On product-sum of triangular fuzzy numbers *

Robert Fullér
rfuller@abo.fi

Abstract
We study the problem: if \( \tilde{a}_i, \ i \in \mathbb{N} \) are fuzzy numbers of triangular form, then what is the membership function of the infinite (or finite) sum \( \tilde{a}_1 + \tilde{a}_2 + \cdots \) (defined via the sup-product-norm convolution)?

Keywords: triangular fuzzy number, product-sum

1 Definitions

A fuzzy number is a convex fuzzy subset of the real line \( \mathbb{R} \) with a normalized membership function. A triangular fuzzy number \( \tilde{a} \) denoted by \((a, \alpha, \beta)\) is defined as

\[
\tilde{a}(t) = \begin{cases} 
1 - \frac{a-t}{\alpha} & \text{if } a - \alpha \leq t \leq a \\
1 & \text{if } a \leq t \leq b \\
1 - \frac{t-b}{\beta} & \text{if } a \leq t \leq b + \beta \\
0 & \text{otherwise}
\end{cases}
\]

where \( a \in \mathbb{R} \) is the centre and \( \alpha > 0 \) is the left spread, \( \beta > 0 \) is the right spread of \( \tilde{a} \).

If \( \alpha = \beta \), then the triangular fuzzy number is called symmetric triangular fuzzy number and denoted by \((a, \alpha)\).

If \( \tilde{a} \) and \( \tilde{b} \) are fuzzy numbers, then their product-sum \( \tilde{a} + \tilde{b} \) is defined as in [2]

\[
(\tilde{a} + \tilde{b})(z) = \sup_{x+y=z} \tilde{a}(x)\tilde{b}(y).
\]

The support \( \text{supp}\tilde{a} \) of a fuzzy number \( \tilde{a} \) is defined as

\[
\text{supp}\tilde{a} = \{ t \in \mathbb{R} | \tilde{a}(t) > 0 \}.
\]

2 Product-sum of triangular fuzzy numbers

In this paper we shall calculate the membership function of the product-sum $\tilde{a}_1 + \tilde{a}_2 + \cdots + \tilde{a}_n + \cdots$ where $\tilde{a}_i, i \in \mathbb{N}$ are fuzzy numbers of triangular form.

The results of this paper are linked with those presented in [1,p.933] and extend them. The following theorem can be interpreted as a central limit theorem for mutually product-related identically distributed fuzzy variables of symmetric triangular form (see [3]).

**Theorem 2.1** Let $\tilde{a}_i = (a_i, \alpha), i \in \mathbb{N}$. If

$$A := \sum_{i=1}^{\infty} a_i$$

exists and it is finite, then with the notations

$$\tilde{A}_n := \tilde{a}_1 + \cdots + \tilde{a}_n, A_n := a_1 + \cdots + a_n, n \in \mathbb{N},$$

we have

$$\left( \lim_{n \to \infty} \tilde{A}_n \right)(z) = \exp(-|A - z|/\alpha), z \in \mathbb{R}.$$  

**Proof.** It will be sufficient to show that

$$\tilde{A}_n(z) = \begin{cases} 
\left[ 1 - \frac{|A_n - z|}{n\alpha} \right]^n & \text{if } |A_n - z| \leq n\alpha \\
0 & \text{otherwise} 
\end{cases} \quad (1)$$

for each $n \geq 2$, because from (1) it follows that

$$\left( \lim_{n \to \infty} \tilde{A}_n \right)(z) = \lim_{n \to \infty} \left[ 1 - \frac{|A_n - z|}{n\alpha} \right]^n = \exp(- | \lim_{n \to \infty} A_n - z | /\alpha) = \exp(- | A - z | /\alpha), z \in \mathbb{R}.$$  

From the definition of product-sum of fuzzy numbers it follows that

$$\text{supp}\tilde{A}_n = \text{supp}(\tilde{a}_1 + \cdots + \tilde{a}_n) = \text{supp}\tilde{a}_1 + \cdots + \text{supp}\tilde{a}_n = \left[ a_1 - \alpha, a_1 + \alpha \right] + \cdots + \left[ a_n - \alpha, a_n + \alpha \right] = [A_n - n\alpha, A_n + n\alpha], n \in \mathbb{N}.$$  

We prove (1) by making an induction argument on $n$. Let $n = 2$. In order to determine $\tilde{A}_2(z), z \in [A_2 - 2\alpha, A_2 + 2\alpha]$ we need to solve the following mathematical programming problem:

$$\left( 1 - \frac{|a_1 - x|}{\alpha} \right) \left( 1 - \frac{|a_2 - y|}{\alpha} \right) \rightarrow \max$$

subject to $|a_1 - x| \leq \alpha$, $|a_2 - y| \leq \alpha, x + y = z$.  

2
By using Lagrange’s multipliers method and decomposition rule of fuzzy numbers into two separate parts (see [2]) it is easy to see that $\tilde{A}_2(z)$, $z \in [A_2 - 2\alpha, A_2 + 2\alpha]$ is equal to the optimal value of the following mathematical programming problem:

$$
\left(1 - \frac{a_1 - x}{\alpha}\right)\left(1 - \frac{a_2 - z + x}{\alpha}\right) \rightarrow \max
$$

subject to $a_1 - \alpha \leq x \leq a_1,$

$$a_2 - \alpha \leq z - x \leq a_2, x + y = z.$$

Using Lagrange’s multipliers method for the solution of (2) we get that its optimal value is

$$\left[1 - \frac{|A_2 - z|}{2\alpha}\right]^2$$

and its unique solution is

$$x = 1/2(a_1 - a_2 + z)$$

(where the derivative vanishes).

