship between c-trioids and concurrent dynamic algebras. Definitions of a domain operation in c-lattices are presented in Section 8. The subalgebras of subidentities and vectors and the associated isomorphisms are studied in Section 9 and 10. Functions separating terminal and nonterminal elements are defined in Section 11 and properties of these functions are presented. Notions of finite and infinite iterations for c-quantales are studied in Section 12 and 13. Section 14 presents some counterexamples; Section 15 sketches how up-closed multirelations arise in our setting. Finally, Section 16 presents a conclusion.
2 Basic Definitions and Properties

This section follows Peleg [28] in defining the operations of sequential and parallel composition on multirelations. Sequential composition is different from the one used for up-closed multirelations introduced by Parikh [23, 24]. We then outline a few set-theoretic properties of multirelations which are important for the algebraic developments in later sections, and we sketch some examples.

A multirelation \( R \) over a set \( X \) is a subset of \( X \times 2^X \). We write \( \mathcal{M}(X) \) for the set of all multirelations over \( X \). The sequential composition of \( R, S \in \mathcal{M}(X) \) is the multirelation

\[
R \cdot S = \{(a, A) \mid \exists B. (a, B) \in R \land \exists f. \forall b \in B. (b, f(b)) \in S) \land A = \bigcup f(B)\}.
\]

The unit of sequential composition is the multirelation

\[
1_\pi = \{(x, \{x\}) \mid x \in X\}.
\]

The parallel composition of \( R \) and \( S \) is the multirelation

\[
R \parallel S = \{(a, A \cup B) \mid (a, A) \in R \land (a, B) \in S\}.
\]

The unit of parallel composition is the multirelation

\[
1_\pi = \{(x, \emptyset) \mid x \in X\}.
\]

The universal multirelation over \( X \) is

\[
U = \{(a, A) \mid a \in X \land A \subseteq X\}.
\]

A pair \((a, B)\) is in \( R \cdot S \) if \( R \) relates \( a \) to some intermediate set \( C \) and \( S \) relates each element \( c \in C \) to a set \( B_c \) in such a way that \( B = \bigcup_{c \in C} B_c \). This can be motivated in various ways, as discussed by [10]. Peleg’s original intended interpretation, for instance, associates \( R \cdot S \) with the sequential composition of parallel programs or concurrent components, such that program \( R \) reaches the concurrent state in \( C \) from \( a \) in a parallel execution, after which program \( S \) reaches the concurrent state \( B \) in parallel executions from each state \( c \in C \), reaching concurrent sub-states \( B_c \) from each of these states. Alternatively, in the context of computation trees based on alternation or dual nondeterminism, component \( R \) makes an internal choice of moving to state \( C \), which resolves the external choices for transitions from state \( a \) and represents one level of the computation tree. After that, component \( S \) makes internal choices of moving from each state \( c \in C \) to a set \( B_c \), thus resolving its external choices. The states in \( \bigcup_{c \in C} B_c \) form the next level of the computation tree. This interpretation reflects, for instance, the construction of a run or computation tree of an alternating automaton whose transitions are modelled by a multirelation.

A pair \((a, B)\) is in \( R \parallel S \) if \( B \) can be decomposed with respect to sets \( C \) and \( D \) such that \((a, C) \in R \) and \((a, D) \in S \), that is, the parallel execution of \( R \) and \( S \) from \( a \) produces the global state \( B \). In other words, the external choices in \( R \parallel S \) arise as the union of the external choices in \( R \) and those in \( S \) from each particular state; whereas the internal choices are not combined. This is dual to \( R \cup S \), where the union of the internal choices of \( R \) and \( S \) is taken while the external choices are not combined.

A multirelation \( R \) is a sequential subidentity if \( R \subseteq 1_\pi \). The sequential subidentities form a boolean algebra with least element \( \emptyset \) and greatest element \( 1_\pi \). Join is \( \cup \) and meet is \( \cdot \). The complement of a subidentity \( R \) is formed by the set \( \{(a, \{a\}) \mid (a, \{a\}) \notin P\} \). A parallel subidentity is a multirelation \( R \subseteq 1_\pi \). We write

\[
\mathcal{I}(X) = \{R \in \mathcal{M}(X) \mid R \subseteq 1_\pi\}, \quad \mathcal{F}(X) = \{R \in \mathcal{M}(X) \mid R \subseteq 1_\pi\},
\]

for the set of all sequential and parallel subidentities of \( X \). The name \( \mathcal{F}(X) \) is justified by the fact that parallel subidentities can be identified with terminal multirelations; see Section [25].
We consider three more sets of multirelations. A multirelation $R$ is a vector if whenever $(a, A) \in R$ for some $A \subseteq X$, then $(a, A) \in A$ for all $A \subseteq X$. A multirelation $R$ is up-closed if $(a, A) \in R$ implies $(a, B) \in R$ for all $B \supseteq A$ and $B \subseteq X$. Finally, we define $T_\pi$ to be the complement of $1_\pi$ in $\mathcal{M}(X)$. We write

\[ Y(X) = \{ R \in \mathcal{M}(X) \mid \exists A \subseteq X. (a, A) \in R \Rightarrow \forall A \subseteq X. (a, A) \in R \}, \]

\[ U(X) = \{ R \in \mathcal{M}(X) \mid \forall A \subseteq X. B \subseteq X. ((a, A) \in R \land A \subseteq B) \Rightarrow (a, B) \in R \}, \]

\[ N(X) = \{ R \in \mathcal{M}(X) \mid R \subseteq \bar{T}_\pi \}. \]

The elements in $N(X)$ are called nonterminal multirelations; see again Section 3.

Multirelations form proto-dioids \[10\], which are defined in Section 5. At this stage it suffices to mention the following laws.

\[ R \cup (S \cup T) = (R \cup S) \cup T, \quad R \cup S = S \cup R, \quad R \cup \emptyset = R, \quad R \cup R = R, \]

\[ (R \cdot S) \cdot T \subseteq R \cdot (S \cdot T), \quad 1_\sigma \cdot R = R, \quad R \cdot 1_\sigma = R, \]

\[ R \cdot S \cup R \cdot T \subseteq R \cdot (S \cup T), \quad (R \cup S) \cdot T = R \cdot T \cup S \cdot T, \quad 0 \cdot R = R, \]

\[ R \Vert (S \Vert T) = (R \Vert S) \Vert T, \quad R \Vert S = S \Vert R, \quad 1_\pi \Vert R = R, \]

\[ R \Vert (S \cup T) = R \cdot S \cup R \cdot T, \quad 0 \Vert R = \emptyset, \]

\[ (R \Vert S) \cdot T \subseteq (R \cdot T) \Vert (S \cdot T). \]

Sequential composition is not associative and $R \cdot \emptyset$ is generally not $\emptyset$. More generally, pairs $(a, \emptyset) \in R$ persist in any sequential composition $R \cdot S$—whence the name terminal.

Sequential subidentities satisfy stronger properties. First of all $(R \cdot S) \cdot T = R \cdot (S \cdot T)$ if one of $R, S, T$ is in $Y(X)$. Second, $R \cdot (S \cup T) = R \cdot S \cup R \cdot T$ if $R \in \mathcal{I}(X)$.

The interaction between sequential and parallel composition is captured by the following properties, among others.

**Lemma 2.1.** Let $R \in \mathcal{I}(X)$ and $S \in \mathcal{I}(X)$. Then

1. $R \Vert R = R$,
2. $S \Vert S = S$,
3. $U \Vert U = U$,
4. $T_\pi \Vert T_\pi = T_\pi$.

The proofs follow immediately from the definitions.

**Lemma 2.2.** Let $R, S, T \in \mathcal{M}(X)$. Then

1. $(R \cdot 1_\pi) \Vert R = R$,
2. $T \Vert T \subseteq T \Rightarrow (R \Vert S) \cdot T = (R \cdot T) \Vert (S \cdot T)$,
3. $(R \Vert S) \cdot T = (R \cdot T) \Vert (S \cdot T)$, if $T \in \mathcal{I}(X) \cup \mathcal{I}(X) \cup \{ U, \bar{T}_\pi \}$,
4. $R \cdot (S \Vert T) \subseteq (R \cdot S) \Vert (R \cdot T)$,
5. $R \cdot (S \Vert T) = (R \cdot T) \Vert (R \cdot S)$, if $R \in \mathcal{I}(X)$,
6. $R \cdot (S \cdot T) = (R \cdot S) \cdot T$, if one of $R, S, T$ is in $\mathcal{I}(X) \cup \mathcal{I}(X)$,
7. $(R \cap S) \cdot T = R \cdot T \cap S \cdot T$, if $R, S \in \mathcal{I}(X)$.

We present one example proof to illustrate the style of reasoning with multirelations.
Proof. (Lemma 2.2(2)) $T\|T \subseteq T$ implies that

$$(a, B) \in T \land (a, C) \in T \Rightarrow (a, B \cup C) \in T$$

holds for all $a \in X$. Therefore

$$(a, A) \in (R \cdot T)\|(S \cdot T) \Rightarrow \exists B, C. A = B \cup C \land (\exists D. (a, D) \in R \land \exists f. (\forall d \in D. (d, f(d)) \in T) \land B = \bigcup f(D)) \land (\exists E. (a, E) \in S \land \exists g. (\forall e \in E. (e, g(e)) \in T) \land C = \bigcup g(E))$$

$$\Rightarrow \exists D, E. (a, D \cup E) \in R\|S \land \exists f, g. (\forall d \in D - E, e \in E - D, x \in D \cap E. (d, f(d)) \in T \land (e, g(e)) \in T) \land A = \bigcup f(D) \cup g(E)$$

$$\Rightarrow \exists D, E. (a, D \cup E) \in R\|S \land \exists h. (\forall b \in D \cup E. (b, h(b)) \in T) \land A = \bigcup h(D \cup E)$$

The third step uses the assumption $T\|T \subseteq T$; the fourth one uses the function

$$h(x) = \begin{cases} f(x) & \text{if } x \in D - E, \\ (f \cup g)(x) & \text{if } x \in D \cap E, \\ g(x) & \text{if } x \in E - D. \end{cases}$$

The converse inclusion has been proved in [10].

$\square$

### 3 Subalgebras and Isomorphisms

This section studies the relationship between the sets of sequential subidentities, parallel subidentities and vectors as well as the special case of nonterminal multirelations. We use these sets and their relationships to extract algebraic axioms for multirelations in later sections. Most of the properties outlined in this section are verified rigorously in the algebraic setting of later sections.

The constants $1_\sigma$, $1_\pi$, $\emptyset$, $U$ and $\bar{T}_\pi$ play an important role in our considerations. The first lemma of this section describes their action on multirelations.

**Lemma 3.1.** Let $R \in \mathcal{M}(X)$. Then

1. $R \cdot 1_\sigma = \{(a, \emptyset) \mid \exists B. (a, B) \in R\}$,
2. $R \cap 1_\sigma = R \cdot \emptyset = \{(a, \emptyset) \mid (a, \emptyset) \in R\}$,
3. $R \cap 1_\pi = \{(a, \{a\}) \mid (a, \{a\}) \in R\}$,
4. $R \cdot U = \{(a, A) \mid A = \emptyset \land (a, A) \in R\} \cup \{(a, A) \mid \exists B \neq \emptyset. (a, B) \in R\}$,
5. $R\|U = \{(a, A) \mid \exists B. (a, B) \in R \land B \subseteq A\}$. 


The proofs are straightforward. Intuitively, \( R \cdot 1_\pi \) replaces every pair \((a, A)\) ∈ \( R \) by \((a, \emptyset)\), overwriting \( A \) by \( \emptyset \), whereas \( R \cap 1_\pi \) and \( R \cdot \emptyset \) both project on the pairs \((a, \emptyset)\) ∈ \( R \). The multirelation \( R \cap 1_\sigma \) projects on the pairs \((a, \{a\})\) ∈ \( R \); and the product \( R \| U \) computes the up-closure of \( R \). Thus

\[
\mathcal{I}(X) = \{ R \in \mathcal{M}(X) \mid R \cap 1_\pi = R \},
\mathcal{J}(X) = \{ R \in \mathcal{M}(X) \mid R \cdot 1_\pi = R \},
\mathcal{V}(X) = \{ R \in \mathcal{M}(X) \mid (R \cdot 1_\pi)\| U = R \},
\mathcal{W}(X) = \{ R \in \mathcal{M}(X) \mid R\| U = R \},
\mathcal{N}(X) = \{ R \in \mathcal{M}(X) \mid R \cap 1_\pi = R \}.
\]

The functions \((\cap 1_\sigma) = \lambda x. x \cap 1_\sigma, (\cdot 1_\pi) = \lambda x. x \cdot 1_\pi, (\| U) = \lambda x. x\| U\), whose fixpoints determine the sets \( \mathcal{I}(X), \mathcal{J}(X) \) and \( \mathcal{V}(X) \), and the map \((\cdot U) = \lambda x. x \cdot U\) play an important structural role. When specialising their sources and targets to \( \mathcal{I}(X), \mathcal{V}(X) \) or \( \mathcal{J}(X) \) they serve as bijective pairs showing that these sets are isomorphic. More precisely, the maps in Figure 1 are isomorphisms, and we verify this fact in Proposition 10.1. Under the source and target restrictions indicated, each function in the diagram is bijective and pairs of functions between the same sets compose to identity maps of the appropriate type. The isomorphism between \( \mathcal{I}(X), \mathcal{J}(X) \) and \( \mathcal{V}(X) \) is well known in the setting of binary relations; the same maps are used for implementing it. The isomorphisms with \( \mathcal{J}(X) \) are particular to multirelations. More generally it can be shown that all triangles in this diagram commute.

