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Topology and Metrizability of Cone Metric Spaces

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Abstract: Replacing the set of real numbers by an ordered Banach space in the definition of a metric, Guang and Xian [5] introduced the concept of a cone metric and obtained some fixed point Theorems for contractive mappings on cone metric spaces. It has been shown that every cone metric space is metrizable [2-4]. In this paper we review and simplify some results of [6] and as a consequence of our earlier results and in a totally different way will show again that every cone metric space is metrizable and finally prove some fixed point theorems.

Key words: Fixed point- Partially order- Topological vector space-cone space- normal cone

PRELIMINARIES

Let (E, τ) be a topological vector space and P a subset of E. Then, P is called a cone whenever

(i) P is closed, non-empty and $P\neq\{0\}$;
(ii) $ax+by\in P$ for all $x,y\in P$ and non-negative real numbers a,b;
(iii) $P\cap(-P)=\{0\}$

For a given cone $P\subseteq E$, we can define a partial ordering $\leq$ with respect to P by $x\leq y$ if and only if $y-x\in P$. $x<y$ will stand for $x\leq y$ and $x\neq y$, while $x<<y$ will stand for $y-x\in intP$, where intP denotes the interior of P. If there is a norm $\|\ \|$ on E, then the cone P is called normal (with respect to this norm) whenever there is a number M>o such that for all $x,y\in E$, $0\leq x\leq y$ implies $\|x\|\leq M\|y\|$. The least positive number satisfying this norm inequality is called the normal constant of P.

Definition 1.1: Let X be a non-empty set. Suppose that the mapping $d: X\times X\to E$ satisfies

(d1) $0\leq d(x,y)$ for all $x,y\in X$ and $d(x,y)=0$ if and only if $x=y$;
(d2) $d(x,y)=d(y,x)$ for all $x,y\in X$;
(d3) $d(x,y)\leq d(x,z)+d(z,y)$ for all $x,y,z\in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

CONE SPACES

The definition of a cone metric is depended on the existence of a closed cone with nonempty interior. The following example shows that such a cone and cone metric space does exist, in general.

Example 2.1: Let $E=(\mathbb{R},\|\ \|)$ and

$$P=\{(x,y)\in\mathbb{R}^2 : x,y\geq 0\}$$

where $\|\ \|$ is the Euclidean norm on E. Let

$$P=\{(x,y)\in\mathbb{R}^2 : x,y\geq 0\}$$

P is closed and $int\ P\neq\phi$. Let (X, d) be a metric space, then for each finite family of metrics $d_1, d_2,\ldots, d_n$ on X and $p_1, p_2,\ldots, p_n$,

$$d(x,y)=\sum_{i=1}^{n} d_i(x,y)p_i\in P$$

defines a cone metric on X.

Proof: Let $x,y\in X$ and d(x,y) = 0. Then

$$-d(x,y)=\sum_{i=1}^{n} d_i(x,y)p_i\in P\cap -P=\{0\}$$

therefore x = y. It is easy to see that if x = y, then $d'(x,y)=0$ and $d'(x,y)=d'(y,x)$ for each $x,y\in X$. Now let $x,y,z\in X$. Define

$$t_i=d_i(x,y)+d_i(y,z)-d_i(x,z)$$

for i = 1,2,…,n. Since $t_i\geq 0$ for i = 1,2,…,n, then $\sum_{i=1}^{n} t_i p_i\in P$ or equivalently

$$d'(x,z)\leq d'(x,y)+d'(y,z)$$

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Definition 2.2: Let (E, p) be a cone space, $x \in E$ and $c >> 0$. A neighborhood with center x and radius $c >> 0$ is the set of all $y \in E$ such that $-c << y-x << c$ and is denoted by N(x,c) or (x-c, x+c).

