On the Oscillation of Certain Neutral Functional Differential Equations

By

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1. Introduction

In this note we consider the neutral functional differential equation

\[
\frac{d^2}{dt^2} (x(t) + \lambda x[t - \tau]) = q(t)x[t - \sigma] + p(t)x[t + \beta],
\]

where \( p, q: [t_0, \infty) \rightarrow [t_0, \infty) \) are continuous and periodic of period \( \tau \); \( \lambda, \tau \) and \( \beta \) are constants, \( \lambda > 0 \) and \( \tau > 0 \). As usual, a solution of equation (1) is called oscillatory if it has arbitrarily large zeros and nonoscillatory if it is eventually positive or eventually negative. Equation (1) is called oscillatory if all of its solutions are oscillatory.

Recently, there has been an increasing interest in establishing computable sufficient conditions for the oscillation of solutions of neutral functional differential equations, see, for example ([1]–[6] and [9]–[12]). Mostly the literature is devoted to the study of first order neutral differential equations, see ([5], [6], [9] and [11]) and the references cited therein, and only a few oscillation results are known for higher order neutral differential equations, see ([1]–[4], [10] and [12]).

The main purpose of this paper is to establish some easily verifiable sufficient conditions for the oscillation of the second order equation (1) with advanced and retarded arguments.

2. Main results

First we discuss the asymptotic behavior of nonoscillatory solutions of the neutral differential equation

\[
\frac{d^2}{dt^2} (x(t) + \lambda x[t - \tau]) = q(t)x[t - \sigma],
\]

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where $q$, $\lambda$, $\tau$ and $\sigma$ are as defined in equation (1).

**Theorem 1.** Every nonoscillatory solution $x$ of equation (2) is such that either $x^{(i)}(t) \to \infty$ or $x^{(i)}(t) \to 0$ as $t \to \infty$, $i = 0, 1$.

**Proof.** Let $x(t)$ be a nonoscillatory solution of equation (2) say

\[ x(t) > 0, \quad x[t - \tau] > 0 \quad \text{and} \quad x[t - \sigma] > 0 \quad \text{for} \quad t \geq t_0. \]

Set

\[ z(t) = x(t) + \lambda x[t - \tau] \quad \text{and} \quad w(t) = z(t) + \lambda z[t - \tau]. \]

Then

\[ z(t) > 0 \quad \text{and} \quad w(t) > 0 \quad \text{for} \quad t \geq t_0 \]

and

\[ z'(t) = q(t)x[t - \tau] \quad \text{for} \quad t \geq t_0, \]

which implies that the function $z'(t)$ is of one sign on $[T, \infty)$ for some $T \geq t_0$. From definition of $w$ and the fact that $q$ is $\tau$-periodic we have

\[ \frac{d^2}{dt^2} (w(t) + \lambda w[t - \tau]) = q(t)w[t - \sigma] \quad \text{(3)} \]

and

\[ w''(t) = q(t)z[t - \sigma]. \quad \text{(4)} \]

Now we consider the following two cases:

**Case 1:** $z'(t) > 0$ on $[T, \infty)$. Then there exists a $T_1 \geq T$ such that $w'(t) > 0$ for $t \geq T_0$. From (4) we get

\[ \frac{d^2}{dt^2} w[t - \tau] = w''[t - \tau] \]

\[ = q(t)z[t - \tau - \sigma] \]

\[ \leq q(t)z[t - \tau] \]

\[ = w''(t). \]

Then

\[ w''(t) \geq \frac{q(t)}{1 + \lambda} w[t - \sigma] \quad \text{for} \quad t \geq T_1. \quad \text{(5)} \]

Integrating (5) from $T_1$ to $t$, we obtain
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\[ w'(t) \geq w'(T_1) + \int_{T_1}^{t} \frac{q(s)}{1 + \lambda} w[s - \sigma] ds \]
\[ \geq w'(T_1) + w[T_1 - \sigma] \int_{T_1}^{t} \frac{q(s)}{1 + \lambda} ds \to \infty \quad \text{as } t \to \infty . \]

Which implies that \( x^{(i)}(t) \to \infty \) as \( t \to \infty \), \( i = 0, 1 \).