The pair \((\| 1_\sigma) \circ (\cdot 1_\pi) = 1_{\mathcal{J}(X)}\) generalises to a map \( \lambda x. (x \cdot 1_\pi)\| 1_\sigma : \mathcal{M}(X) \rightarrow \mathcal{J}(X) \). Applied to a multirelation \( R \), the function \((\cdot 1_\pi)\) overwrites any \( A \) in a pair \((a, A)\) ∈ \( R \) by \( \emptyset \); the function \((\| 1_\sigma)\) further overwrites it by \( \{a\} \). In other words,

\[
(R \cdot 1_\pi)\| 1_\sigma = \{(a, \{a\}) \mid \exists B. (a, B) \in R\},
\]

which represents the domain of \( R \). We can thus define the domain of a multirelation explicitly as

\[
d(R) = (R \cdot 1_\pi)\| 1_\sigma,
\]

that is, the following diagram commutes.

```
M(X) \( (\cdot 1_\pi) \) \mathcal{J}(X) \\
\downarrow d \quad \downarrow (\| 1_\sigma) \\
\mathcal{J}(X)
```
Moreover, \( d(R) = R \cap 1_\sigma \) if \( R \in \mathcal{X}(X) \) and \( d(R) = R \| 1_\sigma \) if \( R \in \mathcal{F}(X) \), such that the maps \((\cap 1_\sigma)\) and \((\| 1_\sigma)\) in Figure 1 can be replaced by \( d \). The result is shown in the left-hand diagram of Figure 2. The right-hand diagram shows the situation restricted to nonterminal multirelations. The sets of sequential subidentities, parallel subidentities and vectors remain isomorphic, but in the bijections, \( \overline{T}_\pi \) replaces \( U \). In addition, vectors are now obtained by \((\overline{T}_\pi) : \mathcal{M}(X) \rightarrow \mathcal{Y}(X)\) similarly to binary relations.

The following properties can be justified from these diagrams; they are needed for verifying soundness of the algebraic axioms in later sections.

**Lemma 3.2.**

1. \((R \cdot 1_\sigma) \| 1_\sigma) : S = (R \cdot 1_\pi) \| S,

2. \(R \cdot 1_\pi \subseteq 1_\pi,

3. \(R \cdot 1_\pi \cup R \cdot 1_\pi = R \cdot U,

4. \(1_\pi \cap (R \cup 1_\pi) = R \cdot \emptyset,

5. \(((R \cap 1_\pi) \cdot 1_\pi) \| 1_\pi = R \cap 1_\sigma,

6. \(((R \cap 1_\pi) \cdot 1_\pi) \| 1_\sigma = 1_\sigma \cap (R \cap 1_\pi) \cdot 1_\pi,

7. \(((R \cap 1_\pi) \cdot 1_\pi) \| T_\pi = (R \cap 1_\pi) \cdot T_\pi.

Equation (1) states that \( d(R) \cdot S = (R \cdot 1_\pi) \| S \), generalising commutation of the diagrams \((U) \circ d = (\| U) \circ (1_\pi)\) and \((T_\pi) \circ d = (\| T_\pi) \circ (1_\pi)\) in Figure 2 to \((S) \circ d = (\| S) \circ (1_\pi)\). For \( S = 1_\sigma \), (1) specialises to the explicit domain definition \( d(R) = (R \cdot 1_\pi) \| 1_\sigma \). Equation (2) reflects the fact that \( R \cdot 1_\sigma \subseteq \mathcal{X}(X)\). Equations (3) and (4) arise from the fact that \( T_\pi \) and \( 1_\pi \) are complements. Equation (5) states that \( d(R) = R \) for all \( R \in \mathcal{X}(X) \), which is the isomorphism condition \((1_\sigma) \circ (1_\pi) = 1_{\mathcal{X}(X)}\). Equation (6) states that \( d(R) = 1_\sigma \cap R \cdot T_\pi \) for all \( R \in \mathcal{N}(X) \), that is, the diagram \( d = d = (\| T_\pi) \circ (1_\pi) \circ (1_\pi) \) commutes. This domain definition is analogous to the relational case; however it does not generalise to \( \mathcal{M}(X) \). Equation (7) states that \( d(R) \cdot T_\pi = R \cdot T_\pi \) for all \( R \in \mathcal{N}(X) \), that is, the diagram \((T_\pi) \circ (T_\pi) \circ (1_\pi) \) commutes. Again, this reflects a relational fact, which does not generalise to \( \mathcal{M}(X) \).

The domain of a binary relation can be defined explicitly as well: either as \( d(R) = 1 \cap R \cdot U \) or as \( d(R) = 1 \cap R \cdot R^- \), where \( 1 \) denotes the identity relation, \( U \) the universal relation and \( R^- \) the converse of \( R \). However, for \( R = \{(a, \emptyset)\} \), we have \( 1_\sigma \cap R \cdot U \emptyset \subseteq \{(a, \{a\}\} \} = d(R) \) and the converse of a multirelation does not seem to make sense.
4 c-Monoids

We now introduce a first axiom system for multirelations in a minimalist bi-monoidal setting, where only sequential and parallel composition and the corresponding units are present. It allows us to use the explicit domain definition

\[ d(x) = (x \cdot 1_\pi) \parallel 1_\sigma, \]

which has been verified in the multirelational model, to derive domain axioms similar to those for domain monoids [6], and to verify some properties of the \( M(X), S(X), T(X) \) triangle.

A proto-monoid is a structure \( (S, \cdot, 1_\sigma) \) such that \( 1_\sigma \cdot x = x = x \cdot 1_\sigma \) holds for all \( x \in S \). Hence composition is not required to be associative. A proto-bi-monoid is a structure \( (S, \cdot, \parallel, 1_\sigma, 1_\pi) \) such that \( (S, \cdot, 1_\sigma) \) is a proto-monoid and \( (S, \parallel, 1_\pi) \) a commutative monoid. A concurrent monoid (c-monoid) is a proto-bi-monoid that satisfies the axioms

\[
\begin{align*}
(x \cdot 1_\pi) \parallel x &= x, & (c1) \\
((x \cdot 1_\pi) \parallel 1_\sigma) \cdot y &= (x \cdot 1_\pi) \parallel y, & (c2) \\
(x \parallel y) \cdot 1_\pi &= (x \cdot 1_\pi) \parallel (y \cdot 1_\pi), & (c3) \\
(x \cdot y) \cdot 1_\pi &= x \cdot (y \cdot 1_\pi), & (c4) \\
1_\pi \parallel 1_\pi &= 1_\sigma. & (c5)
\end{align*}
\]

These axioms are sound with respect to the multirelational model.

Proposition 4.1. \( (M(X), \cdot, \parallel, 1_\sigma, 1_\pi) \) forms a c-monoid.

Proof. The following axioms have been verified in the following Lemmas: (c1) in 2.2(1); (c2) in 3.2(1); (c3) in 2.2(2) and 2.1(2); (c4) in 2.2(6); (c5) in 2.1(1).

Next we prove a property that is crucial for showing closure of subalgebras.

Lemma 4.2. In every c-monoid, the maps \( d \) and \( \cdot 1_\pi \) are retractions, that is, \( d(d(x)) = d(x) \) and \( (x \cdot 1_\pi) \cdot 1_\pi = x \cdot 1_\pi \).

It is a general property of a retraction \( f : X \to X \) that \( x \in f(X) \iff f(x) = x \), where \( f(X) \) is the image of \( S \) under \( f \). For every c-monoid \( S \) we define the sets of domain elements and terminal elements

\[ d(S) = \{ x \in S \mid d(x) = x \}, \quad \mathcal{T}(S) = \{ x \in S \mid x \cdot 1_\pi = x \} \]

and use the fixpoint characterisations \( d(x) = x \) and \( x \cdot 1_\pi = x \) for typing their elements. Sets of sequential subidentities, vectors and nonterminal elements, however, cannot yet be expressed in the c-monoid setting.

Lemma 4.3. In every c-monoid,

1. \( d(x) \cdot y = (x \cdot 1_\pi) \parallel y \),
2. \( d(x \cdot 1_\pi) \cdot y = (x \cdot 1_\pi) \parallel y \),
3. \( d(x) \cdot 1_\pi = x \cdot 1_\pi \),
4. \( d(x \cdot 1_\pi) = d(x) \),
5. \( 1_\pi \cdot 1_\pi = 1_\pi \),
6. \( d(1_\pi) = 1_\sigma \).
Properties (1) and (2) paraphrase axiom (c2). In particular, (2) implies that \(d(w) = w\|1_\sigma\) for each \(w \in \mathcal{T}(S)\). These facts are shown in the left-hand subdiagram of Figure 2 below. Properties (3) and (4) correspond to the right-hand subdiagram.

\[
\begin{array}{cc}
S \xrightarrow{d} d(S) & S \xrightarrow{1_\pi} \mathcal{T}(S) \\
(\cdot 1_\pi) \downarrow & (\cdot y) \\
\mathcal{T}(S) \xrightarrow{(\cdot y)} S & d \xrightarrow{\mathcal{T}(S)} \mathcal{T}(S)
\end{array}
\]

We can now show that the sets \(d(S)\) and \(\mathcal{T}(S)\) are isomorphic.

**Proposition 4.4.** Let \(S\) be a c-monoid. The two functions \(d : \mathcal{T}(S) \to d(S)\) and \((\cdot 1_\pi) : d(S) \to \mathcal{T}(S)\) form a bijective pair; the following diagrams commute.

\[
\begin{array}{cc}
\mathcal{T}(S) \xrightarrow{(\cdot 1_\pi)} d & d(S) \xrightarrow{(\cdot 1_\pi)} \mathcal{T}(S) \\
\mathcal{T}(S) \xrightarrow{d} d(S) & d(S) \xrightarrow{1_\pi} \mathcal{T}(S)
\end{array}
\]

Therefore \(d(S) \cong \mathcal{T}(S)\).

**Proof.** Functions are bijections if and only if they are invertible. It therefore suffices to check that \(d(d(x) \cdot 1_\pi) = d(x)\) and \(d(x \cdot 1_\pi) \cdot 1_\pi = x \cdot 1_\pi\), which follows directly from Lemma 4.3(3) and (4) or from commutation of the associated diagram. \(\square\)

Next we derive the domain axioms proposed in [6].

**Proposition 4.5.** In every c-monoid,

1. \(d(x\|y) = d(x)\|d(y)\),
2. \(d(x\|d(y)) = d(x) \cdot d(y)\),
3. \(d(x) \cdot x = x\),
4. \(d(x \cdot d(y)) = d(x \cdot y)\),
5. \(d(d(x) \cdot y) = d(x) \cdot d(y)\),
6. \(d(x) \cdot d(y) = d(y) \cdot d(x)\),
7. \(d(1_\sigma) = 1_\sigma\).

All axioms (a)-(e) are needed in these proofs. Isabelle/HOL generates counterexamples in their absence. Equations (1) and (2) are domain proto-trioi axioms [10], which are part of the axiomatisation of concurrent dynamic algebras; the others are domain monoid axioms [6].

We can now characterise the subalgebra of domain elements, whereas the c-monoid axioms are too weak to characterise that of parallel subidentities.

**Proposition 4.6.** Let \(S\) be a c-monoid. Then \(d(S)\) forms a sub-semilattice with multiplicative unit \(1_\sigma\) in which sequential and parallel composition coincide.

**Proof.** The closure conditions are verified by checking \(d(d(x) \cdot d(y)) = d(x) \cdot d(y)\), \(d(x\|d(y)) = d(x)\|d(y)\) and \(d(1_\sigma) = 1_\sigma\). Sequential and parallel composition of subidentities coincides due to Lemma 4.5(2). It remains to check that sequential composition of domain elements is associative, commutative and idempotent, and that \(1_\sigma\) is a multiplicative unit. \(\square\)
Depending on the order-dual interpretations of sequential composition as join or meet, \(1_\sigma\) becomes the least or greatest semilattice element. As similar result has been established in [6], but the proof does not transfer due to the lack of associativity of sequential composition. Next we derive three interaction laws.

**Lemma 4.7.** Let \(S\) be a c-monoid. For all \(x, y \in S, w \in d(S)\) and \(z \in \mathcal{T}(S)\),

1. \(z \| z = z\),
2. \(w \| w = w\),
3. \(w \cdot (x \| y) = (w \cdot x) \| (w \cdot y)\).

Finally, we refute additional interaction and associativity laws.

**Lemma 4.8.** There are c-monoids in which, for some elements \(x, y\) and \(z\),

1. \((x \| y) \cdot d(z) \neq (x \cdot d(z)) \| (y \cdot d(z))\),
2. \((x \cdot y) \cdot d(z) \neq x \cdot (y \cdot d(z))\),
3. \(1_\pi \cdot x \neq 1_\pi\).

Isabelle’s counterexample generator Nitpick presents counterexamples [11]. Since they are not very revealing, we do not show them.

## 5 c-Trioids

This section considers variants of dioids, that is, additively idempotent semirings, endowed with a sequential and a parallel composition and with additional interaction laws between these operations. Our intention is to derive the domain axioms of concurrent dynamic logic [28] from the explicit domain definition (d) in a minimalist setting. We cannot expect to prove more properties from Figure 2 since neither a maximal element \(U\) nor a meet operation is available.