Definition 2.3: The subset $\mathcal{U}$ of E is open iff for every x in $\mathcal{U}$ there is $c >> 0$ such that:

$$N(x,c) \subseteq \mathcal{U}$$

Definition 2.4: Let B be a subset of E. $a \in B$ is an interior point of B if there is $c >> 0$ such that $N(x,c) \subseteq B$, and B is open if every point of B is an interior point, an element $b \in B$ is a limit point of B whenever for every $c >> 0$,

$$N(b,c) \cap B \setminus \{b\} \neq \phi$$

B is closed if B contains all of its limit points.

Theorem 2.5: Let (E, P) be a cone space and $x \in E$, for each $c >> 0$, N(x,c) is an open subset of E, for every family $\{G_\alpha\}$ of open subsets of E, $\bigcup G_\alpha$ is an open set in E, intersection of finite numbers of open sets in E is an open set, $B \subseteq E$ is open subset iff B^c is closed subset,

Proof: It is straightforward.

Theorem 2.6: If (E, P) be a cone space then, the topology induced by the subbase $\{N(x,c): x \in E, 0 \ll c\}$, denoted by τ_P is a Hausdorff topology. This topology is called the cone topology on E.

Proof: Let $x,y \in E$ and $N(x,d) \cap N(y,d) \neq \phi$ for any $0 << d$, then $-2d << x-y << 2d$ which implies x = y.

Corollary 2.7: Let (E, τ) be a topological vector space and P be a closed nonempty interior cone in E, then τ_P is weaker than τ.

Proof: By definition, the family {(-c,c): 0<<c} is a local base for the topology τ_P at the origin. To complete the proof it is sufficient to prove that every member of this family is in τ. Therefore let $0 << c$ and $A = \{x \in E : x \ll c\}$, we show that A is open in (E, τ). Let $a \in A$, so c-a int P and there is a balance open subset V in (E, τ) such that $c-a+V \subseteq P$. Let $x \in V$ then

$$c-(a+x) = c-a-x \in c-a+V \subseteq P$$

which implies $a+x \in A$ for every $x \in V$ and consequently $a+V \subseteq A$, so A is open. The similar argument shows that $\{x \in E : x >> -c\}$ is an open set in (E, τ) and it follows that $\{x \in E : -c \ll x \ll c\}$ is open in (E, τ) and this completes the proof.

When E is a real topological vector space, every neighborhood of the origin in the cone topology of the form N(0,c) with $0 << c$ is a balance subset of E, as we will see this is essential to show that E with the cone topology is a topological vector space, to begin with this we need the following Lemma.

Lemma 2.8: For each $a \in E$ and $0 << c$ there exists $n \in \mathcal{N}$ such that for each $t \in \mathbb{R}$ with $|t| < \frac{1}{n}$, $-c \ll ta \ll c$.

Proof: Since $0 << c$, so there exist a balance neighborhood V of origin such that $c+V \subseteq P$. Because V is absorbing, there is $n \in \mathcal{N}$ such that $a \in nV$. Now since V is balance, then $a \in nV \subseteq \frac{1}{t}V$ and hence $ta \in V$, $-ta \in V$ for every t with $0 \leq |t| < \frac{1}{n}$

Therefore for any $t \in \mathbb{R}$ with $|t| < \frac{1}{n}$,

$$c + ta \in c + V \subseteq \text{int}P, c - ta \in c + V \subseteq \text{int}P$$

consequently, $-c << ta$ and $ta << c$ for all $t \in \mathbb{R}$ with $|t| < \frac{1}{n}$, and the proof is complete.

Inspired by [7] we get the following Lemma which is helpful in the sequel.

Lemma 2.9: Let (E, P) be a cone space with $\text{int}P \neq \phi$, for any $x \in X$ there exists $0 << d$ such that $-d << x << d$.

Proof: Let $c \in \text{int}P$ and V be an absorbing balance neighborhood of the origin that $C+V \subseteq \text{int}P$. Assume m be a positive integer that $x \in mV$. Since $mc+mV \subseteq \text{int}P$, then $mc-x \in \text{int}P$ and $mc+x \in \text{int}P$. Taking d = mc we have $-d << x << d$.