**Case 2.** \( z'(t) < 0 \) on \([T, \infty)\). There exists a \( T_2 \geq T \) such that \( w'(t) < 0 \) for \( t \geq T_2 \). Now from (4) we have

\[ w''[t - \tau] = q(t)z[t - \tau - \sigma] \geq q(t)z[t - \sigma] = w''(t). \]

Thus,

\[ w''[t - \tau] \geq \frac{q(t)}{1 + \lambda} w[t - \sigma] \]

or

\[ w''(t) \geq \frac{q(t)}{1 + \lambda} w[t - (\sigma - \tau)] \quad \text{for } t \geq T_2 . \]

Since

\[ w''(t) > 0 , \quad w'(t) < 0 \quad \text{and } w(t) > 0 \quad \text{for } t \geq T_1 \]

one can easily see that

\[ w^{(i)}(t) \to 0 \quad \text{as } t \to \infty , \quad i = 0, 1 \]

which implies

\[ x^{(i)}(t) \to 0 \quad \text{as } t \to \infty , \quad i = 0, 1 . \]

This completes the proof.

The following example is illustrative.

**Example 1.** Consider the neutral delay differential equation

\[ \frac{d^2}{dt^2} (x(t) + \lambda x[t - \tau]) = qx[t - 3] , \]

where \( \lambda \) and \( q \) are positive constants. By Theorem 1, every solution \( x \) of equation (7) is such that either \( x^{(i)}(t) \to 0 \) as \( t \to \infty \) or \( x^{(i)}(t) \to \infty \) as \( t \to \infty \),
We note that if $q = e^3 + \lambda e$, equation (7) has a nonoscillatory solution $x(t) = e^t$ and if $q = e^{-3} + \lambda e^{-1}$, equation (7) has a nonoscillatory solution $x(t) = e^{-t}$.

Theorems 2 and 3 below provide sufficient conditions for the oscillation and convergence to zero of every solution of equation (2).

**Theorem 2.** If $\sigma < 0$ and

$$\limsup_{t \to \infty} \int_{t-\sigma}^{t} (s-t)^k(t-s-\tau)^{1-k} \frac{q(s)}{1+\lambda} ds > 1$$

for some $k = 0, 1$, then every solution of equation (2) either oscillates or $x^{(i)}(t) \to 0$ as $t \to \infty$, $i = 0, 1$.

**Theorem 3.** If $\sigma < 0$, $q(t) \geq c > 0$ for $t \geq t_0$ and

$$\sqrt{\frac{c}{1+\lambda}} \frac{\sigma}{2} e > 1,$$

then the conclusion of Theorem 2 holds.

**Proof of Theorems 2 and 3.** Let $x(t)$ be a nonoscillatory solution of equation (2) and assume that $x(t) > 0$ for $t \geq t_0$. As in the proof of Theorem 1, the function $z$ is of one sign. Now we consider the case: $z'(t) > 0$ for $t \geq t_0$. Proceeding as for the case 1 in Theorem 1 we obtain inequality (5) where every solution of (5) satisfies:

$$w(t) > 0, \quad w'(t) > 0 \quad \text{and} \quad w''(t) > 0 \quad \text{for} \quad t \geq T_1 \geq T.$$

But, in view of Corollary 4.3 in [7] and condition (8) or Theorem 5 in [8] and condition (9), inequality (5) has no solution such that (10) holds. The proof of the case when $z'(t) < 0$ for $t \geq t_0$ is similar to that of case 2 of Theorem 1 and hence is omitted.

In the following two theorems we establish the conditions under which every solution of equation (2) is either oscillatory or it diverges to $\infty$ together with its first derivative.

**Theorem 4.** If $\sigma > \tau$ and

$$\limsup_{t \to \infty} \int_{t-(\sigma-\tau)}^{t} (s-t+\sigma-\tau)^{1-k}(t-s)^k \frac{q(s)}{1+\lambda} ds > 1,$$

for some $k = 0, 1$, then every solution $x$ of equation (2) is either oscillatory or $x^{(i)}(t) \to \infty$ as $t \to \infty$, $i = 0, 1$. 


Theorem 5. If \( \sigma > \tau \), \( q(t) \geq 0 \) for \( t \geq t_0 \) and
\[
\frac{c}{\sqrt{1 + \lambda}} \frac{\sigma - \tau}{2} > 1,
\]
then the conclusion of Theorem 4 holds.