A proto-dioid [10] is a structure \((S, +, \cdot, 0, 1_\sigma)\) such that \((S, +, 0)\) is a semilattice with least element 0, \((S, \cdot, 1_\sigma)\) is a proto-monoid, and the following axioms hold:

\[
x \cdot y + x \cdot z \leq x \cdot (y + z), \quad (x + y) \cdot z = x \cdot z + y \cdot z, \quad 0 \cdot x = 0.
\]

The relation \(\leq\) is the standard semilattice order defined as \(x \leq y \iff x + y = y\). A dioid is a proto-dioid in which multiplication is associative and the left distributivity law \(x \cdot (y + z) = x \cdot y + x \cdot z\) holds. A dioid is commutative if multiplication is. A proto-trioid is a structure \((S, +, \cdot, \|, 0, 1_\sigma, 1_\pi)\) such that \((S, +, 0, 1_\sigma)\) is a proto-dioid and \((S, +, \|, 0, 1_\pi)\) a commutative dioid. In every proto-dioid, multiplication is left-isotone, that is, \(x \leq y \Rightarrow z \cdot x \leq z \cdot y\). Moreover, every proto-trioid is a proto-bi-monoid.

A concurrent trioid (c-trioid) is a proto-trioid in which the c-monoid axioms and

\[
x \cdot 1_\pi \leq 1_\pi
\]

hold. Independence of the axioms has been checked with Isabelle/HOL’s automated theorem provers and counterexample generators.

**Proposition 5.1.** \((\mathcal{M}(X), \cup, \cdot, \emptyset, 0, 1_\sigma, 1_\pi)\) forms a c-trioid.

**Proof.** The proto-trioid axioms have been verified in [10]; the c-monoid axioms in Proposition 4.1 and Axiom (c6) in Lemma 5.2(2).
Figure 3: Class inclusions for proto-monoids (pM), proto-bi-monoids (pbM), c-monoids (cM), proto-dioids (pD), proto-trioids (pT) and c-trioids (cT)

In the setting of a c-trioid $S$ we can now define

$$\mathcal{S}(S) = \{x \in S \mid x \leq 1_\sigma\}, \quad \mathcal{N}(S) = \{x \in S \mid x \cdot 0 = 0\}.$$ 

It turns out that $d(S) \subseteq \mathcal{S}(S)$, whereas the converse inclusion need not hold. For $S = \emptyset$ and trioid operations defined by $0 < 1_\sigma < 1_\pi$ and $1_\pi \cdot 1_\pi = 1_\pi$ (remember that $1_\sigma \parallel 1_\sigma = 1_\sigma$ is an axiom) we have $1_\pi \leq 1_\sigma$, but $d(1_\pi) = (1_\pi \cdot 1_\pi) \parallel 1_\sigma = 1_\sigma \parallel 1_\sigma = 1_\sigma \neq 1_\pi$.

Similarly, $\mathcal{S}(S) \subseteq \{x \in S \mid x \leq 1_\pi\}$, whereas the converse inclusion need not hold. For $S = \emptyset$ and trioid operations defined by $0 < 1_\pi < 1_\sigma$ and $1_\pi \cdot 1_\sigma = 1_\pi$ we have $1_\pi \leq 1_\sigma$, but $1_\pi \cdot 1_\sigma = 1_\pi \neq 1_\pi$.

This shows that, for c-trioids, the relationships between subidentities, domain elements and fixpoints of $\cdot (1_\pi)$ are not as tight as expected. The set $\mathcal{N}(S)$ of terminal elements is studied in more detail in later sections. However we can derive the domain proto-trioid axioms of [10].

**Proposition 5.2.** Every c-trioid satisfies the domain axioms of domain proto-trioids.

**Proof.** Every c-trioid is a c-monoid, hence the properties from Section 4 hold. The following domain axioms of domain proto-dioids must be verified:

- $x \leq d(x) \cdot x$, which holds by Proposition 4.5(3);
- $d(x \cdot d(y)) = d(x) \cdot y$, which holds by Proposition 4.5(4);
- $d(x \parallel y) = d(x) \parallel d(y)$, which holds by Proposition 4.5(1);
- $d(x) \parallel d(y) = d(x) \cdot d(y)$, which holds by 4.5(2);
- $d(x + y) = d(x) + d(y)$, which holds by right distributivity of $\cdot$ and $\parallel$;
- $d(x) \leq 1_\sigma$, which follows from (c6);
- $d(0) = 0$, which is immediate from the domain definition.

This does not mean, however, that every c-trioid is a domain proto-trioid: additional axioms are assumed in the latter class. The relationship between c-trioids and concurrent dynamic algebra is discussed in detail in Section 6.

**Proposition 5.3.** Let $S$ be a c-trioid. Then $d(S)$ forms a bounded distributive lattice with least element $0$ and greatest element $1_\sigma$, and in which sequential and parallel composition coincide.
Proof. First, by Proposition 4.6, \( d(S) \) forms a meet semilattice, and it is clear that \( 1_s \) is the greatest element with respect to the semilattice order.

Second, the closure condition for 0 has already been checked in the proof of Proposition 5.2 and \( d(x + y) = x + y \) holds for all \( x, y \in d(S) \) by domain additivity and idempotency. Thus \( d(S) \) is a join semilattice with respect to \( + \) and 0 is the least element with respect to the semilattice order.

Third, for all \( x, y, z \in d(S) \), the absorption laws and distributivity laws \( x + x \cdot y = x, x \cdot (x + y) = x, (x + y) \cdot z = x \cdot z + y \cdot z \) and \( x + y \cdot z = (x + y) \cdot (x + z) \) must be verified.

Once more, this result is similar to that for domain proto-trioids [10], but proofs need to be revised due to different axioms.

Finally sequential composition of domain elements and parallel composition of parallel subidentities are greatest lower bound operations, hence meets.

Lemma 5.4. Let \( S \) be a c-trioid. Then

1. \( z \leq x \land z \leq y \Leftrightarrow z \leq x \cdot y \), for all \( x, y, z \in d(S) \),
2. \( z \leq x \land z \leq y \Leftrightarrow z \leq x \parallel y \), for all \( x, y, z \in \mathcal{I}(S) \).

6 Concurrent Dynamic Algebra

This section explains the relationship between c-trioids and concurrent dynamic algebras, as formalised in [10]. In every proto-monoid or proto-trioid \( S \) with a domain operation, a modal diamond operation can be defined as

\[
\langle x \rangle p = d(x \cdot p)
\]

for all \( x \in S \) and \( p \in d(S) \). In the setting of c-trioids, some, but not all, of the concurrent dynamic algebra axioms can be derived.

Here we call strong c-trioid a c-trioid \( S \) which satisfies, for all \( x, y \in S \) and \( p \in \mathcal{I}(S) \),

\[
\langle x \rangle \parallel y \cdot p = \langle x \rangle (y \cdot p), \quad (p \cdot x) \cdot y = p \cdot (x \cdot y), \quad (x \cdot p) \cdot y = x \cdot (p \cdot y), \quad (x \cdot y) \cdot p = x \cdot (y \cdot p).
\]

A strong c-Kleene algebra is a strong c-trioid expanded by a star operation which satisfies, for all \( x, y \in S \) and \( p \in \mathcal{I}(S) \),

\[
1_s + x \leq x^*, \quad p + x \leq y \Rightarrow x^* \cdot p \leq y, \quad (x \cdot p) \cdot z = x \cdot (p \cdot z).
\]

Soundness has been shown in [10].

Lemma 6.1. In every strong c-trioid, the following concurrent dynamic algebra axioms are derivable.

1. \( \langle x + y \rangle p = \langle x \rangle p + \langle y \rangle p \),
2. \( \langle x \cdot y \rangle p = \langle x \rangle \langle y \rangle p \),
3. \( \langle p \rangle q = p \cdot q \),
4. \( \langle x \parallel y \rangle p = \langle x \rangle p \cdot \langle y \rangle p \).

Lemma 6.2. In every strong c-Kleene algebra, the remaining concurrent dynamic algebra axioms are derivable as well.

1. \( p + \langle x \rangle x^* \rangle p = \langle x^* \rangle p \),
2. \( \langle x \rangle p \leq p \Rightarrow \langle x^* \rangle p \leq p \).
Figure 4: Class inclusions including domain proto-dioids (dpD), domain proto-trioids (DpT), strong c-trioids (cT\(^+\)), domain proto-Kleene algebras (dpKA), domain proto-bi-Kleene algebras (dpbKA) and strong c-Kleene algebras (cKA\(^+\)).

Hence every strong c-Kleene algebra is a concurrent dynamic algebra. The resulting subclass relationships are shown in Figure 4.

The following counterexample shows that the additional axioms are necessary.

**Lemma 6.3.** In some c-troid,

1. \((x∥y) \cdot d(z) \neq (x \cdot d(z))∥(y \cdot d(z))\),
2. \((x \cdot y) \cdot d(z) \neq x \cdot (y \cdot d(z))\),
3. \((x \cdot y)p \neq (x)⟨y⟩p\),
4. \((x∥y)p \neq (x)p ∥ (y)p\).

**Proof.**

1. Let \(S = \{a\}\) and let the trioid operations be defined by \(0 < 1_\pi < 1_\sigma < a\) and the tables for \(∥\) and \(\cdot\); from which \(d\) can be computed.

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1_\pi</th>
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<td>1_\pi</td>
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\(d\)

<table>
<thead>
<tr>
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<th>0</th>
<th>1_\pi</th>
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<td>1_\pi</td>
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<td>a</td>
</tr>
</tbody>
</table>

Then \((a∥1_\pi) \cdot d(0) = a \cdot 0 = 1_\pi \neq 0 = 1_\pi∥0 = (a \cdot 0)\|(1_\pi \cdot 0) = (a \cdot d(0))\|(1_\pi \cdot d(0))\).

2. With the same counterexample as in (1), \((a \cdot 1_\pi) \cdot 0 = 1_\pi \cdot 0 = 0 \neq 1_\pi = a \cdot 0 = a \cdot (1_\pi \cdot 0)\).

3. Again, with the same counterexample, \(⟨a\cdot 1_\pi⟩0 = d((a\cdot 1_\pi)\cdot 0) = d(0) = 0 \neq 1_\sigma = d(1_\pi) = d(a\cdot d(1_\pi \cdot 0)) = (a)⟨1_\sigma⟩0\).

4. Once more with the same counterexample, \(⟨a∥1_\pi⟩0 = d((a∥1_\pi) \cdot 0) = d(1_\pi) = 1_\sigma \neq 0 = d(0) = d((a \cdot 0)\|(1_\pi \cdot 0)) = ⟨a⟩0 \cdot ⟨1_\pi⟩0\).

\(\square\)
7 c-Lattices

We have seen that c-monoids and c-trioids do not capture all isomorphisms between sequential subidentities, parallel subidentities and vectors outlined in Section 3. In addition, they are too weak to link the c-monoid-based domain definition with the alternative ones presented in the same section. A meet operation, and more specifically a bounded distributive lattice structure, is needed for this purpose.

A c-lattice is a structure \((S, +, \cap, \cdot, ||, 0, 1_\sigma, 1_\pi, U, \mathcal{T}_\pi)\) such that \((L, +, \cap, 0, U)\) is a bounded distributive lattice with least element 0 and greatest element \(U\), \((S, +, \cdot, ||, 0, 1_\sigma, 1_\pi)\) is a proto-trioid and the following axioms hold.

\[
x \cdot 1_\pi + x \cdot \mathcal{T}_\pi = x \cdot U, \quad (cl1)
\]
\[
1_\pi \cap (x + \mathcal{T}_\pi) = x \cdot 0, \quad (cl2)
\]
\[
x \cdot (y || z) \leq (x \cdot y) || (x \cdot z), \quad (cl3)
\]
\[
z || z \Rightarrow (x || y) \cdot z = (x \cdot z) || (y \cdot z), \quad (cl4)
\]
\[
x \cdot (y \cdot (z \cdot 0)) = (x \cdot y) \cdot (z \cdot 0), \quad (cl5)
\]
\[
(x \cdot 0) \cdot y = x \cdot (0 \cdot y), \quad (cl6)
\]
\[
1_\pi \mathcal{T}_\pi = 1_\pi, \quad (cl7)
\]
\[
(x \cdot 1_\pi) || 1_\pi \cdot y = (x \cdot 1_\pi) || y, \quad (cl8)
\]
\[
(x \cap 1_\sigma) \cdot 1_\pi || 1_\sigma = x \cap 1_\sigma, \quad (cl9)
\]
\[
(x \cap \mathcal{T}_\pi) \cdot 1_\pi || 1_\sigma = 1_\sigma \cap (x \cap \mathcal{T}_\pi) \cdot \mathcal{T}_\pi, \quad (cl10)
\]
\[
(x \cap \mathcal{T}_\pi) \cdot 1_\pi || \mathcal{T}_\pi = (x \cap \mathcal{T}_\pi) \cdot \mathcal{T}_\pi. \quad (cl11)
\]

Axiom (cl11) and (cl2) imply that \(1_\pi\) and \(\mathcal{T}_\pi\) are complements, that is,

\[
1_\pi + \mathcal{T}_\pi = U, \quad 1_\pi \cap \mathcal{T}_\pi = 0.
\]

In addition, (cl2) implies that \(x \cap 1_\pi = x \cdot 0\).