Let (E, τ) be a real topological vector space and P be a closed nonempty interior cone in E then, (E, τ_p) is a locally convex topological vector space.

Proof: Let $a,b \in E$, V be an open subset of E which contains a+b and $N(a+b,c) \subseteq V$. If

$$W_1 = N(a,\frac{c}{2}), W_2 = N(b,\frac{c}{2})$$

then $W_1 + W_2 \subseteq V$, and the continuity of addition follows.

Lemma 2.10: Let $x \in E$ and $\alpha \in \mathbb{R}$, to prove that scaler multiplication in continuous we must show that for each $0 \ll c$, there are $0<\delta$ and $0 \ll d$ such that $|\beta-\alpha|<\delta$ and $-d \ll y-x \ll d$ imply $-c \ll \beta y-\alpha x \ll c$.

There is $0 \ll d_1$ such that $-d \ll x \ll d_1$, choose $\delta>0$ such that

$$\delta.d_1 \ll \frac{c}{2}, \quad d_2 = \frac{c}{2(\delta+|\alpha|)} \tag{1}$$

Now suppose that $|\beta-\alpha|<\delta$ and

$$-d_2 \ll y-x \ll d_2 \tag{2}$$

since $(-d,d)$ is a balanced neighborhood for each $0 \ll d$, then for each $\beta \neq \alpha$ we have:

$$-d_1 \ll \frac{(\beta-\alpha)x}{|\beta-\alpha|} \ll d_1 \tag{3}$$

and

$$-d_2 \ll \frac{\beta}{|\beta|}(y-x) \ll d_2 \tag{4}$$

combining above inequalities we get:

$$\beta y-\alpha x = (\beta-\alpha)x+\beta(y-x)$$
$$\ll |\beta-\alpha|\, d_1 + |\beta|\, d_2 \ll \frac{c}{2}+\frac{c}{2} = c$$

similarly

$$\beta y-\alpha x = (\beta-\alpha)x+\beta(y-x)$$
$$>> -|\beta-\alpha|\, d_1 - |\beta|\, d_2 >> -\frac{c}{2}-\frac{c}{2} = -c$$

and the proof is complete.

Corollary 2.11: If (E, τ) be a real topological vector space of finite dimension then $\tau_p = \tau$.

Teorem 2.12: The topological vector space (E, τ_p) is normable.

Proof: Suppose that $0 \ll c$, $V = (-c,c)$ and W be a neighborhood of the origin, so there exists $0 \ll d$ such that $(-d, d) \subseteq W$. By Lemma 2.2 of [7] there is $m>1$ such that $c \ll md$ thus,

$$V = (-c,c) \subseteq (-md,md) \subseteq m.W$$

so V is convex, bounded and balance neighborhood of the origin in the topological vector space (E, P) with the cone topology. Now by Theorem 1.39 of [8] the proof is complete.

Remark 2.13: The norm on the topological vector space (E, P) is denoted by $\|\ \|_p$, or briefly by $\|\ \|$ when there is no ambiguity.

If E is a topological vector space, P be a closed cone in E, $0 \ll c$ then (E, τ_p) is normable and again by 1.39 of [8], we know that for any $r>0$:

$$\{x \in E : \|x\|_p < r\} = rV \tag{5}$$

Theorem 2.14: Let E be a topological vector space and P be a closed cone in E, then $(E, P, \|\ \|_p)$ is a normal cone with constant $k = 1$.

Proof: Let $0 \ll c$, $v = (-c, c)$ and $0 \leq a \leq b$. Let $\varepsilon>0$, since $\|b\|_p < (\|b\|_p+\varepsilon)$, thus by equation (5) $b \in (|b|+\varepsilon)V$. Now $0 \leq a \leq b$ and $b \in (\|b\|_p+\varepsilon)V$. Since $a \leq b$ and $b \ll (|b|+\varepsilon)c$, then $a \ll (|b|+\varepsilon)c$. And by $a \geq 0$ and $0 >> -(\|b\|_p+\varepsilon)c$ it follows that $-(\|b\|_p+\varepsilon)c \ll a$.