Indeed, it can be easily checked that the inequality

$$\left[1 - \frac{|A_2 - z|}{2\alpha}\right]^2 \geq 1 - \frac{A_2 - z}{\alpha}$$

holds for each $z \in [A_2 - 2\alpha, A_2]$.

In order to determine $\tilde{A}_2(z)$, $z \in [A_2, A_2 + 2\alpha]$ we need to solve the following mathematical programming problem:

$$
\left(1 + \frac{a_1 - x}{\alpha}\right)\left(1 + \frac{a_2 - z + x}{\alpha}\right) \rightarrow \max
$$

subject to $a_1 \leq x \leq a_1 + \alpha,$ $a_2 \leq z - x \leq a_2 + \alpha.$

In a similar manner we get that the optimal value of (3) is

$$\left[1 - \frac{|z - A_2|}{2\alpha}\right]^2.$$

Let us assume that (1) holds for some $n \in \mathbb{N}$. By similar arguments we obtain

$$\tilde{A}_{n+1}(z) = (\tilde{A}_n + \tilde{a}_{n+1})(z) =$$

$$\sup_{x+y=z} \tilde{A}_n(x) \cdot \tilde{a}_{n+1}(y) = \sup_{x+y=z} \left(1 - \frac{|A_n - x|}{n\alpha}\right)\left(1 - \frac{|a_{n+1} - y|}{\alpha}\right) =$$

$$\left[1 - \frac{|A_{n+1} - z|}{(n+1)\alpha}\right]^{n+1}, z \in [A_{n+1} - (n+1)\alpha, A_{n+1} + (n+1)\alpha],$$

and

$$\tilde{A}_{n+1}(z) = 0, z \notin [A_{n+1} - (n+1)\alpha, A_{n+1} + (n+1)\alpha].$$

This ends the proof.

Figure 2 gives a graphical illustration of the limiting distribution.

The proof of the following theorem carried out analogously to the proof of the preceding theorem.
Figure 2: The limit distribution of the product-sum $\tilde{a}_1 + \cdots + \tilde{a}_n + \cdots$.

**Theorem 2.2** Let $\tilde{a}_i = (a_i, \alpha, \beta)$, $i \in \mathbb{N}$ be fuzzy numbers of triangular form. If $A := \sum_{i=1}^{\infty} a_i$ exists and it is finite, then with the notations of Theorem 1 we have

$$
\left( \lim_{n \to \infty} \tilde{A}_n \right)(z) = \begin{cases} 
\exp \left( -\frac{|A - z|}{\alpha} \right) & \text{if } z \leq A \\
\exp \left( -\frac{|A - z|}{\beta} \right) & \text{if } z \geq A
\end{cases}
$$

Figure 3: Product-sum of $n$ triangular fuzzy numbers.

3 Question

Let $\tilde{a}_i = (a_i, \alpha, \beta)_{LR}$, $1 \leq i \leq n$ be fuzzy numbers of LR-type. On what condition will the membership function of the product-sum $\tilde{A}_n$ have the following form

$$
\tilde{A}_n(z) = \begin{cases} 
L^n \left( \frac{A_n - z}{n\alpha} \right) & \text{if } A_n - n\alpha \leq z \leq A_n, \\
R^n \left( \frac{z - A_n}{n\beta} \right) & \text{if } A_n \leq z \leq A_n + n\beta
\end{cases}
$$

4 Follow ups

The results of this paper have been extended and improved in the following papers.

References


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in journals


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In 1991, Fullér calculated the membership function of the product-sum of triangular fuzzy numbers, and he asked for conditions on which the product-sum of L-R fuzzy numbers has the same membership function. The answer for this question was given by Triesch [2] and Hong [3], which is the conditions that log L and log R are concave functions. Recently, Houg and Hwang [1] determined the exact membership function of the t-norm-based sum of fuzzy numbers, in the case of Archimedean t-norm having convex additive generator function and fuzzy numbers with concave shape functions, which is the generalization of Fullér and Keresztfalvi’s result [4]. The purpose of this paper is to study the membership function of the t-norm-based sum of fuzzy numbers on Banach spaces, which generalizes earlier results by Fullér [5] and Hong and Hwang [1]. The idea follows from Hong and Hwang’s paper [1]. (page 129)
Triesch (1993) provided a partial answer to Fullér’s (1991) question about the membership function of the finite sum (defined via sup-product-norm convolution) of L-R fuzzy numbers. In this short note, we prove the other half. (page 381)

Fullér [A32] asks for conditions on L-R fuzzy numbers \( \tilde{a}_i = (a_i, \alpha, \beta)_{LR}, i = 1, 2, \ldots, n \) which imply that partial product sums are given by the formula

\[
\tilde{A}_n(z) = \begin{cases} 
L^n \left( \frac{A_n - z}{n\alpha} \right) & \text{if } A_n - n\alpha \leq z \leq A_n, \\
R^n \left( \frac{z - A_n}{n\beta} \right) & \text{if } A_n \leq z \leq A_n + n\beta
\end{cases}
\]

where \( A_n = \sum_{i=1}^{n} a_i \). (page 381)
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