

# BENDING VIBRATIONS OF ROTATING NON-UNIFORM COMPOSITE TIMOSHENKO BEAMS WITH AN ELASTICALLY RESTRAINED ROOT

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**SUMMARY:** The bending vibrations of a rotating non-uniform composite Timoshenko beam with attachment and elastically restrained root are studied. The beam considered is an in-extensional laminated angle ply structure with doubly symmetric cross section such that the centroidal axis and the neutral axis are coincident. It is a common concept that the natural frequencies of a rotating beam will increase as the rotating speed is increased. However, this is not the case for the rotating beam with setting angle and elastically restrained root. A dynamic instability due to rotating, namely rotating instability phenomenon (divergence instability) may occur. The influence of lamination angle, rotating speed, setting angle, hub radius, tip mass, taper ratio and elastic root restraints on the rotating instability phenomenon is investigated.

**KEYWORDS:** bending vibration, rotating composite beam, Timoshenko beam, non-uniform beam.

## INTRODUCTION

Rotating beams have importance in many practical applications such as turbine blades, helicopter rotor blades, robot arms, flexible rotating space booms. An interesting review of the subject can be found in the review papers by Leissa [1], Ramamurti et al. [2] and Rao [3]. Recently, the exact solutions for the bending vibrations of rotating nonuniform isotropic Timoshenko beams with bending rigidity and mass density varying in arbitrary polynomial forms were derived by Lee and Lin [4]. The influence of the coupling effect of the rotating speed and the mass moment of inertia, the setting angle, the rotating speed and taper ratio on the natural frequencies, and the phenomenon of divergence instability (rotating instability) were explored.

For symmetrically laminated non-rotating beams, a theoretical analysis of the torsional and flexural vibrations of fiber reinforced composite beams was given by Teoh and Huang [5]. Including the shear deformation effect, a formulation and a microcomputer program for an anisotropic symmetrically laminated beam finite element was given by Chen and Yang [6]. Singh and Abdelnaser [7] solved the equations of motion of a cross-ply symmetric laminated beam with third-order shear deformation theory. The equations of motion for a symmetrically laminated beam with tip mass, taking account the Poisson effect were derived by

Chandrashekhara and Bangera [8].

Most of the study on the rotating composite beams was based on the finite element method. Ko [9] [10] studied the flexural vibration of a rotating sandwich tapered beam without considering the effect of setting angle, hub radius and centrifugal stiffening effect. Based on finite element procedures, Rand [11] presented a theoretical modeling of slender composite beams with emphasis on the ability to predict the behavior of rotating helicopter blades operating under combined transverse, torsional and centrifugal loads. Details of fabrication and experimental investigations on glass-epoxy composite blades were presented by Seshu et al. [12], and supporting finite element results were also given. The natural frequencies and mode shapes of rotating thin-walled composite beams were presented by Rand [13]. From the existing literature, it reveals that no exact solutions were given to the bending vibrations of a rotating nonuniform composite Timoshenko beam.

In the paper, we consider the pure bending vibrations of a rotating nonuniform composite Timoshenko beam with attachments. The effect of Coriolis force is not considered. The beam considered is inextensional and doubly symmetric such that the centroidal axis and the neutral axis are coincident. The two coupled governing differential equations are uncoupled into one complete fourth-order governing differential equation with variable coefficients in the flexural displacement. The explicit relation between the angle of rotation due to bending and the flexural displacement is established. The frequency equations of the beam are expressed in terms of the four normalized fundamental solutions of the associated uncoupled governing differential equations. Finally, the influence of rotating speed, hub radius, setting angle, rotating spring constant, translational spring constant, taper ratio and tip mass on the natural frequencies of the beam is studied. The phenomenon of divergence instability (rotating instability) is investigated.