Therefore $-(\|b\|_p+\varepsilon)c \ll a \ll (\|b\|_p+\varepsilon)c$, that is $a \in (\|b\|_p+\varepsilon)V$ and again by equation (5) we get

$$\|a\|_p \leq \|b\|_p + \varepsilon$$

since ε is arbitrary the proof is complete.

Theorem 2.15: Suppose that (X, d) is a cone metric space, $\{x_n\}$, $\{y_n\}$ are two sequences in X and $x,y \in X$ Then:

(i) $x_n \to x$ in (X,d) iff $\|d(x_n, x)\|_p \to 0$,
(ii) if $x_n \to x$, $y_n \to y$ then $\|d(x_n, y_n) - d(x,y)\|_p \to 0$.

Proof:

(i) Let $x_n \to x$ and $0 \ll c$, $\delta>0$. There is $m \in \mathbb{N}$ such that if $n \geq m$ then $d(x_n, x) \ll \delta c$. Therefore $\|d(x_n, x)\|_p \leq \delta \|c\|_p$, since δ and c is arbitrary this follows that $\|d(x_n, x)\|_p\, 0$. By definition of convergence in (E, τ_p) the converse implication is also true.
(ii) Let $0 \ll c$. $d(x,y) - t_n \leq d(x_n, y_n) \leq t_n + d(x,y)$ where $t_n = d(x_n,x)+d(y_n,y)$, but $t_n \to 0$ as $n \to \infty$, so there is $n \in \mathbb{N}$ such that for $n \geq N$, $-c \ll t_n \ll c$, thus we have $d(x,y) - c \ll d(x_n, y_n) \ll c + d(x,y)$ for each $n \geq N$ and this completes the proof.

The following Lemma will help us to take limit of inequalities.

Theorem 2.15: Let $\{x_n\}$ and $\{y_n\}$ be sequences in E. Then if $x_n \to x$ and $x_n \geq 0$ for each $n \in \mathbb{N}$, then $x \geq 0$,

Proof: (i) Let c>>0, so there is a positive integer N such that for every n≥N, $x + c \gg x_n \geq 0$. Therefore for every c>>0, x+c>>0. But P is closed cone and this follows that x≥0.

Let E be a topological vector space and P be a closed nonempty interior cone in E then by Theorem 1 the cone space $(E, P, \|\ \|_p)$ is a normal cone space with normal constant k = 1, therefore we have the following Theorem.

Theorem 2.16: Suppose (X, d, E, P) be a cone metric space. Define

$$\delta(x,y) := \|d(x,y)\|_p$$

for all $x, y \in X$ then: δ is a metric on X, for each generalized sequence $\{x_\alpha\}$ in X, $x_\alpha \to x$ in (X, d) iff $x_\alpha \to x$ in (X, δ), let $\{x_n\}_{n\geq 1}^{\infty}$ be any sequence in X, then $\{x_n\}_{n\geq 1}^{\infty}$ is Cauchy in (X, d) if and only if $\{x_n\}_{n\geq 1}^{\infty}$ is Cauchy in (X, δ).

Proof: Recall that the cone P is normal w.r.t the induced norm $\|\ \|_P$ on this cone space. To show that δ is a metric on X let x,y,z∈X,

$$\delta(x,y) = \|d(x,y)\|_p \leq \|d(x,z) + d(z,y)\|_p$$
$$\leq \|d(x,z)\|_p + \|d(z,y)\|_p$$

So the triangle inequality hold. It is easy to see that the other properties to show that δ is a metric on X hold. Now we show that every net is convergent in (X, δ) iff it is convergent in (X, d).

Let $\{x_\alpha\}_{\alpha\in I}$ converges to x∈X in (X, δ) and ε>0. Choose $0 \ll c \in E$ such that $\|c\|_p \leq \varepsilon$, there is j∈I such that for each α≥j, $d(x_\alpha, x) \ll c$. By Theorem 1, $\|d(x_\alpha, x)\|_p \leq \|c\|_p < \varepsilon$, therefore $\{x_\alpha\}_{\alpha\in I}$ also converges to x∈X in (X, δ).