The axioms (cl3)-(cl6) express associativity and interaction properties. Further associativity properties for sequential and parallel subidentities as well as for \(U\) hold in the multirelational model, but those are either derivable or not directly needed for the main results in this article. We mention them explicitly whenever they occur. Axioms (cl7) and (cl8) are taken from c-monoids. Axiom (cl9) can be written as \(d(x \cap 1_\sigma) = x \cap 1_\sigma\); it expresses one of the isomorphism conditions from Figure 2. Axiom (cl10) can be written as \(d(x \cap \mathcal{T}_\pi) = 1_\sigma \cap (x \cap \mathcal{T}_\pi) \cdot \mathcal{T}_\pi\). Axiom (cl11) can be rewritten as \(d(x \cap \mathcal{T}_\pi) \cdot \mathcal{T}_\pi = (x \cap \mathcal{T}_\pi) \cdot \mathcal{T}_\pi\). These domain properties are reminiscent of the relational case and have been motivated in Section 3.

We have used Isabelle’s counterexample generators to analyse irredundancy of these axioms. However, due to their large number, this was not always successful. Whether the set of axioms can be compacted further remains to be seen.

Proposition 7.1. Every c-lattice is a c-trioid.

It follows that every c-lattice is a c-monoid. Next we prove a soundness result.

Proposition 7.2. \(\mathcal{M}(X, \cap, \cdot, ||, \emptyset, 1_\sigma, 1_\pi, U, \mathcal{T}_\pi)\) forms a c-lattice.

Proof. We have verified the proto-trioid axioms in Proposition 5.1. Axioms (cl3)-(cl6) in Lemma 2.2. Axioms (cl7) and (cl8) in Proposition 4.1, and the remaining axioms in Lemma 3.2. □

We call a c-lattice boolean if its lattice reduct forms a boolean algebra. It is easy to see that multirelations form in fact boolean c-lattices. Moreover, infima and suprema of arbitrary sets exist in the algebra of multirelations, and an infinite left distributivity law for sequential composition with respect to suprema and infinite left and right distributivity laws for parallel composition with respect to suprema hold. Multirelations therefore form quantale-like algebras. This is further explored in Sections 12 and 13.
For a c-lattice $S$ we can now define also
\[ V(S) = \{ x \in S \mid (x \cdot 1_{\pi}) \parallel U = x \}, \quad \mathcal{U}(S) = \{ x \in S \mid x \parallel U = x \}. \]

**Lemma 7.3.** Let $S$ be a c-lattice. Then
1. $d(S) = \mathcal{I}(S)$,
2. $\mathcal{I}(S) = \{ x \in S \mid x \leq 1_{\pi} \} = \{ x \in S \mid x \cdot 0 = x \}$,
3. $\mathcal{I}(S) = \{ x \in S \mid d(x) \cdot U = x \}$,
4. $\mathcal{N}(S) = \{ x \in S \mid x \cap 1_{\pi} = 0 \} = \{ x \in S \mid x \leq \overline{1_{\pi}} \}$.

More specifically, $d(x) = x$ if and only if $x \leq 1_{\sigma}$, $x \cdot 1_{\pi} = x$ is equivalent to each of $x \leq 1_{\pi}$ and $x \cdot 0 = x$, $(x \cdot c) \parallel U = x$ if and only if $d(x) \cdot U = x$, and $x \cdot 0 = 0$ is equivalent to each of $x \cap 1_{\pi} = 0$ and $x \leq \overline{1_{\pi}}$.

This correspondence between subidentities, domain elements and terminal elements captures that in the multirelational model.

The next three lemmas collect some basic properties of c-lattices.

**Lemma 7.4.** Let $S$ be a c-lattice. Then
1. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$, if $z \in \mathcal{I}(S)$,
2. $(x \cdot z) \parallel (y \cdot z) = (x\parallel y) \cdot z$, if $z \in \mathcal{I}(S) \cup \mathcal{I}(S)$,
3. $x \leq x \parallel y$,
4. $x \cap y \leq x \parallel y$.

**Lemma 7.5.** Let $S$ be a c-lattice. Then
1. $x = (x \cap \overline{1_{\pi}}) + x \cdot 0$,
2. $x \leq \overline{1_{\pi}}$, if $x \in \mathcal{I}(S)$,
3. $(x \cdot 1_{\pi}) \cap \overline{1_{\pi}} = (x \cdot 0) \cap \overline{1_{\pi}} = (x \cap \overline{1_{\pi}}) \cdot 0 = 0$.

**Lemma 7.6.** In every c-lattice,
1. $1_{\pi} = U \cdot 0$,
2. $U \parallel U = U \cdot U = U \cdot \overline{1_{\pi}} = \overline{1_{\pi}} \cdot U = U$,
3. $1_{\pi} \cdot x = \overline{1_{\pi}} \cdot 1_{\pi} = U \cdot 1_{\pi} = 1_{\pi}$,
4. $\overline{1_{\pi}} \parallel 1_{\pi} = U \parallel 1_{\pi} = \overline{1_{\pi}} \cdot 1_{\pi} = \overline{1_{\pi}}$.

Property (1) gives an explicit definition of $1_{\pi}$. Some of the other properties have already been stated in Lemma 3.1 in the multirelational model. It follows that all constants are sequential and parallel idempotents.

The final lemma of this section collects further simplification and decomposition properties that are interesting for domain definitions.

**Lemma 7.7.** In every c-lattice,
1. $x \cdot y = (x \cap \overline{1_{\pi}}) \cdot y + x \cdot 0$,
2. $1_{\sigma} \cap (x \cap \overline{1_{\pi}}) \cdot y = 1_{\sigma} \cap x \cdot y$,
3. $1_{\sigma} \cap x \cdot \overline{1_{\pi}} = 1_{\sigma} \cap x \cdot U$. 

\[ \]
4. \(1_\sigma \sqcap x \parallel \bar{T}_\pi = 1_\sigma \sqcap x \parallel U\),

5. \((x \cdot 1_\pi) \parallel \bar{T}_\pi = (x \sqcap \bar{T}_\pi) \cdot \bar{T}_\pi + (x \cdot 0) \parallel \bar{T}_\pi\),

6. \((x \cdot 1_\pi) \parallel U = x \cdot U + (x \cdot 0) \parallel U\).

Equations (2), (3) and (4) can be visualised by the following subdiagrams of Figure 2.

8 Domain in c-Lattices

This section presents explicit domain definitions and related properties for general and nonterminal elements. First, using \(d(x) = (x \cdot 1_\pi) \parallel 1_\pi\), recall that we can rewrite some of the c-lattice axioms:

\[
\begin{align*}
d(x) \cdot y &= (x \cdot 1_\pi) \parallel y, \quad (cl8) \\
d(x \sqcap 1_\pi) &= x \sqcap 1_\pi, \quad (cl9) \\
d(x \sqcap T_\pi) &= 1_\pi \sqcap (x \sqcap T_\pi) \cdot \bar{T}_\pi, \quad (cl10) \\
d(x \sqcap T_\pi) \cdot \bar{T}_\pi &= (x \sqcap T_\pi) \cdot \bar{T}_\pi. \quad (cl11)
\end{align*}
\]

Properties (cl10) and (cl11) are visualised by the following subdiagram of Figure 2.

These two identities admit the following variations.

**Lemma 8.1.** In every c-lattice,

1. \(d(U) = d(T_\pi) = 1_\pi\),
2. \((x \sqcap T_\pi) = 1_\pi \sqcap x \cdot T_\pi\),
3. \(d(x \sqcap T_\pi) = 1_\pi \sqcap (x \sqcap T_\pi) \cdot U\),
4. \(d(x \sqcap T_\pi) = 1_\pi \sqcap x \cdot U\),
5. \(d(x \sqcap T_\pi) = 1_\pi \sqcap ((x \sqcap T_\pi) \cdot 1_\pi) \parallel T_\pi\),
6. \(d(x \sqcap T_\pi) = 1_\pi \sqcap ((x \sqcap T_\pi) \cdot 1_\pi) \parallel U\).

Identity (5) corresponds to another subdiagram of Figure 2.
The remaining properties are obtained by combining the diagrams for (cl10), (cl11) and (5) with those for Lemma 7.7. Equation (3), for instance, corresponds to the following commuting diagram.

Since $d(x) = d(x \cap T_\pi) + d(x \cdot 0)$ by Lemma 7.4 and Proposition 5.2, we can generate explicit definitions of $d(x)$ by inserting those for $d(x \cap T_\pi)$ and $d(x \cdot 0)$, for instance

$$d(x) = (1_\sigma \cap x \cdot U) + (x \cdot 0)\|1_\sigma.$$ 

This explains, in particular, the difference between relational and multirelational domain definitions. We can also use the various definitions of $d(x \cap T_\pi)$ and $d(x \cdot 0)$ to generate more compact definitions.

**Lemma 8.2.** In every c-lattice,

1. $d(x) = 1_\sigma \cap (x \cdot 1_\pi)\|T_\pi$,
2. $d(x) = 1_\sigma \cap (x \cdot 1_\pi)\|U$,
3. $d(x) = (1_\pi \cap x \cdot U)\|1_\sigma$.

These identities correspond to the following commuting subdiagrams of Figure 2.

Finally we mention some variations of Axiom 3.11.

**Lemma 8.3.** In every c-lattice,

1. $d(x \cap T_\pi) \cdot U = (x \cap T_\pi) \cdot U$,
2. $d(x) \cdot T_\pi = (x \cap T_\pi) \cdot T_\pi + (x \cdot 0)\|T_\pi$,
3. $d(x) \cdot U = (x \cap T_\pi) \cdot U + (x \cdot 0)\|U$,
4. $d(x) \cdot U = x \cdot U + (x \cdot 0)\|U$,
5. $x \cdot T_\pi = d(x \cap T_\pi) \cdot T_\pi + x \cdot 0$.
6. $x \cdot U = d(x \cap T_\pi) \cdot U + x \cdot 0$,
7. $d(x \cdot U) = d(x \cdot T_\pi) = d(x)$.
9 Subalgebras of c-Lattices

This section studies the structure of the subalgebras of sequential subidentities, parallel subidentities, vectors and nonterminal elements. First we consider the set $\mathcal{I}(S)$ of sequential subidentities of a c-lattice $S$. We can freely identify subidentities with domain elements and use results for $d(S)$ from Section 8.

Proposition 9.1. Let $S$ be a c-lattice. The set $\mathcal{I}(S)$ forms a distributive lattice bounded by 0 and $1_\pi$, and in which sequential and parallel composition are meet. It forms a boolean algebra if $S$ is boolean.

Proof. Relative to Proposition 5.3 it remains to check that $\mathcal{I}(S)$ is closed under meets and that meet coincides with multiplication in this subalgebra. Meet closure follows from $d(p \cap q) = p \cap q$ for all $p, q \in \mathcal{I}(S)$. Sequential composition of domain elements is a greatest lower bound operation in $\mathcal{I}(S)$ by Lemma 5.4(4). Hence it coincides with meet and parallel composition. Finally, if $S$ is a boolean algebra with complement $\overline{x}$ for each $x \in S$, we define complementation on $\mathcal{I}(S)$ by $x' = 1_\pi \cap \overline{x}$. □

Proposition 9.2. Let $S$ be a c-lattice. The set $\mathcal{V}(S)$ forms a sub-c-lattice bounded by 0 and $1_\pi$, in which parallel composition is meet, and in which all $x \in \mathcal{V}(S)$ are right units of sequential composition. It is boolean whenever $S$ is.

Proof. First, parallel composition of terminal elements is a greatest lower bound operation on $\mathcal{V}(S)$ by Lemma 5.3(7). It therefore coincides with meet. Second, the c-lattice operations are closed: $0 \cdot 0 = 0$, $1_\pi \cdot 0 = 1_\pi$, $x \cdot 0 + y \cdot 0 = (x + y) \cdot 0$, $(x \cdot 0) \cap (y \cdot 0) = (x \cap y) \cdot 0$, $(x \cdot 0) \parallel (y \cdot 0) = (x \parallel y) \cdot 0$ and $(x \cdot 0) \parallel (y \cdot 0) = x \cdot 0$.

Thus $\mathcal{V}(S)$ forms a c-lattice bounded by 0 and $1_\pi$. By $(x \cdot 0) \parallel (y \cdot 0) = x \cdot 0$, sequential composition is a projection, hence every terminal element a right identity. Finally, for every boolean c-lattice $S$, the set $\mathcal{V}(S)$ forms a boolean subalgebra with complementation defined similarly to the sequential case. □

The subset $\mathcal{V}(S)$ of vectors of a c-lattice $S$ does not have such pleasant properties. In particular, vectors are not closed under sequential composition. With $x \in \mathcal{V}(S)$ if and only if $d(x) \cdot U = x$ it is easy to see that $U \in \mathcal{V}(S)$ and $0 \notin \mathcal{V}(S)$, whereas $U \cdot 0 \notin \mathcal{V}(S)$. In addition, vectors in c-lattices need not be closed under meets, whereas this is the case in the multirelational model, where $(R \cap S) \cdot T = R \cdot T \cap S \cdot T$ holds for $R, S \in \mathcal{I}(X)$. At least we have the following closure properties.

Lemma 9.3. In every c-lattice,

1. $d(0) \cdot U = 0$ and $d(U) \cdot U = U$.
2. $d(x) \cdot z + d(y) \cdot z = d(x + y) \cdot z$.
3. $(d(x) \cdot U) \parallel (d(y) \cdot U) = d(x \parallel y) \cdot U$.