## ANALYSIS

Consider the vibration of an untwisted and doubly symmetric non-uniform composite Timoshenko beam, elastically restrained mounted with setting angle  $\theta$  on a hub with radius  $R$ , rotating with constant angular velocity  $\Omega$ , as shown in Figure 1. For time harmonic vibration with angular frequency  $\omega$ , one assumes

$$\begin{aligned} w(x, t) &= w_1(x)e^{i\omega t} \\ \psi(x, t) &= \Psi(x)e^{i\omega t} \end{aligned} \quad (1)$$

where  $w$  is the flexural displacement and  $\psi$  is the angle of rotation due to bending of the beam. In terms of following dimensionless quantities,

$$\begin{aligned} s(\xi) &= \frac{A_{55}(\xi)}{A_{55}(0)} \quad , \quad b(\xi) = \frac{\bar{D}_{11}(\xi)}{\bar{D}_{11}(0)} \quad , \quad m(\xi) = \frac{I_1(\xi)}{I_1(0)} \quad , \quad q(\xi) = \frac{I_3(\xi)}{I_3(0)} \quad , \\ W(\xi) &= \frac{w_1(\xi)}{L} \quad , \quad \beta_T = \frac{K_T L^3}{B\bar{D}_{11}(0)} \quad , \quad \mu = \frac{\bar{D}_{11}(0)}{A_{55}(0)L^2} \quad , \quad \beta_\theta = \frac{K_\theta L}{B\bar{D}_{11}(0)} \quad , \end{aligned}$$

$$\eta = \frac{I_3(0)}{I_1(0)L^2}, \quad \xi = \frac{x}{L}, \quad \Lambda^2 = \frac{I_1(0)\omega^2 L^4}{\overline{D}_{11}(0)}, \quad (2)$$

$$n(\xi) = \alpha^2 \int_{\xi}^1 m(\chi)(r_0 + \chi)d\chi + \beta_1 \alpha^2 (1 + r_0), \quad r_0 = \frac{R}{L}, \quad \beta_1 = \frac{M}{BI_1(0)L},$$

$$\alpha^2 = \frac{I_1(0)\Omega^2 L^4}{\overline{D}_{11}(0)},$$

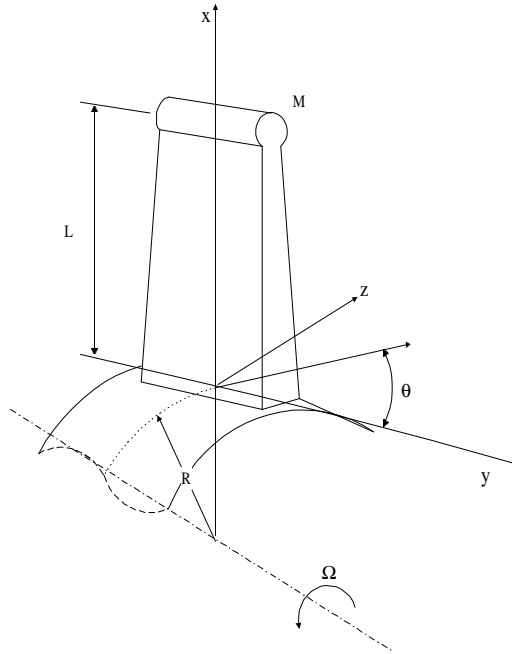


Fig. 1: Geometry and coordinate system of a rotating nonuniform composite beam with an elastically restrained root

the governing characteristic differential equations of the system are

$$\frac{d}{d\xi} \left[ n \frac{dW}{d\xi} + \frac{s}{\mu} \left( \frac{dW}{d\xi} - \Psi \right) \right] + m(\alpha^2 \sin^2 \theta + \Lambda^2)W = 0, \quad (3)$$

$$\frac{d}{d\xi} \left( b \frac{d\Psi}{d\xi} \right) + \frac{s}{\mu} \left( \frac{dW}{d\xi} - \Psi \right) + q\eta(\alpha^2 + \Lambda^2)\Psi = 0, \quad \xi \in (0,1) \quad (4)$$

the associated boundary conditions are at  $\xi=0$ :

$$\beta_T W - n \frac{dW}{d\xi} - \frac{1}{\mu} \left( \frac{dW}{d\xi} - \Psi \right) = 0, \quad (5)$$

$$\beta_\theta \Psi = \frac{d\Psi}{d\xi}, \quad (6)$$

at  $\xi=1$ :

$$\frac{d\Psi}{d\xi} = 0, \quad (7)$$

$$\beta_1(\alpha^2 \sin^2 \theta + \Lambda^2)W - n \frac{dW}{d\xi} - \frac{s}{\mu} \left( \frac{dW}{d\xi} - \Psi \right) = 0. \quad (8)$$

