Singular Perturbations and Singular Arcs—Part I

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***Abstract*—Singular perturbation theory is applied to obtain the asymptotic solution for the nearly singular optimal control of a constant linear system on a finite time interval. In the limit as the control cost is reduced to zero, the initial control is found to have an impulse-like behavior, while the outer solution agrees asymptotically with the familiar solution for a singular arc. The detailed structure of the impulse is provided by the asymptotic solution.**

I. Introduction

CONSIDER the linear, time-invariant state regulator problem consisting of the differential equation

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1.1}$$

for the n-dimensional state vector x on $0 \leqslant t \leqslant 1$ (or any other closed, bounded interval); the initial state

$$x(0) \quad \text{being prescribed;} \tag{1.2}$$

and the scalar cost functional

$$J(\epsilon) = \frac{1}{2}\int_0^1 [x^T(t)Qx(t) + \epsilon^2 u^T(t)Ru(t)]\,dt \tag{1.3}$$

to be minimized by selection of the r-dimensional control vector u. Here the superscript T denotes transposition, Q and R are symmetric matrices, Q is positive semidefinite, R is positive definite, and ϵ is a small positive parameter.

For each fixed $\epsilon > 0$, a unique optimal solution can be readily obtained through the classical calculus of variations (cf., e.g., Anderson and Moore [1]) while singular arcs occur when $\epsilon = 0$ (cf., e.g., Bryson and Ho [2] and Ho [3]) and impulse controls are needed to get on and off the singular arcs. By obtaining the asymptotic solution of the problem as $\epsilon \to 0$ (through the use of singular perturbations theory), we shall find how this comes about. We note that substantial technical difficulties would result if one tried to asymptotically evaluate the usual solution (valid for each fixed $\epsilon > 0$) in the limit $\epsilon \to 0$.

Such problems are important in studying such singular arcs, and also arise in analyzing the limiting possibilities for regulators (cf., Kwakernaak and Sivan [4] and Friedland [5]), the inverse problem (Anderson and Moore [1]), and "cheap control" (Lions [6]). We observe that Jacobson and others (cf., Jacobson and Speyer [7] and Jacobson, Gershwin, and Lele [8]) have introduced a cheap control cost to advantage for both theoretical and computational study of singular arc problems.

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To obtain necessary conditions for optimality, we introduce the Hamiltonian

$$H(x,p,u,\epsilon) = \tfrac{1}{2}(x^TQx + \epsilon^2 u^TRu) + p^T(Ax + Bu) \tag{1.4}$$

where the n-dimensional costate vector $p(t)$ satisfies the terminal value problem

$$\dot{p} = -\frac{\partial H}{\partial x} = -Qx - A^Tp, \qquad p(1) = 0. \tag{1.5}$$

Along an optimal path, we must have

$$\frac{\partial H}{\partial u} = \epsilon^2 Ru + B^Tp = 0,$$

so we can solve for the optimal control

$$u = -\frac{1}{\epsilon^2}R^{-1}B^Tp. \tag{1.6}$$

Substituting for u in (1.1) then implies the two-point problem

$$\begin{aligned} \epsilon^2\dot{x} &= \epsilon^2 Ax - BR^{-1}B^Tp, \qquad && x(0) \ \text{prescribed} \\ \dot{p} &= -Qx - A^Tp, && p(1) = 0 \end{aligned} \tag{1.7}$$

[cf., also (1.2) and (1.5)]. Note that this system is singularly perturbed since its differential order reduces from $2n$ when $\epsilon > 0$ to n when $\epsilon = 0$.

The asymptotic theory of singularly perturbed linear boundary value problems is fairly well established (cf., e.g., Wasow [9], O'Malley [10], [11] and Harris [12]). As $\epsilon \to 0$ the limiting solution to such a problem converges (under appropriate hypotheses) to an outer solution within $0 < t < 1$, while regions of nonuniform convergence (boundary layers) occur near one or both endpoints. The limiting outer solution will satisfy the system obtained when $\epsilon = 0$. In the closed interval $0 \leqslant t \leqslant 1$, the asymptotic solution will be expressed as the sum of the outer solution plus boundary layer corrections at $t = 0$ and $t = 1$. These boundary layer corrections depend on some stretched variable (like $\tau = t/\epsilon^\alpha$ at $t = 0$ and $\sigma = (1-t)/\epsilon^\alpha$ at $t = 1$, for some $\alpha > 0$) and tend to zero as the appropriate variable tends to infinity. Here, the singular arc will be provided by the reduced problem (limiting outer solution) of singular perturbations theory, while the impulse at $t = 0$ will be given by the limiting initial boundary layer correction. Our detailed singular perturbation results should be useful in indicating the nature of impulsive control in this and other more general situations. Indeed, this (rather than the asymptotic calculation of the singular arc) is our primary objective. In practical applications, however, one should realize that higher order asymptotic approximations are frequently not too important. The authors give formulas for the general terms of the asymptotic solution because this has been accomplished with so little additional effort.