Finally we consider the subset $\mathcal{N}(S)$ of nonterminal elements, where the analogy with binary relations is more striking.

Proposition 9.4. Let $S$ be a c-lattice. The set $\mathcal{N}(S)$ forms a sub-c-lattice of $S$ without parallel unit in which 0 is a right annihilator of sequential composition and $\overline{T_\pi}$ the maximal element. It is boolean whenever $S$ is.

Proof. It needs to be checked that $0 \cap \overline{T_\pi} = 0$, $1_\pi \cap \overline{T_\pi} = 1_\pi$, and that $(x + y) \cap \overline{T_\pi} = x + y$, $(x \cap y) \cap \overline{T_\pi} = x \cap y$, $(x \cdot y) \cap \overline{T_\pi} = x \cdot y$, $(x \parallel y) \cap \overline{T_\pi} = x \parallel y$ hold for all $x, y \in \mathcal{N}(S)$.

Also, in every non-trivial algebra, $1_\pi \notin \mathcal{N}(S)$. Finally, $(x \cap \overline{T_\pi}) \cdot 0 = 0$ by Lemma 7.3(3) and $\overline{T_\pi}$ is the maximal element by definition. □

Next we consider the subalgebras $\mathcal{I}(\mathcal{N}(S))$ and $\mathcal{V}(\mathcal{N}(S))$. First, $\mathcal{I}(S) \subseteq \mathcal{N}(S)$, hence $\mathcal{I}(\mathcal{N}(S)) = \mathcal{I}(S)$ and Proposition 9.1 applies without modification to $\mathcal{N}(S)$.

Corollary 9.5. Let $S$ be a c-lattice. The set $\mathcal{I}(\mathcal{N}(S)) = \mathcal{I}(S)$ forms a bounded distributive sublattice of $\mathcal{N}(S)$. It is a boolean algebra whenever $S$ is.

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Next we consider the subalgebra of vectors. First we derive a variant of Tarski’s rule from relation algebra (cf. [17]).

**Lemma 9.6.** Let \( R \in \mathcal{M}(X) \). Then \( R \cap T_\pi \neq \emptyset \Rightarrow T_\pi \cdot ((R \cap T_\pi) \cdot T_\pi) = T_\pi. \)

As common in relation algebra we keep Tarski’s rule separate from the other axioms since it is not even a quasi-identity. We can use it to prove the following fact.

**Lemma 9.7.** Let \( S \) be a c-lattice in which Tarski’s rule and \( d(x) \cdot (y \cdot z) = (d(x) \cdot y) \cdot z \) hold. Let \( x, y \in \mathcal{V}(\mathcal{N}(S)) \). Then

\[
    x \cdot y = \begin{cases} 
        0, & \text{if } y = 0, \\
        x, & \text{if } y \neq 0.
    \end{cases}
\]

**Proposition 9.8.** Let \( S \) be a c-lattice in which Tarski’s rule and

\[
    (d(x) \cap d(y)) \cdot z = d(x) \cdot z \cap d(y) \cdot z, \quad (d(x) \cdot y) \cdot z = d(x) \cdot (y \cdot z)
\]

hold. Then \( \mathcal{V}(\mathcal{N}(S)) \) is a sub-c-lattice of \( \mathcal{N}(S) \) bounded by 0 and \( \overline{T_\pi} \), in which 0 is a left annihilator and parallel composition is meet.

**Proof.** We need to verify the closure conditions \( 0 \cdot \overline{T_\pi} = 0 \) and \( \overline{T_\pi} \cdot \overline{T_\pi} = \overline{T_\pi} \) as well as \( (x + y) \cdot \overline{T_\pi} = x + y \), \( (x \cap y) \cdot \overline{T_\pi} = x \cap y \), \( (x \cdot y) \cdot \overline{T_\pi} = x \cdot y \) and \( (x \parallel y) \cdot \overline{T_\pi} = x \parallel y \) for all \( x, y \in \mathcal{V}(\mathcal{N}(S)) \), and \( x \parallel y = x \cap y \) for all \( x, y \in \mathcal{V}(\mathcal{N}(S)) \).

Note that meet-closure is enforced by assuming \( (d(x) \cap d(y)) \cdot z = d(x) \cdot z \cap d(y) \cdot z \). This and all additional assumptions on multirelations used in this section have, of course, been verified in the multirelational model. Whether stronger properties hold in situations where sequential composition is associative remains to be seen.

## 10 Isomorphisms in c-Lattices

This section finally verifies the isomorphisms between sequential subidentities, parallel subidentities and vectors from Figure 1 and 2 in the context of c-lattices and for their nonterminal elements.

We also characterise the structure that is preserved by these mappings. Given the results on subalgebras from the previous section it cannot be expected that sequential composition is preserved. This is indeed confirmed by the multirelational counterexamples in this section—apart from one exception. The other c-lattice operations are preserved. In particular, all isomorphisms are constructed from the operations and constants of c-lattices, so that their properties can be checked within the c-lattice setting by simple equational reasoning.

**Proposition 10.1.** Let \( S \) be a c-lattice.

1. The maps \((-U)\) and \( d \) as well as \((\parallel U)\) and \((-1_\pi)\) in the following diagrams form bijective pairs; the diagrams commute.

\[
\begin{array}{ccc}
    \mathcal{V}(S) & \xrightarrow{\mathcal{F}(S)} & \mathcal{F}(S) \\
    (-U) & \downarrow & d \\
    \mathcal{V}(S) & \xrightarrow{\mathcal{F}(S)} & \mathcal{F}(S) \\
    \parallel(U) & \downarrow & -1_\pi \\
    \mathcal{V}(S) & \xrightarrow{\mathcal{F}(S)} & \mathcal{F}(S) \\
    \end{array}
\]

\[
\begin{array}{ccc}
    \mathcal{V}(S) & \xrightarrow{\mathcal{F}(S)} & \mathcal{F}(S) \\
    d & \downarrow & (-U) \\
    \mathcal{V}(S) & \xrightarrow{\mathcal{F}(S)} & \mathcal{F}(S) \\
    \parallel(U) & \downarrow & 1_\pi \\
    \mathcal{V}(S) & \xrightarrow{\mathcal{F}(S)} & \mathcal{F}(S) \\
    \end{array}
\]

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2. Therefore $\mathcal{I}(S) \cong \mathcal{I}(S) \cong \mathcal{V}(S)$.

Proof. The isomorphism between $d(S)$ and $\mathcal{I}(S)$ has been verified in Proposition 4.4, moreover $d(S) = \mathcal{I}(S)$ by Lemma 7.3. It remains to check that

$$
(1) d(d(x) \cdot U) = d(x), \quad \quad \quad d(1_x \cdot U) \cdot 1_\pi = x \cdot 1_\pi, \quad \quad \quad d((x \cdot 1_x) \cdot U) \cdot 1_\pi = x \cdot 1_\pi
$$

Proposition 10.2. Let $S$ be a c-lattice.

1. The maps $\cdot 1_x$ and $d$, $\cdot 1_\pi$ and $d$, and $\cdot 1_\pi$ in the following diagrams form bijective pairs; the diagrams commute.

2. Therefore $\mathcal{I}(\mathcal{N}(S)) \cong \mathcal{I}(\mathcal{N}(S)) \cong \mathcal{V}(\mathcal{N}(S))$.

Proof. The following conditions must be checked.

$$
(2) d(d(x \sqcap 1_\pi) \cdot 1_\pi) = d(x \sqcap 1_\pi), \quad \quad \quad d((x \sqcap 1_\pi) \cdot 1_\pi) = (x \sqcap 1_\pi) \cdot 1_\pi
$$

We now investigate structure preservation of these bijections.

Proposition 10.3. Let $S$ be a c-lattice. The maps

$$
(1) (\cdot 1_x) : \mathcal{I}(S) \to \mathcal{I}(S), \quad \quad \quad d : \mathcal{I}(S) \to \mathcal{I}(S),
$$

$$
(2) (\cdot 1_\pi) : \mathcal{I}(S) \to \mathcal{I}(S), \quad \quad \quad d : \mathcal{I}(S) \to \mathcal{I}(S),
$$

$$
(3) (\cdot 1_\pi) : \mathcal{I}(S) \to \mathcal{I}(S), \quad \quad \quad d : \mathcal{I}(S) \to \mathcal{I}(S)
$$

preserve addition, meet and parallel composition, minimal elements and maximal elements of the subalgebras. The last map also preserves sequential composition.
Proposition 10.4. Let\( S \) be a c-lattice. The maps 
\[
\begin{align*}
(1_\pi) : & \mathcal{F}(\mathcal{N}(S)) \to \mathcal{F}(\mathcal{N}(S)) \\
(T_\pi) : & \mathcal{F}(\mathcal{N}(S)) \to \mathcal{V}(\mathcal{N}(S)) \\
([T_\pi] : & \mathcal{F}(\mathcal{N}(S)) \to \mathcal{V}(\mathcal{N}(S)) \\
(1_\pi) : & \mathcal{V}(\mathcal{N}(S)) \to \mathcal{F}(\mathcal{N}(S))
\end{align*}
\]
preserve addition, meet and parallel composition, minimal elements and maximal elements of the subalgebras.

We complement these results by refuting preservation of sequential composition for the remaining maps between sequential identities, subidentities and vectors.

Lemma 10.5. \(\mathbf{1.}\) No isomorphism in \(\mathcal{M}(X)\) except \((1_\pi)\) preserves sequential composition.

\(\mathbf{2.}\) No isomorphism in \(\mathcal{N}(X)\) preserves sequential composition.

Proof. \(\bullet\) Let \(R = \{(a, \{a\}\})\) and \(S = \emptyset\), both of which are in \(\mathcal{F}(\mathcal{N}(X))\) and \(\mathcal{V}(\mathcal{N}(X))\). Then
\[
(R \cdot S) \cdot 1_\pi \subset \{(a, \emptyset)\} = (R \cdot 1_\pi) \cdot (S \cdot 1_\pi),
\]
\[
(R \cdot S) \cdot U = \emptyset \subset \{(a, \emptyset)\} = (R \cdot U) \cdot \emptyset = (R \cdot U) \cdot (S \cdot U).
\]

\(\bullet\) Let \(R = \{(a, \emptyset)\}\) and \(S = \emptyset\), both of which are in \(\mathcal{F}(\mathcal{N}(X))\) and \(\mathcal{V}(X)\). Then
\[
d(R \cdot S) = d(R) = \{(a, \{a\}\)} \supset \emptyset = d(R) \cdot \emptyset = d(R) \cdot d(S),
\]
\[
(R \cdot S) \| U = \{(a, \emptyset), (a, \{a\}\} = R = (R \| U) \cdot \emptyset = (R \| U) \cdot (S \| U).
\]

\(\bullet\) Let \(R = \{(a, \{a\}), (a, \{b\}), (a, \{a, b\}\}), S = \{(b, \{a\}), (b, \{b\}), (b, \{a, b\}\)}\), both of which are in \(\mathcal{V}(\mathcal{N}(X))\). Then
\[
d(R \cdot S) = d(R) = \{(a, \{a\}\)} \supset \emptyset = d(R) \cdot d(S).
\]

\(\square\)

11 Terminal and Nonterminal Elements

Algebras of multirelations share some features with algebras of languages with finite and infinite words. Both form trioids with parallel composition corresponding to shuffle in the language case. However, \((X \| Y) \cdot Z \subseteq (X \cdot Z) \| (Y \cdot Z)\) does not generally hold in languages (consider \(X = \{a\}, Y = \{b\}\) and \(Z = \{cd\}\)), the algebra of sequential subidentities is trivial (it consists only of the empty and the empty word language), and the notion of vector does not seem to make sense.

For an alphabet \(\Sigma\), a finite word language is a subset of \(\Sigma^*\), the set of all finite words over \(\Sigma\). An infinite word language is a subset of \(\Sigma^\omega\), the set of all strictly infinite words over \(\Sigma\). Languages in which finite and infinite words are mixed are subsets of \(\Sigma^\omega = \Sigma^* \cup \Sigma^\omega\). One can then use \(\text{fin} : 2^{\Sigma^\omega} \to 2^{\Sigma^*}\) and \(\text{inf} : 2^{\Sigma^\omega} \to 2^{\Sigma^\omega}\) to project on the finite and infinite words in a language.

Analogously, the maps \(\text{fin}\) to \(\tau : \mathcal{M}(X) \to \mathcal{F}(X)\) and \(\text{inf}\) to \(\nu : \mathcal{M}(X) \to \mathcal{N}(X)\) defined by
\[
\tau = \lambda x.\ x \cdot 0, \quad \nu = \lambda x.\ x \cap T_\pi
\]
project on the terminal and the nonterminal part of a multirelation. More abstractly, we define such functions \(\tau : S \to \mathcal{F}(S)\) and \(\nu : S \to \mathcal{N}(S)\) on a c-lattice \(S\).

Many properties of \(\text{fin}\) and \(\text{inf}\) (cf. [25]) are shared with \(\tau\) and \(\nu\), but there are also differences.

Lemma 11.1. In every c-lattice,

1. the functions \(\tau\) and \(\nu\) are interior operators, and therefore retractions,

2. \(\tau(x) + \nu(x) = x\) and \(\tau(x) \cap \nu(x) = 0\),
3. \( \tau(\nu(x)) = 0 \) and \( \nu(\tau(x)) = 0 \).