Proof of Theorems 4 and 5. Let \( x(t) \) be a solution of equation (2). Assume that \( x(t) > 0 \) for \( t \geq t_0 \). As in the proof of Theorem 1, the function \( z \) is of one sign. Next, we consider the following two cases:

Case 1. \( z'(t) > 0 \) for \( t \geq T \). The proof of this case is similar to that of case 1 in Theorem 1.

Case 2. \( z'(t) < 0 \) for \( t \geq T \). As in the proof of case 2 in Theorem we obtain inequality (6) and note that every solution of (6) satisfies
\[
w(t) > 0, \quad w'(t) < 0 \quad \text{and} \quad w''(t) > 0 \quad \text{for} \quad t \geq T_0.
\]
Now we observe that condition (11) (respectively (12)) implies that (6) has no solution satisfying (13) (see [7, corollary 4.2] (respectively [8, Theorem 2])), which is a contradiction. This completes the proof.

Finally we present our main result which is concerned with the oscillatory behavior of equation (1) of mixed arguments.

Theorem 6. If \( \sigma > \tau \), \( \beta > 0 \),
\[
\lim_{t \to \infty} \sup_{t \infty} \int_{t}^{t+\beta} (s - t)^k(t - s + \beta)^{1-k} \frac{p(s)}{1 + \lambda} ds > 1
\]
and
\[
\lim_{t \to \infty} \sup_{t \infty} \int_{t-(\sigma-\tau)}^{t} (s - t + \sigma - \tau)^{1-k}(t - s)^k \frac{q(s)}{1 + \lambda} ds > 1
\]
for some \( k = 0, 1 \), then equation (10) is oscillatory.

Theorem 7. If \( \sigma > \tau \), \( \beta > 0 \) \( p(t) \geq c_1 > 0 \), \( q(t) \geq c_2 > 0 \) for \( t \geq t_0 \),
\[
\frac{c_1}{\sqrt{1 + \lambda}} \frac{\beta}{2} e > 1
\]
and
\[
\frac{c_2}{\sqrt{1 + \lambda}} \frac{\sigma - \tau}{2} e > 1,
\]
then equation (1) is oscillatory.
Proof of Theorems 6 and 7. Let \( x(t) \) be a nonoscillatory solution of equation (1), say \( x(t) > 0 \) for \( t \geq t_0 \). Define functions \( z \) and \( w \) as in the proof of Theorem 1. Then

\[
\frac{d^2}{dt^2}(w(t) + \lambda w[t - \tau]) = q(t)w[t - \sigma] + p(t)w[t + \beta]
\]

and

\[
w''(t) = q(t)z[t - \sigma] + p(t)z[t + \beta].
\]

Next, we consider the cases: \( z'(t) > 0 \), and \( z'(t) < 0 \) for \( t \geq T \geq t_0 \), and as in the proof of Theorem 1, we can easily obtain the inequalities

\[
w''(t) \geq \frac{p(t)}{1 + \lambda}w[t + \beta], \tag{18}
\]

and

\[
w''(t) \geq \frac{q(t)}{1 + \lambda}w[t - (\sigma - \tau)]. \tag{19}
\]

Now, by applying Theorems 2 and 3 to (18) and Theorems 4 and 5 to (19) we obtain the desired contradictions.

For illustration we consider the following examples:

Example 2. The neutral differential equation

\[
\frac{d^2}{dt^2}(x(t) + 3x[t - 1]) = e^{-t}x[t - 2] + 3e^2x[t + 1], \quad t \geq 0,
\]

has a nonoscillatory solution \( x(t) = e^{-t} \). Only condition (17) of Theorem 7 is violated.

Example 3. Consider the neutral differential equation

\[
\frac{d^2}{dt^2}\left(x(t) + x\left[t - \frac{\pi}{2}\right]\right) = x[t - 2\pi] + x\left[t + \frac{\pi}{2}\right], \quad t \geq 0.
\]

All the conditions of Theorem 7 are satisfied and hence equation (21) is oscillatory. One such solution is \( x(t) = \sin t \). We believe that none of the known criteria in [1]–[4], [10] and [12] are applicable to equation (21).

Remarks.
1. Some of the existing oscillation criteria are based on the specific values of \( \lambda \) however, we impose no such restriction except that \( \lambda > 0 \).
2. The results of this paper are extendable to higher order equations of the form

\[ \frac{d^n}{dt^n}(x(t) + \lambda x(t - \tau)) = q(t)x[t + \sigma] + p(t)x[t + \beta], \]

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