Conversely, let $\{x_\alpha\}_{\alpha\in I}$ converges to some x∈X in (X, δ) and $0 \ll c \in E$. So by the definition of δ, $\|d(x_\alpha, x)\|_p$ converges to 0. Since the cone topology of τ_P is induced by $\|.\|_P$, therefore $d(x_\alpha, x) \xrightarrow{\tau_P} 0$. Now from the definition of convergence in (E, τ_P), there is j∈I such that for a≥j, $d(x_\alpha, x) \ll c$.

FIXED POINT RESULTS

As a consequence of our previous results now we generalize Theorem 2.1 of [1] to cone metric spaces as follow.

Theorem 3.1: Let $(X, \sqsubseteq)$ be a partially ordered set and there is a cone metric d on X such that (X, d, E, P) is a complete metric space. Assume there is a nondecreasing function φ: P→P with $\lim_{n\infty} \varphi^n(c) = 0$ for each c>0 and also suppose that *f*:X→X is a nondecreasing map with respect to $\sqsubseteq$ which satisfies:

$$d(fx, fy) \leq \varphi(d(x,y)) \text{ for all } x, y \in X \text{ with } x \sqsubseteq y$$

and one of the following conditions hold. *f* is continuous, if a nondecreasing sequence $\{x_n\}$ converges to x∈X, then $x_n \sqsubseteq x$ for all $n \in \mathbb{N}$.

If there exists x_0∈X with $x_0 \sqsubseteq fx_0$, then f has a fixed point.

Proof: Since *f* is nondecreasing and $x_0 \sqsubseteq fx_0$, then we have:

$$x_0 \sqsubseteq fx_0 \sqsubseteq \ldots \sqsubseteq f^n x_0 \sqsubseteq \ldots \quad (6)$$

Define $x_{n+1} = fx_n$ for all n≥0, by (6), x_n is a nondecreasing sequence. If $x_{n+1} = x_n$ for some positive integer n, then x_n is a fixed point of *f* and the proof is complete. So suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$, since x_n is nondecreasing,

$$d(x_{n+2}, x_{n+1}) = d(fx_{n+1}, fx_n) \leq \varphi(d(x_{n+1}, x_n))$$

by induction

$$d(x_{n+1}, x_n) \leq \varphi^n(d(x_1, x_0))$$

by taking the cone norm we get,

$$\|d(x_{n+1}, x_n)\|_p \leq \|\varphi^n(d(x_1, x_0))\|_p \quad (7)$$

By Theorem 1, since $\varphi^n(d(x_1, x_0))0$ as n→∞, then $\|\varphi^n(d(x_1, x_0))\|_p \to 0$ as n→∞. Now by (7) we have $\|d(x_{n+1}, x_n)\|_p \to 0$ as n→∞. Again by Theorem 1 this follows that $d(x_{n+1}, x_n) \to 0$ as n→∞.

Let 0<<c, choose the positive integer $n_0 \in \mathbb{N}$ such that:

$$d(x_{n+1}, x_n) \leq c - \varphi(c) \text{ for all } n \geq n_0$$

let $n \geq n_0$,

$$\begin{aligned} d(x_{n+2}, x_n) &\leq d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n) \\ &\leq \varphi(d(x_{n+1}, x_n)) + d(x_{n+1}, x_n) \\ &\leq \varphi(c - \varphi(c)) + c - \varphi(c) < c \end{aligned}$$

by induction for all positive integer k,

$$d(x_{n+k}, x_n) < c \text{ with } n \geq n_0$$

Therefore $\{x_n\}$ is Cauchy and since (X, d) is complete, it converges to some point $x \in X$. If f is continuous then it is clear that $fx = x$. So let condition (iii) hold.

$$\begin{aligned} d(x, fx) &\leq d(x, x_{n+1}) + d(x_{n+1}, fx) \\ &= d(x, x_{n+1}) + d(fx_n, fx) \\ &\leq d(x, x_{n+1}) + \varphi(d(x_n, x)) \\ &\leq 2d(x, x_{n+1}) \end{aligned}$$

Now by letting $n \to \infty$ we get $d(x, fx) \leq 0$ and since $d(x, fx) \in P$, it follows that $d(x, fx) = 0$.