The properties \( \tau(x) \leq x, \tau(\tau(x)) = \tau(x) \) and \( x \leq y \Rightarrow \tau(x) \leq \tau(y) \) must be verified to show that \( \tau \) is an interior operator, and likewise for \( \nu \). The next lemmas are essentially transcriptions of properties verified in the proofs of Proposition 9.2 and 9.3.

**Lemma 11.2.** In every c-lattice,

1. \( \tau(0) = 0 \) and \( \nu(0) = 0 \),
2. \( \tau(1) = 0 \) and \( \nu(1) = 1 \),
3. \( \tau(1) = 1 \) and \( \nu(1) = 0 \),
4. \( \tau(T) = 0 \) and \( \nu(T) = T \),
5. \( \tau(U) = U \) and \( \nu(U) = U \).

**Lemma 11.3.** In every c-lattice,

1. \( \tau(x + y) = \tau(x) + \tau(y) \) and \( \nu(x + y) = \nu(x) + \nu(y) \),
2. \( \tau(x \sqcap y) = \tau(x) \sqcap \tau(y) \) and \( \nu(x \sqcap y) = \nu(x) \sqcap \nu(y) \),
3. \( \tau(x \parallel y) = \tau(x) \parallel \tau(y) \) and \( \nu(x \parallel y) = d(\tau(x)) \cdot \nu(y) + d(\tau(y)) \cdot \nu(x) + \tau(x) \parallel \nu(y) \),
4. \( \tau(x \cdot y) = \tau(x) + \nu(x) \cdot \tau(y) \).

**Lemma 11.4.**

1. There are \( R, S \in \mathcal{M}(X) \) such that \( \tau(R \cdot S) \neq \tau(R) \cdot \tau(S) \).
2. There are \( R, S \in \mathcal{M}(X) \) such that \( \nu(R \cdot S) \neq \nu(R) \cdot \nu(S) \).
3. There are \( R, S \in \mathcal{M}(X) \) such that \( \nu(R \| S) \neq \nu(R) \| \nu(S) \).

**Proof.**

1. If \( R = \{(a, \emptyset), (b, \{a\})\} \) and \( S = \{(a, \emptyset)\} \), then \( \tau(R) \cdot \tau(S) = S \subset R \cdot 1_\pi = \tau(R \cdot S) \).
2. If \( R = \{(a, \{a, b\})\} \) and \( S = \{(a, \emptyset), (b, \{a, b\})\} \), then \( \nu(R \cdot S) = R \neq \emptyset = \nu(R) \cdot \nu(S) \).
3. If \( R = \{(a, \{a\})\} \) and \( S = \{(a, \emptyset)\} \), then \( \nu(R \| S) = R \neq \emptyset = \nu(R) \| \nu(S) \).

We do not have a compositional characterisation of \( \nu(x \cdot y) \). On the one hand, \( \nu(x \cdot y) = \nu(\nu(x) \cdot y) \), but on the other hand, without left distributivity, this cannot easily be decomposed further. In the multirelational model, elements \((b, \emptyset) \in S\) can obviously contribute to pairs \((a, A) \in R \cdot S\) with \( A \neq \emptyset \). This makes the situation different from the language case, where \( \text{fin}(X \cdot Y) = \text{fin}(X) \cdot \text{fin}(Y) \). Interestingly, however, this does not rule out simple decomposition theorems for sequential and parallel composition.

**Lemma 11.5.** In every c-lattice,

1. \( x \cdot y = \tau(x) + \nu(x) \cdot y \),
2. \( x \parallel y = \nu(x) \parallel \nu(y) + d(\nu(x)) \cdot \tau(y) + d(\nu(y)) \cdot \tau(x) + \tau(x) \parallel \tau(y) \).

It seems natural to identify elements of c-lattices if they coincide on their terminal or their nonterminal parts.

**Lemma 11.6.** Let \( S \) be a c-lattice. Then \( \mathcal{N}(S) \) and \( \mathcal{F}(S) \) form order ideals.
Lemma 11.7. In every c-lattice,
1. $x \in \mathcal{N}(S)$ implies $x \parallel y \in \mathcal{N}(S)$,
2. $x \in \mathcal{F}(S)$ implies $x \cdot y \in \mathcal{F}(S)$ and $y \cdot x \in \mathcal{F}(S)$.

Lemma 11.8. There are $R \in \mathcal{F}(X)$ and $S \in \mathcal{N}(X)$ with $R \cdot S \not\in \mathcal{N}(X)$, $S \cdot R \not\in \mathcal{N}(X)$ and $R \parallel S \not\in \mathcal{F}(X)$.

Proof. Let $R = \{(a, \emptyset)\}$ and $S = \{(a, \{a\})\}$. Then $R \cdot S = S \cdot R = R$ is not in $\mathcal{N}(X)$ and $R \parallel S = S$ is not in $\mathcal{F}(X)$. □

Define the relations on a c-lattice $S$ by
\[ x \sqsubseteq_\tau y \iff \tau(x) \leq \tau(y), \quad x \sqsubseteq_\nu y \iff \nu(x) \leq \nu(y). \]

Lemma 11.9. In every c-lattice,
1. the relations $\sqsubseteq_\tau$ and $\sqsubseteq_\nu$ are partial orders,
2. $x \sqsubseteq_\tau y$ implies $x + z \sqsubseteq_\tau y + z$, $x \cap z \sqsubseteq_\tau y \cap z$, $x \parallel z \sqsubseteq_\tau y \parallel z$ and $z \cdot x \sqsubseteq_\tau z \cdot y$,
3. $x \sqsubseteq_\nu y$ implies $x + z \sqsubseteq_\nu y + z$, $x \cap z \sqsubseteq_\nu y \cap z$ and $z \cdot x \sqsubseteq_\nu z \cdot y$.

The missing precongruence properties are justified by the following counterexamples.

Lemma 11.10. 1. There are $R, S, T \in \mathcal{M}(X)$ such that $\tau(R) \sqsubseteq \tau(S)$ and $\tau(R \cdot T) \not\sqsubseteq \tau(S \cdot T)$.

2. There are $R, S, T \in \mathcal{M}(X)$ such that $\nu(R) \sqsubseteq \nu(S)$ and $\nu(R \parallel T) \not\sqsubseteq \nu(S \parallel T)$.

3. There are $R, S, T \in \mathcal{M}(X)$ such that $\nu(R) \sqsubseteq \nu(S)$ and $\nu(T \cdot R) \not\sqsubseteq \nu(T \cdot S)$.

Proof. 1. Let $R = \{(a, \{a\})\}$, $S = \emptyset$ and $T = \{(a, \emptyset)\}$. Then
\[ \tau(R) = \tau(S) = \tau(S \cdot T) = \emptyset \subseteq T = \tau(R \cdot T). \]

2. Let $R = \{(a, \emptyset)\}$, $S = \emptyset$ and $T = \{(a, \{a\})\}$. Then
\[ \nu(R) = \nu(S) = \nu(S \parallel T) = \emptyset \subseteq \nu(R \parallel T) = T. \]

3. Let $R = \{(a, \emptyset), (b, \{b\})\}$, $S = \{(b, \{b\})\}$ and $T = \{(a, \{a, b\})\}$. Then $\nu(R) = \nu(S) = S$, but $\nu(T \cdot R) = \{(a, \{b\})\} \supset \emptyset = \nu(T \cdot B)$. □

Similarly, we can define the relations
\[ x \sim_\tau y \iff \tau(x) = \tau(y) \quad \text{and} \quad x \sim_\nu y \iff \nu(x) = \nu(y). \]

Obviously, therefore, $x \sim_\tau y \iff \tau(x) = \tau(y)$ and $x \sim_\nu y \iff \nu(x) = \nu(y)$. The following facts then follow immediately from Lemma 11.7 and 11.10.

Corollary 11.11. In every c-lattice,
1. the relations $\sim_\tau$ and $\sim_\nu$ are equivalences,
2. $x \sim_\tau y$ implies $x + z \sim_\tau y + z$, $x \cap z \sim_\tau y \cap z$, $x \parallel z \sim_\tau y \parallel z$ and $z \cdot x \sim_\tau z \cdot y$, 

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Corollary 11.12. 1. There are $R, S, T \in \mathcal{M}(X)$ such that $\tau(R) = \tau(S)$ and $\tau(R \cdot T) \neq \tau(S \cdot T)$.

2. There are $R, S, T \in \mathcal{M}(X)$ such that $\nu(R) = \nu(S)$ and $\nu(R \parallel T) \neq \nu(S \parallel T)$.

3. There are $R, S, T \in \mathcal{M}(X)$ such that $\nu(R) = \nu(S)$ and $\nu(T \cdot R) \neq \nu(T \cdot S)$.

These results confirm our intuition about terminal and nonterminal elements. First, parallel composition with a nonterminal element yields a nonterminal element and sequential composition with a terminal element yields a terminal element, whereas sequential composition with a nonterminal element does not necessarily yield a nonterminal element and parallel composition with a terminal element does not necessarily yield a terminal element. Second, if two multirelations agree on their terminal elements, then their parallel compositions with a third element and their sequential composition from the left with a third element also yield a nonterminal element whereas sequential composition with a terminal element does not necessarily yield a nonterminal element, whereas sequential composition with a nonterminal element and parallel composition with a terminal element does not necessarily yield a terminal element.

12 c-Quantales and Finite Iteration

Iteration is best studied in a quantale setting where various fixpoints exist. In fact, in our Isabelle formalisation, many of the results in this and the following section are obtained in the weaker settings of c-Kleene algebras and c-$\omega$-algebras [11], but quantales provide a unifying generalisation. In addition, the least and greatest fixpoints corresponding to finite and infinite iteration exist in quantales, whereas they need to be postulated in the weaker algebras.

Let $(L, \leq)$ be a complete lattice. We write $\sum X$ for the supremum of the set $X \subseteq L$ and $\prod X$ for its infimum. In particular, we write $x + y$ for the binary supremum and $x \sqcap y$ for the binary infimum of $x, y \in L$. We write $U = \sum L$ for the greatest and $0 = \sum \emptyset$ for the least element of the lattice.

A proto-quantale is a structure $(Q, \leq, \cdot)$ such that $(Q, \leq)$ is a complete lattice and

$$x \leq y \Rightarrow z \cdot x \leq z \cdot y, \quad (\sum_{i \in I} x_i) \cdot y = \sum_{i \in I} (x_i \cdot y).$$

A proto-quantale is unital if it has a unit of multiplication $1$ which satisfies $1 \cdot x = x$ and $x \cdot 1 = x$. It is commutative if multiplication is: $x \cdot y = y \cdot x$ and distributive if the underlying lattice is distributive.

A quantale is a proto-quantale with associative multiplication, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$, and the following distributivity law holds:

$$x \cdot (\sum_{i \in I} y_i) = \sum_{i \in I} (x \cdot y_i).$$

A proto-bi-quantale is a structure $(Q, \leq, \cdot, \|, 1_\sigma, 1_\pi)$ such that $(Q, \leq, \cdot, 1_\sigma)$ is a unital proto-quantale and $(Q, \leq, \|, 1_\pi)$ a unital commutative quantale. Obviously, every pb-quantale is a proto-triod. A c-quantale is a proto-bi-quantale which is also a c-lattice.

Theorem 12.1. $(\mathcal{M}(X), \subseteq, \cdot, \|, 1_\sigma, 1_\pi)$ forms a boolean c-quantale.

All axioms except for the infinite distributivity laws with respect to sequential and parallel composition have either been verified in Proposition 12.2 or they follow directly from the underlying set structure. Verification of these distributivity laws is straightforward.

It follows from general fixpoint theory that all isotone functions on a quantale (as a continuous lattice) have least and greatest fixpoints. In addition, least fixpoints can be iterated from the least element of the quantale up to the first infinite ordinal whenever the underlying function is continuous, that is, it distributes with arbitrary suprema. Similarly, greatest fixpoints can be iterated from the greatest element of the quantale whenever the underlying function is co-continuous, which, however, is rarely the case.
Furusawa and Struth [10] have studied the least fixpoints of the function

\[ F_{RS} = \lambda X. S \cup R \cdot X \]

and its instance \( F_R = F_{R\lambda} = \lambda X. 1_\sigma \cup R \cdot X \) in the concrete case of multirelations. Here we study them abstractly in c-quantales. We write \( x^* = \mu F_x \) and \( x^* y = \mu F_{xy} \).

Our first statement expresses \( x^* \) in terms of its terminal and nonterminal parts, at least in a special case.

**Lemma 12.2.** In every c-quantale in which \( x \cdot (y \cdot z) = (x \cdot y) \cdot z \) holds for all elements,

\[ x^* = \nu(x)^* \cdot (1_\sigma + \tau(x)). \]

**Proof.** For \( x^* \leq \nu(x)^*(1_\sigma + \tau(x)) \) it suffices to show that that \( \nu(x)^* \cdot (1_\sigma + \tau(x)) \) is a pre-fixpoint of \( F_x \). In addition, \( \nu(x)^*(1_\sigma + \tau(x)) \leq x^* \) follows from results from [10], namely that \((x^* x^*) \leq x^*\) (because \( x^* + x \cdot x^* \leq x^* \)) and that \( x^* \cdot x^* \leq (x^* x^*) \) (by fixpoint fusion), whence \( x^* \cdot x^* \leq x^* \).

Next we characterise the terminal and nonterminal parts of \( x^* \).