Here, all the material constants of the composite material are followed the notations used in reference [14],  $k$  is the shear correction factor.  $B$ ,  $H(x)$  and  $L$  are the breadth, the height and the length of the beam, respectively.  $K_\theta$  and  $K_T$  are the rotating and the translational spring constants at root of the beam, respectively.  $t$  is the time,  $\rho(x)$  is the mass density per unit volume and  $M$  is the tip mass. In this paper the tip mass is regarded as a particle.  $I_1$ ,  $I_3$ ,  $\bar{D}_{11}$  [8] and  $N$  are defined by

$$(I_1, I_3) = \int_{-\frac{H}{2}}^{\frac{H}{2}} \rho(1, z^2) dz \quad (9)$$

$$\bar{D}_{11} = D_{11} - \frac{D_{12}^2}{D_{22}} + \frac{(D_{12}D_{26} - D_{16}D_{22})^2}{D_{22}(D_{26}^2 - D_{22}D_{66})} \quad (10)$$

$$N = BA_{11} \frac{\partial u_0}{\partial x} \quad (12)$$

Equation (3) can be written as

$$\frac{d\Psi}{d\xi} = \frac{1}{s} \left\{ \frac{d}{d\xi} \left[ (s + \mu n) \frac{dW}{d\xi} \right] + \mu m (\alpha^2 \sin^2 \theta + \Lambda^2) W - \frac{ds}{d\xi} \Psi \right\} \quad (13)$$

Upon substituting equation (13) into equation (4) and multiplying it by a factor  $\mu s^2$ , the angle of rotation due to bending is explicitly in terms of the dimensionless flexural displacement

$$\begin{aligned} \Psi = \frac{1}{\zeta} \left\{ \mu \left( 2 \frac{ds}{d\xi} b - s \frac{db}{d\xi} \right) \left[ \frac{d}{d\xi} \left[ (s + \mu n) \frac{dW}{d\xi} \right] + \mu m (\alpha^2 \sin^2 \theta + \Lambda^2) W \right] \right. \\ \left. - \mu b s \frac{d}{d\xi} \left[ \frac{d}{d\xi} \left[ (s + \mu n) \frac{dW}{d\xi} \right] + \mu m (\alpha^2 \sin^2 \theta + \Lambda^2) W \right] - s^3 \frac{dW}{d\xi} \right\} \end{aligned} \quad (14)$$

where

$$\zeta = \mu \left[ 2 \left( \frac{ds}{d\xi} \right)^2 b - s \frac{d}{d\xi} \left( b \frac{ds}{d\xi} \right) + s^2 q \eta (\alpha^2 + \Lambda^2) \right] - s^3 \quad (15)$$

Substituting equation (14) into equation (3), the governing characteristic differential equation

can be expressed in terms of the dimensionless flexural displacement

$$\begin{aligned}
& \frac{d}{d\xi} \left\{ n \frac{dW}{d\xi} - \frac{s}{\zeta} \left[ \left( 2 \frac{ds}{d\xi} b - \frac{db}{d\xi} s \right) \left( \frac{d}{d\xi} \left[ (s + \mu n) \frac{dW}{d\xi} \right] \right. \right. \right. \\
& + \mu m (\alpha^2 \sin^2 \theta + \Lambda^2) W \left. \left. \left. - bs \frac{d}{d\xi} \left( \frac{d}{d\xi} \left[ (s + \mu n) \frac{dW}{d\xi} \right] \right) \right. \right. \right. \\
& + \mu m (\alpha^2 \sin^2 \theta + \Lambda^2) W \left. \left. \left. \right] + \frac{s}{\zeta} \left[ 2 \left( \frac{ds}{d\xi} \right)^2 b - s \frac{d}{d\xi} \left( \frac{ds}{d\xi} b \right) \right. \right. \right. \\
& \left. \left. \left. + s^2 q \eta (\alpha^2 + \Lambda^2) \right] \frac{dW}{d\xi} \right\} + m (\alpha^2 \sin^2 \theta + \Lambda^2) W = 0 \quad , \quad \xi \in (0,1)
\end{aligned} \tag{16}$$

The boundary conditions for the beam in terms of the flexural displacement can be obtained by substituting the explicit relations between them, equations (13) and (14), into the boundary conditions (5)-(8).