The asymptotic solution of the two-point problem (1.7) will be found in two cases, namely;

Case 0: B of rank n, Q positive definite; and

Case 1: B of rank $r \leqslant n$, B^TQB positive definite. (The problem where $r=n$ is common to Cases 0 and 1.)

The distinction between these two cases was previously noted by Ho [3]. Elsewhere we shall discuss the following.

Case k:

$$B^T(A^T)^m QA^mB=0, \qquad m<k-1$$

$$B^T(A^T)^{k-1}QA^{k-1}B>0.$$

We shall not examine intermediate possibilities. In the cases studied, the limiting solution within (0,1) follows a singular arc while the boundary layer behavior is impulse-like (specifically, the optimal control is like $(1/\epsilon)e^{-t/\epsilon^\alpha}$, $\alpha>0$, at $t=0$).

Before proceeding, we note that many other regulator problems could be solved in a similar manner. In particular, problem (1.1)–(1.4) could be modified to allow certain terminal costs, time-varying coefficients, and fixed terminal states. The infinite interval problem and some quasilinear problems are also solvable, in addition to some examples with bounded control (cf., O'Malley [13]). Related control problems solved through singular perturbation theory are discussed in American Society of Mechanical Engineers [14], Collins [15], Sannuti and Reddy [16], and Wilde and Kokotović [17].

II. Two Examples

Example 1: Consider the scalar problem

$$\dot{x}(t)=u(t), \qquad 0\leqslant t\leqslant 1$$

$x(0)$ prescribed with

$$J(\epsilon)=\frac{1}{2}\int_0^1 (x^2(t)+\epsilon^2u^2(t))\,dt. \tag{2.1}$$

The linear system corresponding to (1.7) is

$$\epsilon^2\dot{x}=-p, \qquad x(0) \text{ prescribed}$$

$$\dot{p}=-x, \qquad p(1)=0$$

with $u=-p/\epsilon^2$. Solving, we obtain the trajectory

$$x(t,\epsilon)=\left(\frac{x(0)}{1+e^{-2/\epsilon}}\right)\left(e^{-t/\epsilon}+e^{-1/\epsilon}e^{-(1-t)/\epsilon}\right)$$

and the optimal control

$$u(t,\epsilon)=-\frac{1}{\epsilon}\left(\frac{x(0)}{1+e^{-2/\epsilon}}\right)\left(e^{-t/\epsilon}-e^{-1/\epsilon}e^{-(1-t)/\epsilon}\right).$$

Since $e^{-1/\epsilon}$ is asymptotically negligible (less than ϵ^j for any $j>0$), the asymptotic solution is

$$x(t,\epsilon)\sim e^{-t/\epsilon}x(0)$$

$$u(t,\epsilon)\sim -\frac{1}{\epsilon}e^{-t/\epsilon}x(0) \tag{2.2}$$

for $0\leqslant t\leqslant 1$. Thus, the solution features nonuniform convergence at $t=0$ as $\epsilon\to 0$ (like the function $e^{-t/\epsilon}$) unless $x(0)=0$. Away from $t=0$, the optimal control and the resulting trajectory are asymptotically negligible, i.e., the outer solution and the boundary layer correction at $t=1$ are both trivial. At $t=0$, the optimal control is, however, unbounded like $1/\epsilon$. The corresponding limiting cost

$$\int_0^1 e^{-2t/\epsilon}x^2(0)\,dt,$$

however, tends to zero as $\epsilon\to 0$.

We note that the limit of

$$\frac{1}{\epsilon}e^{-t/\epsilon}, \qquad 0\leqslant t\leqslant 1,$$

as $\epsilon\to 0$ behaves like a delta function $\delta(t)$. That is, for any differentiable function $f(t)$

$$\int_0^1 f(t)\frac{e^{-t/\epsilon}}{\epsilon}\,dt=f(0)-f(1)e^{-1/\epsilon}+\int_0^1 f'(t)e^{-t/\epsilon}\,dt$$

$$=f(0)+0(\epsilon).$$

Thus, we seem to have $u(t)\sim -\delta(t)x(0)$, which is in agreement with Ho [3].