**Lemma 12.3.** In every c-quantale,

1. \( \tau(x)^* = 1_\sigma + \tau(x) \),
2. \( \tau(x^*) = \nu(x^*) \cdot \tau(x) \) if \( x \cdot (y \cdot z) = (x \cdot y) \cdot z \),
3. \( \nu(x)^* \leq \nu(x^*) \),
4. \( \nu(\tau(x)^*) = 0 \) and \( \nu(\tau(x^*)) = \nu(x)^* \),
5. \( \tau(\tau(x)^*) = 1_\sigma \) and \( \tau(\tau(x^*)) = \tau(x) \).

Of course the lack of characterisation for \( \nu(x^*) \) owes to that for \( \nu(x \cdot y) \).

**Lemma 12.4.** There is a \( R \in \mathcal{M}(X) \) such that \( \nu(R^*) \not\subseteq \nu(R)^* \).

**Proof.** Let \( R = \{(a, \{b, c\}), (b, \emptyset), (c, \{d\})\} \). Then \((a, \{d\}) \in R^* \) and \((a, \{d\}) \in \nu(R^*) \), but \((a, \{d\}) \not\in \nu(R)^* \).

Next we relate \( x^* \) with notions of finite iteration. For multirelations, Peleg [28] has shown that \( F_R \) and \( F_{RS} \) are not necessarily continuous, whereas in the externally image finite case, where for each \((a, A) \in R\) the set \( A \) has finite cardinality,

\[ R^* = \mu F_R = F_R^*(\emptyset) = \bigcup_{i \in \mathbb{N}} F^i(\emptyset). \]

In addition, Furusawa and Struth [10] have defined

\[ R^{(0)} = \emptyset, \quad R^{(i+1)} = 1_\sigma \cup R \cdot R^{(i)}, \quad R^{(\ast)} = \bigcup_{i \in \mathbb{N}} R^{(i)} \]

and shown that, in the externally image finite case, \( R^* = R^{(\ast)} \). Finally, Goldblatt [14] has defined

\[ R^{[0]} = 1_\sigma, \quad R^{[n+1]} = 1_\sigma \cup R \cdot R^{[n]}, \quad R^{[\ast]} = \bigcup_{n \in \mathbb{N}} R^{[n]} \]

We now compare these iterations in the c-quantale setting. We call a c-quantale externally image finite if \( x^* = x^{(\ast)} \). First we prove a technical lemma.

**Lemma 12.5.** In every proto-oid,

1. \( x^{(n)} \leq x^{(n+1)} \),
2. \( x^{[n]} \leq x^{[n+1]} \),
3. \( (1_\sigma + x)^n \leq (1_\sigma + x)^{n+1} \),
4. \( (1_\sigma + x)^{n+1} = 1_\sigma + x \cdot (1_\sigma + x)^n \).

The next lemma shows that Goldblatt’s iteration coincides with Peleg’s, and it gives a simpler characterisation of the former.

**Lemma 12.6.** In every c-quantale,

1. \( x^{(*)} = x^{[\ast]} \),
2. \( x^{(*)} = \sum_{n \in \mathbb{N}} (1_\sigma + x)^n \),
3. \( \nu(x)^{(*)} = \sum_{n \in \mathbb{N}} (1_\sigma + \nu(x))^n \).

Obviously, (3) follows from (1) and (2). Therefore, the characterisation of \( x^* \) can sometimes be simplified in the externally image finite case.

**Corollary 12.7.** In every c-quantale in which \( x \cdot (y \cdot z) = (x \cdot y) \cdot z \) holds for all elements,

\[
x^{(*)} = \left( \sum_{n \in \mathbb{N}} (1_\sigma + \nu(x))^n \right) \cdot (1_\sigma + \tau(x)) = (1_\sigma + \tau(x))^\omega.
\]

### 13 c-Quantales and Infinite Iteration

This section studies three additional notions of iteration for multirelations: a unary strictly infinite iteration \( R^\omega \), a possibly infinite iteration \( R^\infty \) and a binary possibly infinite iteration \( R^\omega S \). These arise as the greatest fixpoints of the function

\[
F_{RS} = \lambda X. \ S \cup R \cdot X
\]

and its instances \( F_R = F_{RB} = \lambda X. \ R \cdot X \) and \( F_{RL} = \lambda X. \ 1_\sigma \cup R \cdot X \). More precisely, \( R^\omega = \nu F_R \), \( R^\infty = \nu F_{RL} \), and \( R^\omega S = \nu F_{RS} \) and these fixpoints exist due to the complete lattice structure of \( \mathcal{M}(X) \) and the fact that the three functions under consideration are isotone on that lattice.

As in the cases of the least fixpoints \( \mu F_R \) and \( \mu F_{RS} \), we wish to relate the greatest fixpoints. We can use the following fusion law for greatest fixpoints.

**Theorem 13.1.**

1. Let \( f \) and \( g \) be isotone functions; let \( h \) be a co-continuous function over a complete lattice. If \( h \circ g \geq f \circ h \), then \( h(\nu g) \geq \nu f \).

2. Let \( f, g \) and \( h \) be isotone functions over a complete lattice. If \( f \circ h \geq h \circ g \), then \( \nu f \geq h(\nu g) \).

**Corollary 13.2.** Let \( R, S \in \mathcal{M}(X) \). Then \( R^\omega \cup \mu F_{RS} \subseteq \nu F_{RS} \).

The proof uses Theorem 13.1. In fact, we have also proved a corresponding abstract statement in the context of proto-quantales with Isabelle/HOL [1]. However, the inequality is strict.

**Lemma 13.3.** There are \( R, S \in \mathcal{M}(X) \) such that \( \nu F_{RS} \neq R^\omega \cup \mu F_{RS} \).

**Proof.** Consider the multirelations \( R = \{(a, \{b, c\}), (b, \{a\})\} \) and \( S = \{(c, \{a\})\} \) over the set \( X = \{a, b, c\} \). Then \( R^\omega = \emptyset \) can be checked easily.

Moreover, \( R^\infty S = \{(a, \{a\}), (a, \{b, c\}), (b, \{a\}), (b, \{b, c\}), (c, \{a\})\} \). This can be checked by verifying \( S \cup R \cdot (R^\omega S) = (R^\infty S) \). It follows that

\[
R \cdot (R^\omega S) = \{(a, \{a\}), (a, \{b, c\}), (b, \{a\}), (b, \{b, c\})\}.
\]

Finally, \( R^\omega S = \{(a, S), (b, S), (c, a)\} \) for all \( S \subseteq X \), which can be checked by verifying \( S \cup R \cdot (R^\omega S) = (R^\infty S) \). In particular, \( R \cdot (R^\omega S) = \{(a, S), (b, S)\} \).
This counterexample is not related to the absence of associativity, but to the lack of left distributivity. There is therefore no hope that the situation can be resurrected for tests and modalities, as in the case of the star \[10\].

As a consequence, the greatest fixpoint \(R^\omega S\) cannot be reduced to a formula involving the greatest fixpoint \(R^\infty\). At the level of c-quantales we obtain the fixpoint axioms
\[
x^\omega y \leq y + x \cdot (x^\omega y), \quad z \leq y + x \cdot z \Rightarrow z \leq x^\omega y.
\]
Since \(F_R = F_{R^\emptyset}\) by definition, this yields \(x^\omega = x^\omega 0\) and the following unary \(\omega\)-unfold and \(\omega\)-coinduction axioms as special cases:
\[
x^\omega \leq x \cdot x^\omega, \quad y \leq x \cdot y \Rightarrow y \leq x^\omega.
\]
These can be used for deriving the following properties.

**Lemma 13.4.** For every c-quantale,
1. \(x^\omega = x \cdot x^\omega\),
2. \(x \leq y \Rightarrow x^\omega \leq y^\omega\).

**Lemma 13.5.** For every c-quantale,
1. \(0^\omega = 0\),
2. \(1^\omega_\tau = 1_\tau\),
3. \(1^\omega_\sigma = T^\omega_\sigma = U^\omega = U\).

The fact that \(1^\omega_\tau\) and \(T^\omega_\tau\) are \(U\), instead of \(T\), is not entirely satisfactory: one would not assume that the iteration of an element \(M(S)\) leads outside of this set.

The characterisation of terminal parts of elements is still satisfactory, whereas for nonterminal elements, the lack of characterisation for products makes the situation less pleasant.

**Lemma 13.6.** In every c-quantale,
1. \(\tau(x) \leq \tau(x^\omega)\),
2. \(\tau(x)^\omega = \tau(x)\),
3. \(\tau(x)^\omega \leq \tau(x^\omega)\).

**Lemma 13.7.** In every c-quantale, \(\nu(x)^\omega + \nu(x)^* \cdot \tau(x) \leq x^\omega\).

The converse implication is ruled out by the counterexample in Lemma \[13.3\]

Next we briefly compare \(F_{R^\infty}\) with \(F_{R^S}\), that is, \(R^\infty\) with \(R^\omega S\).

**Lemma 13.8.** There are \(R, S \in \mathcal{M}(X)\) such that \((R^\omega S) \neq R^\infty \cdot S\).

*Proof.* In the above counterexample, \(R^\omega S = \{(a, S), (b, S), (c, a)\}\) for all \(S \subseteq X\). However, \(R^\infty = R^* = \{(a, \{a\}), (a, \{b, c\}), (b, \{b\}), (b, \{a\}), (b, \{b, c\})\}\) and therefore \(R^\infty \cdot S = R^* \cdot S = \emptyset\). \(\square\)

Here we cannot even obtain \(R^\infty \cdot S \subseteq (R^\infty S)\) by fixpoint fusion, since this requires co-continuity of \(\lambda x. x \cdot S\), that is, \((\bigcap_{i \in I} R_i) \cdot S \subseteq \bigcap_{i \in I} (R_i \cdot S)\), which does not hold. Hence this time the failure of equality owes to the lack of associativity and co-continuity. Note that \((R^\infty S)\) need not be equal to \(R^\infty \cdot S\) for binary relations for similar reasons. In this case, \(x^\infty = x^\omega 1_\infty\) by definition of \(F_{R^\infty}\) and \(F_{R^S}\), which yields the unary laws
\[
x^\infty \leq 1_\infty \cdot x \cdot x^\infty, \quad y \leq 1_\infty + x \cdot y \Rightarrow y \leq x^\infty.
\]

Finally, following \[7\], we study a notion of greatest fixpoint on c-quantales which models the set of all states in \(x\) from which either \(\emptyset\) is reachable or infinite \(x\)-chains start. Obviously, the set of domain elements

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is the complete distributive lattice or boolean subalgebra of the sequential subidentities. The function \( \lambda p. q + \langle x \rangle p \) is isitone and thus has a least (binary) fixpoint \( \langle x \rangle^* q \) which, by the results of [7], is equal to \( \langle x \rangle^* q \). In the context of c-monoids, \( \langle x \rangle p = d(x \cdot p) \).

For the same reasons, the function \( \lambda p. \langle x \rangle p \) has a greatest fixpoint which we denote \( \nabla x \). It satisfies the unfold and coinduction axiom
\[
\nabla x \leq \langle x \rangle \nabla x, \quad p \leq \langle x \rangle p \Rightarrow p \leq \nabla x.
\]

We can use fixpoint fusion again and try to derive the rule
\[
p \leq \langle x \rangle p + q \Rightarrow p \leq \nabla x + \langle x \rangle^* q.
\]

We must instantiate \( f = \lambda y. \langle x \rangle y + q, \ g = \lambda y. \langle x \rangle y \) and \( h = \lambda y. \langle x \rangle^* q \). Then
\[
h(g(y)) = \langle x \rangle y + \langle x \rangle^* q = \langle x \rangle y + q + \langle x \rangle \langle x \rangle^* q \leq \langle x \rangle (y + \langle x \rangle^* q) + q = f(h(y)).
\]

Then greatest fixpoint fusion yields once more
\[
\nabla x + \langle x \rangle^* q \leq \nu f,
\]

but the above counterexample excludes equality without \( \langle x \rangle (p + q) = \langle x \rangle p + \langle x \rangle q \).

In general, \( \nabla x \) is the largest subidentity \( p \) which satisfies \( p = d(x \cdot p) \). Hence \( p \) models the largest set of states from which executing \( x \) either leads to \( \emptyset \), or it leads back into \( p \) in the following sense. For each element in \( p \) there exists a set \( A \) which is reachable via \( X \), and all elements of \( A \) are in \( p \). This means that from all states in \( p \) indeed either infinite executions with \( x \) are possible or \( \emptyset \) is reachable.

In this case, at least an explicit definition of \( \nu(x)^\omega \) is possible.

**Proposition 13.9.** Let \( S \) be a c-quantale. If \( x \cdot (d(y) \cdot z) = (x \cdot d(y)) \cdot z \) for \( x, y, z \in S \), then
\[
\nu(x)^\omega = \nabla(\nu(x)) \cdot U.
\]

As in Lemma [3.3], the infinite iteration of nonterminal elements contains terminal parts, which seems undesirable, but unavoidable in this context.

**Corollary 13.10.** Let \( S \) be a c-quantale. If \( x \cdot (d(y) \cdot z) = (x \cdot d(y)) \cdot z \) for all \( x, y, z \in S \), then
\[
\nabla(\nu(x)) \cdot U + \nu(x)^\omega \cdot \tau(x) \leq x^\omega.
\]

At least for nonterminal multirelations, the situation is the same. In the general case, terminal parts contribute to infinite iteration, too. The absence of associativity and distributivity makes the situation more complicated. Separation of infinite iterations into terminating and nonterminating parts yields at least an underapproximation. Using \( \nabla \) also yields sharper properties for nonterminal elements.