### FREQUENCY EQUATION

The four linearly independent fundamental solutions  $V_i(\xi)$ ,  $i = 1, 2, 3, 4$  of the corresponding governing characteristic differential equation can be chosen such that they satisfy the following normalization conditions at the origin of the coordinate system

$$\begin{bmatrix} V_1(0) & V_2(0) & V_3(0) & V_4(0) \\ V_1'(0) & V_2'(0) & V_3'(0) & V_4'(0) \\ V_1''(0) & V_2''(0) & V_3''(0) & V_4''(0) \\ V_1'''(0) & V_2'''(0) & V_3'''(0) & V_4'''(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{17}$$

where primes indicate differentiation with respect to dimensionless spatial variable  $\xi$ . If the coefficients of the differential equation are in polynomial forms, then the closed form power series solutions of the four fundamental solutions can be obtained by following the method given by Lee and Lin [15].

Substituting the homogeneous solution, which is a linear combination of the four fundamental solutions into the associated boundary conditions, the frequency equation of the system can be obtained

### NUMERICAL RESULTS

In the following numerical analysis, a graphite/epoxy AS4/3501-6 is used. The material properties are given as:  $E_{11} = 144.80\text{GPa}$ ,  $E_{12} = 9.65\text{GPa}$ ,  $G_{12} = 4.14\text{GPa}$ ,  $G_{13} = 4.14\text{GPa}$ ,  $G_{23} = 3.45\text{GPa}$ ,  $\nu_{12} = 0.3$  and  $\rho = 1389.23\text{kg/m}^3$  [8]. The lamina thickness of a symmetrically laminated beam is the same, the shear correction factor  $k$  is taken as  $5/6$  [16] and the  $L/H(0)$  ratio is 15. The width of the beam  $B$  is unity and the height of the beam varies linearly with taper ratio  $\lambda$ . As a result,  $m(\xi) = s(\xi) = 1 + \lambda\xi$ , and  $b(\xi) = q(\xi) = (1 + \lambda\xi)^3$

In Table 1, the first five dimensionless frequencies for an uniform composite Timoshenko beam

are shown. While comparing the natural frequencies of the composite beam with those given by Chandrashekhara and Bangera [8], it shows that the results are very consistent.

Table 1: First five dimensionless frequencies of a uniform composite Timoshenko beam

$$\bar{\omega} = \omega L^2 \sqrt{\rho/E_{11}H^2}, \alpha = 0, r_0 = 0, \theta = 0^\circ, \lambda = 0, \beta_1 = 0, \beta_\theta \rightarrow \infty, \beta_T \rightarrow \infty$$

$\bar{\omega}$	[0/90/90/0]		[45/-45/-45/45]	
	**	#	**	#
$\bar{\omega}_1$	0.92307	0.92305	0.31443	0.31434
$\bar{\omega}_2$	4.88899	4.88844	1.91823	1.91769
$\bar{\omega}_3$	11.43510	11.43313	5.16100	5.15965
$\bar{\omega}_4$	18.69322	18.68896	9.59864	9.59627
$\bar{\omega}_5$	26.21001	26.20329	14.95412	14.91965

\*\* : in the dimensionless flexural displacement

# : given by Chandrashekhara and Bangera [8]

In Figure 2, the influence of the ply angle on the first natural frequencies of the beam with different setting angles is shown. It can be observed that the natural frequencies increase as the ply angle is increased.