Remark 3.2: Let (E, P) is cone a space (E is a real topological vector space and P is a closed nonempty interior cone in E) and $k \in (0,1)$. Define $\varphi: P \to P$ by $\varphi(c) = kc$ for each $c \in P$. Then φ is a nondecreasing self map on P and for each $c>0$ $\varphi^n(c) = k^n.c \to 0$ as $n \to \infty$. Therefore defining $\varphi(c) = kc$ for all $c \in P$, Banach contraction Theorem is deduced as a special case of above Theorem.

The following Theorem generalize Theorem (2) in [5] by omitting the assumptions of normality and regularity of the cone space but we replace < with <<.

Theorem 3.3: Let (X, d, P, E) be a sequentially compact cone metric space. Suppose $T: X \to X$ satisfies the contraction condition:

$$d(Tx, Ty) \ll d(x, y)$$

for all x,y in E with $x \neq y$. Then T has a unique fixed point in X

Proof: First we show that $0 \leq a \ll b$ implies $\|a\| < \|b\|$ for every a,b in E. To show this let $0 \leq a \ll b$. There is a balance absorbing neighborhood V of the origin such that $b + V - a \in P$. There is $n \in \mathbb{N}$ such that $\frac{-1}{n} b \in V$. Therefore

$$a \leq b - \frac{1}{n} b = \frac{n-1}{n} b \tag{8}$$

taking the cone norm of the above inequality we get

$$\| a \|_p \leq \frac{n-1}{n} \| b \|_p < \| b \|_p \tag{9}$$

therefore by assumption

$$\| d(Tx, Ty) \|_p \ll \| d(x, y) \|_p \quad (x, y \in X, x \neq y)$$

Now by Theorem 1 $(X, \| d \|_p)$ is a metric space. It is easy to see that $d: X \times X \to E$ and $\| \ \| : (E, \tau_p) \to \mathbb{R}$ are continuous functions, so $(X, \| d \|_p)$ is a compact Hausdorff metric space. Define $g: X \to \mathbb{R}$ by $g(x) = \| d(x, Tx) \|_p$ for every $x \in X$, since g is continuous on $(X \| . \|_p)$, then g attains its minimum at some point $x_0 \in X$. We must have $Tx_0 = x_0$, since otherwise:

$$\| d(TTx_0, Tx_0) \| < \| d(Tx_0, x_0) \|$$

a contradiction.

It is worth to mention that some results in metric fixed point theory are generalized to cone metric spaces without much difficulty. As an example we extend the Banach contraction Theorem to cone metric spaces as follow.

Theorem 3.4: Let (X, d, E, P) be complete cone metric space, suppose the mapping $T: X \to X$ satisfies the contraction condition:

$$d(Tx, Ty) \leq \alpha d(x, y), \text{ for all } x, y \in X,$$

where $\alpha \in [0,1)$ is a constant. Then T has a unique fixed point in X. And for any $x \in X$ the iterative sequence $\{T^n(x)\}$ converges to a fixed point.

Proof: By the assumption it follows that: $|d(Tx, Ty)| \leq \alpha |d(x, y)|$ for all $x, y \in X$, so $D(Tx, Ty) \leq \alpha D(x, y)$ for all $x, y \in X$, where $D(x, y) := |d(x, y)|$, but D is a metric on X and (X,D) is also a complete metric space that satisfies the contraction condition

$$D(Tx, Ty) \leq \alpha D(x, y)$$

Now by Banach contraction Theorem the proof is followed.

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