**Lemma 13.11.** Let \( S \) be a c-quantale. If \( x \cdot (d(y) \cdot z) = (x \cdot d(y)) \cdot z \) for all \( x, y, z \in S \), then
\[
\begin{align*}
1. \quad & \nu(\nu(x)^\omega) = \nabla(\nu(x)) \cdot T_x, \\
2. \quad & \tau(\nu(x)^\omega) = \nabla(\nu(x)) \cdot 1_x.
\end{align*}
\]

Lemma [3.11(2)] clearly shows that the infinite iteration of a nonterminal multirelation has a terminal and a nonterminal part. Hence there is no direct relationship between strictly infinite iteration and terminal or nonterminal elements. This is in contrast to the language case, where \( \inf \) models infinite or divergent behaviour. The latter, according to Lemma [3.4], is captured by \( \nabla \) and or \( U \) in the multirelational model, which is similar to the relational case. As suggested by alternating automata, terminal parts of a multirelation rather model success or failure states, or winning or loosing states in a game based scenario, but not nontermination.

The final part of this section sets up the correspondence between infinite iteration and notions of deflationarity and wellfoundedness, as they have been studied in the relational model by [31]. We call an element \( x \) \( \omega \)-trivial if \( x^\omega = 0 \), deflationary if \( \forall y. (y \leq x \cdot y \Rightarrow y = 0) \) and wellfounded if \( \forall y. (d(y) \leq d(x \cdot y) \Rightarrow d(y) = 0) \).
Proposition 13.12. Let \( x \) be an element of a c-quantale. The following statements are equivalent:

1. \( x \) is wellfounded;
2. \( x \) is deflationary;
3. \( x \) is \( \omega \)-trivial.

In this respect, the behaviour of relations and multirelations is the same.

14 Counterexamples

This section collects some counterexamples for multirelations. The second set, in particular, rules out variants of interchange laws, as they arise for instance in the context of monoidal categories, in shuffle languages or for partially ordered multisets [2, 13, 16]. More abstractly, such laws have been considered in concurrent Kleene algebras [15].

Lemma 14.1. There exist multirelations \( R, S, T \in \mathcal{M}(X) \) such that

1. \( R \parallel R \not\subseteq R \),
2. \( R \not\subseteq R \parallel S \),
3. \( R \parallel S \cap R \parallel T \not\subseteq R \parallel (S \cap T) \),
4. \( (R \cdot S) \parallel T \not\subseteq R \cdot (S \cdot T) \),
5. \( (R \cdot T) \parallel (S \cdot T) \not\subseteq (R \parallel S) \cdot T \),
6. \( (R \cdot S) \parallel (R \cdot T) \not\subseteq R \cdot (S \parallel T) \), even for \( R \parallel R \subseteq R \), \( S \parallel S \subseteq S \) and \( T \parallel T \subseteq T \).

Proof. 1. Let \( R = \{(a, \{a\}), (b, \{b\})\} \). Then \( R \parallel R = \{(a, \{a\}), (b, \{b\}), (a, \{a, b\})\} \not\subseteq R \).

2. Let \( R = \{(a, \{a\})\} \) and \( S = \{(a, \{a, b\})\} \). Then \( R \not\subseteq S = R \parallel S \).

3. Let \( R = \{(a, \{b, c\})\}, S = \{(a, \{b\})\} \) and \( T = \{(a, \{c\})\} \). Then
\[
R \parallel S \cap R \parallel T = R \parallel S = R \parallel T = R \cap \emptyset = S \cap T = R \parallel (S \cap T).
\]

4. A counterexample has again been given in [10].

5. A counterexample has again been given in [10].

6. Let \( R = \{(a, \{a\}), (a, \emptyset)\}, S = \{(a, \{a\})\} \) and \( T = \emptyset \). Then
\[
R \cdot (S \parallel T) = \{(a, \emptyset)\} \subset \{(a, \{a\}), (a, \emptyset)\} = (R \cdot S) \parallel (R \cdot T).
\]

It is straightforward to check that \( R \parallel R \subseteq R \), \( S \parallel S \subseteq S \) and \( T \parallel T \subseteq T \).

Our next counterexamples explains the difference between algebras of multirelations and concurrent Kleene algebras. The latter are based on a full sequential dioid and a commutative one, with shared units of sequential and concurrent composition. The sequentiality-concurrency interaction is captured by the interchange law
\[
(w \parallel x) \cdot (y \parallel z) \leq (w \cdot y) \parallel (x \cdot z).
\]

From this law, the small interchange laws \( x \parallel (y \cdot z) \leq x \parallel ((y \cdot z)) \), \( x \cdot (y \parallel z) \leq (x \cdot y) \parallel z \) and \( x \cdot y \leq x \parallel y \) are derivable in the presence of a shared unit for sequential and concurrent composition. These laws hold, in particular, in certain pomset languages and in word languages under the regular operations and with concurrent composition interpreted as shuffle. However, the next lemma refutes all variants of interchange between sequential and concurrent composition. We write \( R \succ S \) if \( R \not\subseteq S \) and \( R \not\supseteq S \).
Lemma 14.2. There exist \( R, S, T, U \in \mathcal{M}(X) \) such that

1. \((R \parallel S) \cdot (T \parallel U) \simeq (R \cdot T) \parallel (S \cdot U)\),
2. \((R \parallel S) \cdot T \simeq R \parallel (S \cdot T)\),
3. \(R \cdot (S \parallel T) \simeq (R \cdot S) \parallel T\),
4. \(R \cdot S \simeq R \parallel S\).

Proof. 1. Let \( R = \{(a, \{a\}), (b, \{a, b\})\} \) and \( S = \{(a, \{a\}), (b, \{a\})\} \). Then
\[
(R \parallel S) \cdot (R \parallel S) = \{(a, \{a, b\}), (b, \{a, b\})\} \simeq \{(a, \{a\})\} = (R \cdot R) \parallel (S \cdot S).
\]

2. Let \( R = \{(a, \{a, b\})\} = S \) and \( T = \{(a, \{a\}), (b, \{a\})\} \). Then
\[
(R \parallel R) \cdot T = R \cdot T = \{(a, \{a\})\} \simeq R = R \parallel (R \cdot T).
\]

3. Let \( R = \{(a, \{a, b\})\}, S = \{(a, \{a\}), (b, \{a\})\} \) and \( T = \{(a, \{a\}), (b, \{a, b\})\} \). Then
\[
R \cdot (S \parallel T) = R \cdot T = R \simeq \{(a, \{a\})\} = (R 
\cdot S) \parallel T.
\]

4. Let \( R = \{(a, \{a, b\})\} \) and \( S = \{(a, \{a\}), (b, \{a\})\} \). Then
\[
R \cdot S = \{(a, \{a\})\} \simeq \{(a, \{a, b\})\} = R \parallel S.
\]

\[\square\]

15 Remarks on Up-Closed Multirelations

This section briefly discusses the relationship between general multirelations à la Peleg and the up-closed multirelations studied by Parikh and others. As explained in Section 3, a multirelation is up-closed if \((a, A) \in R \) and \( A \subseteq B \) imply \((a, B) \in R\). This is the case if and only if \( R = R \parallel U \).

For this special case, Parikh has defined sequential composition as
\[(a, A) \in R; S \Leftrightarrow \exists B. (a, B) \in R \land \forall b \in B. (b, A) \in S.\]
This corresponds to Peleg’s definition with \( f \) instantiated to \( \lambda x. A \). Peleg’s definition, however, does not preserve up-closure. Let \( R' = \{(a, \emptyset)\}, R = R' \parallel U \) and \( S = \emptyset \). Then \( R, S \in \mathcal{U}(X) \), whereas \( R \cdot S = R' \notin \mathcal{U}(X) \). Hence Peleg’s definition does not simply specialise to Parikh’s when multirelations are up-closed. The next lemma captures the precise relationship.

Lemma 15.1. Let \( R, S \in \mathcal{M}(X) \). Then \( R; (S \parallel U) = (R \cdot S) \parallel U \).

\[
\mathcal{M}(X) \times \mathcal{M}(X) \xrightarrow{\cdot} \mathcal{M}(X) \xrightarrow{\parallel} \mathcal{M}(X) \xrightarrow{\parallel} \mathcal{U}(X)
\]

Consequently, Parikh’s sequential composition preserves up-closure,
\[(R; (S \parallel U)) \parallel U = (R \cdot S) \parallel U \parallel U = (R \cdot S) \parallel U = R; (S \parallel U),\]
and join. Moreover, Parikh composition of up-closed multirelations can be simulated by Peleg’s as \((R\parallel U); (S\parallel U) = ((R\parallel U) \cdot (S\parallel U))\parallel U\).

Parikh composition of up-closed multirelations is associative. Hence it might be possible to derive this property in c-lattices, but it does not seem straightforward, and specific associativity laws for Peleg’s composition in the presence of up-closed elements might be needed. A deeper investigation is left for future work.

The case of parallel composition in the up-closed case is very simple in comparison. Rewitzky and Brink [30] have studied parallel composition of up-closed multirelations under the name power union and shown that it simply yields a contrived definition of meet. We can reproduce this result in the setting of c-lattices.

**Lemma 15.2.** In every c-lattice,

1. \(x\parallel U \cap y\parallel U = (x\parallel U)\parallel (y\parallel U)\),
2. \((x\parallel U)\parallel (x\parallel U) = x\parallel U\),
3. \((x\parallel y)\cdot (z\parallel U) = (x\cdot (z\parallel U))\parallel (y\cdot (z\parallel U))\).

**Corollary 15.3.** If \(R, S \in \mathcal{M}(X)\) are up-closed, then \(R\parallel S = R \cap S\).

By this result, up-closed multirelations are also meet closed, and therefore closed under the usual operations, which is well known.

Our results for c-lattices and c-quantales hold automatically in the up-closed case, interpreting parallel composition as intersection and translating Peleg’s sequential composition into Parikh’s. In the up-closed case, in particular, the interaction of sequential and parallel composition becomes a sub-distributivity law for sequential composition over meet, which follows directly from the greatest lower bound properties of meet and isotonicity of sequential composition.

However there are differences as well. First, \(1_\pi\parallel U = U\), hence the up-closure of the unit of parallel composition is \(U\), which is consistent with \(U\) being the unit of meet. More generally, up-closure turns parallel subidentities into vectors and therefore \(\mathcal{F}(X) = \mathcal{V}(X)\) in that context. Second, the definitions of subidentities, of domain as \(d(R) = (R\cdot 1_\pi)\parallel 1_\sigma\), and of the corresponding box and diamond modalities, do not carry over directly to the up-closed case. In particular, the definitions of the sequential unit and of sequential subidentities are now based on the \(\varepsilon\)-relation instead of the function \(\lambda x.\{x\}\). We leave a reconstruction of the subalgebra of up-closed elements in the context of c-lattices for future work as well. In particular, Parikh’s game logic can be based on the domain and antdomain axioms for pre-dioids given in [8] and linked with concepts from previous sections.

A duality between up-closed multirelations and sets of isotone predicate transformers has already been noticed by Parikh [24]. By this isomorphism, sequential composition of up-closed multirelations is associated with function composition of monotone predicate transformers. Obviously, a similar isomorphism between Peleg’s multirelations and a class of predicate transformers cannot exist since sequential composition of multirelations is not associative. Otherwise, a non-associative operation would have to be defined on predicate transformers, which may not lead to a natural concept.

### 16 Conclusion

We have investigated the structure and algebra of multirelations, which model alternation or dual nondeterminism in a relational setting, and which form the semantics of Peleg’s concurrent dynamic logic, extending a previous algebraic approach to concurrent dynamic algebra. Apart from the derivation of a considerable number of algebraic properties which arise from the union, intersection, sequential and parallel composition, and finite and infinite iteration of multirelations, we have also studied the structure of various subalgebras and the relationships between them, as illustrated in the diagrams of Figure 2. In particular we found that a domain operation, which is important for these investigations, can be defined explicitly in this setting.
The operations on multirelations are rather complex; their interactions are intricate. Sequential composition, for instance, requires a higher-order definition, its manipulation often depends on the axiom of choice. Algebraic axioms similar to Tarski's relation algebra are therefore desirable to hide this complexity. To address this we have developed algebraic axiom systems ranging from c-monoids to c-quantales and carried out most of our work at that level. At the moment, these axiom systems are less compact than those for up-closed multirelations or even relation algebra. It seems that much of the power of relation algebra comes from the operation of conversion, to which we do not know a multirelational counterpart. There is certainly scope for completing, revising and perhaps simplifying our axioms.

Hence, from a mathematical point of view, more concise and comprehensive axiom systems seem desirable and questions such as representability, axiomatisability, and decidability at least of fragments are interesting—the class of representable relation algebras, which are isomorphic to algebras of binary relations, is not finitely axiomatisable [19] and a similar result might be expected here. Other directions for research include the investigation of the up-closed case in relationship with c-lattices and c-quantales, the study of other classes of multirelations in which sequential composition is associative, the algebraic reconstruction of Parikh’s game logic and the association of multirelations with predicate transformer algebras. Beyond that, a reference formalisation of multirelational algebras in Isabelle, including their modal variants such as dynamic and game logics, has been developed in parallel to this article [11]. Its use in the formal analysis of games and the verification of computing systems with dual nondeterminism are next steps towards taming multirelations.

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