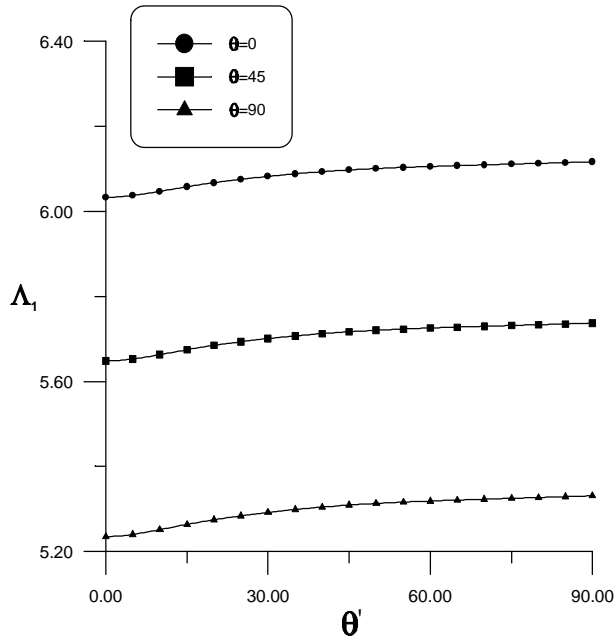


Fig. 2: Influence of ply angle on the first natural frequency of a composite Timoshenko beam with different setting angles

$$\left[ \text{stacking sequence } \theta' / -\theta' / -\theta' / \theta', L/H(0) = 15, \alpha = 3, r_0 = 1, \beta_\theta \rightarrow \infty, \beta_T \rightarrow \infty, \lambda = -0.2, \beta_1 = 0 \right]$$

In Figure 3, the influence of setting angle on the first natural frequencies of the beam with different stacking sequences and tip mass is shown. It can be observed that the natural frequencies decrease as the setting angle is increased. For  $\beta_1 = 0.2$ , the phenomenon of rotating instability occurs as the setting angle is greater than the critical value. Therefore, the tip mass is an important parameter for the rotating instability even though the translational spring constant is large.

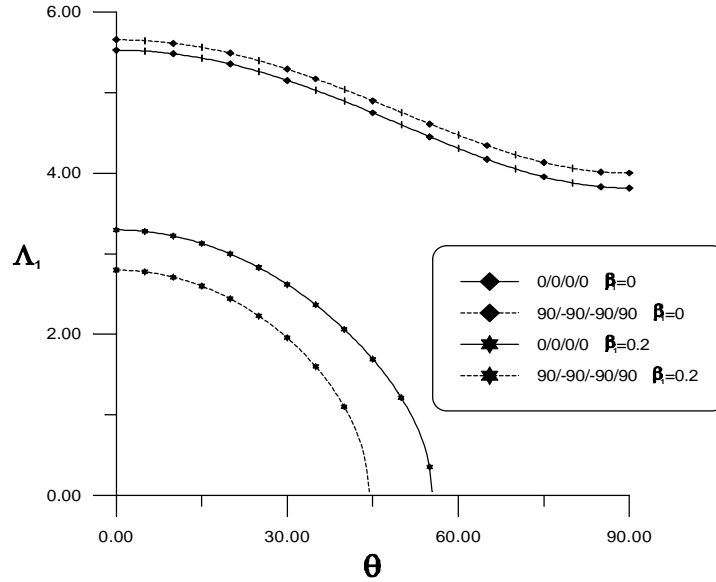


Fig. 3: Influence of setting angle on the first natural frequency of a composite Timoshenko beam with different stacking sequences and tip mass

$$\left[ L/H(0) = 15, \alpha = 4, r_0 = 0, \beta_\theta \rightarrow \infty, \beta_T \rightarrow \infty, \lambda = -0.2 \right]$$

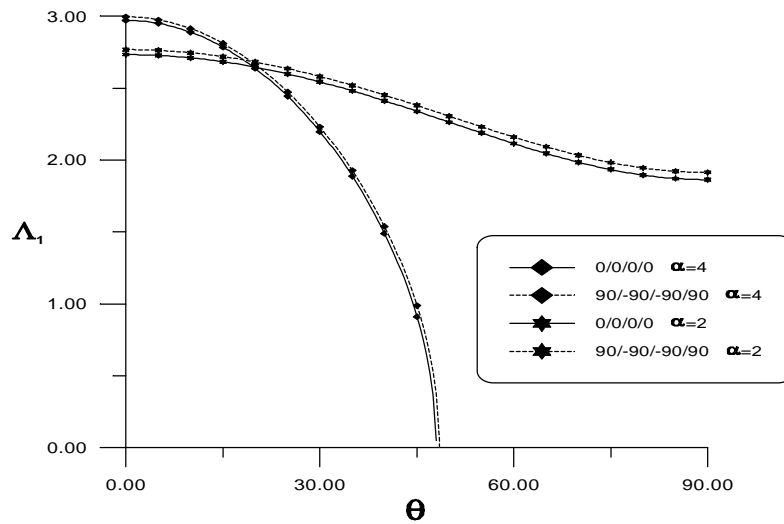


Fig. 4: Influence of setting angle on the first natural frequency of a composite Timoshenko beam with different stacking sequences and rotating speeds

$$\left[ L/H(0) = 15, r_0 = 0, \beta_\theta \rightarrow \infty, \beta_T = 10, \lambda = -0.2, \beta_1 = 0 \right]$$

Figure 4 shows the influence of setting angle on the first natural frequency of a composite Timoshenko beam with different stacking sequences and rotating speeds. It is observed that the influence of setting angle on the natural frequency at higher rotating speed is more significant than that at lower rotating speed. For  $\alpha = 4$ , the phenomenon of divergence instability may occur as the translational rigidity of the root is relatively low and the setting angle is greater

than the critical value.

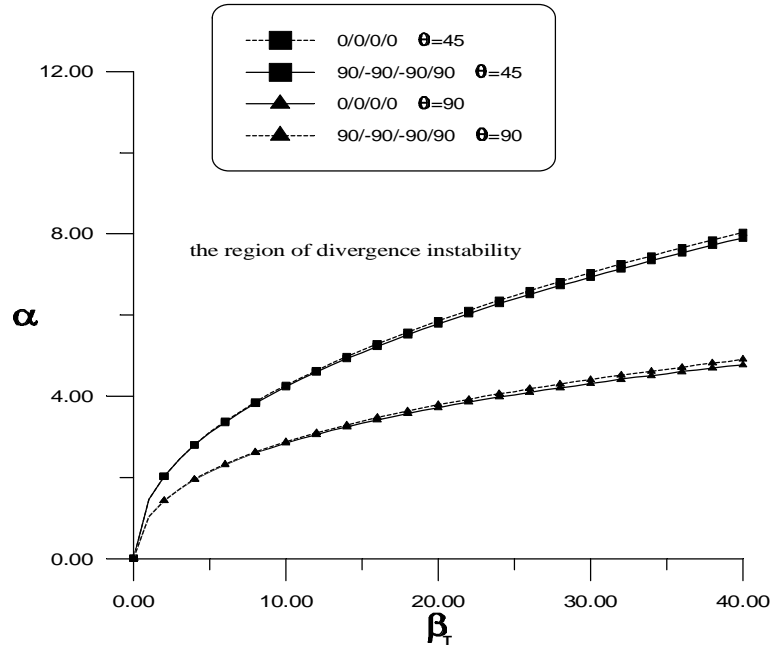


Fig. 5: The region of divergence instability of rotating speed and translational spring constant with different stacking sequences and setting angles

$$\left[ L/H(0) = 15, r_0 = 0, \beta_\theta \rightarrow \infty, \lambda = -0.2, \beta_1 = 0 \right]$$

In Figure 5, the regions of divergence instability of rotating speed and translational spring constant with different stacking sequences and setting angles are shown. It is found that when the translational spring constant is increased, the critical rotating speed for the occurrence of divergence instability increases. The region of divergence instability increases as the setting angle is increased.

## CONCLUSIONS

In this paper, the exact solutions for the bending vibrations of a rotating non-uniform composite Timoshenko beam with attachment and elastically restrained root are given. The numerical results are consistent with those in the literatures. It is shown that the common concept that the natural frequencies of a rotating beam will increase as the rotating speed is increased should be revised. In addition, one has the following physical observations:

- (1) The natural frequencies increase as the ply angle is increased.
- (2) The natural frequencies decrease as the setting angle is increased.
- (3) The influence of setting angle on the natural frequency at higher rotating speed is more significant than that at lower rotating speed.
- (4) When the translational spring constant is increased, the critical rotating speed for the occurrence of divergence instability increases. The region of divergence instability increases as the setting angle is increased.

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