Example 2: Now consider the following problem for control of a harmonic oscillator:

$$\ddot{y}+y=u$$

$y(0)$, $\dot{y}(0)$ prescribed with

$$J(\epsilon)=\frac{1}{2}\int_0^1 (\dot{y}^2(t)+\epsilon^2u^2(t))\,dt. \tag{2.3}$$

Introducing $x_1=y$ and $x_2=\dot{y}$, we put the problem in the form (1.1)–(1.3) and obtain necessary conditions for an optimum as follows [cf., (1.7)]:

$$\begin{aligned} &\dot{x}_1=x_2, && x_1(0)=y(0)\\ &\epsilon^2\dot{x}_2=-\epsilon^2x_1-p_2, && x_2(0)=\dot{y}(0)\\ &\dot{p}_1=p_2, && p_1(1)=0\\ &\dot{p}_2=-x_2-p_1, && p_2(1)=0. \end{aligned} \tag{2.4}$$

The general solution of this linear system is

$$x_1(t,\epsilon)=k_1e^{\epsilon\mu t}+k_2e^{-\epsilon\mu t}+\epsilon k_3e^{-\lambda t/\epsilon}+\epsilon^2k_4e^{(\lambda/\epsilon)(t-1)}$$

$$x_2(t,\epsilon)=\epsilon\mu k_1e^{\epsilon\mu t}-\epsilon\mu k_2e^{-\epsilon\mu t}-\lambda k_3e^{-\lambda t/\epsilon}+\epsilon\lambda k_4e^{(\lambda/\epsilon)(t-1)}$$

$$p_1(t,\epsilon)=-\epsilon\mu(1-\epsilon^2\gamma)k_1e^{\epsilon\mu t}+\epsilon\mu(1-\epsilon^2\gamma)k_2e^{-\epsilon\mu t}$$
$$+\lambda(1-\beta)k_3e^{-\lambda t/\epsilon}-\epsilon\lambda(1-\beta)k_4e^{(\lambda/\epsilon)(t-1)}, \quad \text{and}$$

$$p_2(t,\epsilon)=-\epsilon^2k_1\gamma e^{\epsilon\mu t}-\epsilon^2k_2\gamma e^{-\epsilon\mu t}-\epsilon k_3\beta e^{-\lambda t/\epsilon}$$
$$-\epsilon^2k_4\beta e^{(\lambda/\epsilon)(t-1)}, \tag{2.5}$$

where

$$\mu(\epsilon^2)=\sqrt{\frac{1-2\epsilon^2-\sqrt{1-4\epsilon^2}}{2\epsilon^4}},$$

$$\lambda(\epsilon^2)=\sqrt{\frac{1-2\epsilon^2+\sqrt{1-4\epsilon^2}}{2}},$$

$$\gamma(\epsilon^2)=1+\epsilon^2\mu^2 \quad \text{and} \quad \beta(\epsilon^2)=\lambda^2+\epsilon^2$$

are all $1+0(\epsilon^2)$. The boundary conditions provide four linear equations for the unknown $k_i(\epsilon)$'s. Up to asymptotically negligible terms, then,

$$y(0)=k_1+k_2+\epsilon k_3$$
$$\dot{y}(0)=\epsilon\mu k_1-\epsilon\mu k_2-\lambda k_3$$
$$0=-\mu(1-\epsilon^2\gamma)e^{\epsilon\mu}k_1+\mu(1-\epsilon^2\gamma)e^{-\epsilon\mu}k_2-\lambda(1-\beta)k_4$$
$$0=\gamma e^{\epsilon\mu}k_1+\gamma e^{-\epsilon\mu}k_2+\beta k_4. \tag{2.6}$$

This implies that the $k_i(\epsilon)$'s have asymptotic series expansions in ϵ with leading terms

$$k_1(0)=k_2(0)=-2k_4(0)=\frac{y(0)}{2} \quad \text{and} \quad k_3(0)=-\dot{y}(0).$$

We note that the optimal control $u(t,\epsilon)=-p_2(t,\epsilon)/\epsilon^2$ and the corresponding trajectories $x_1(t,\epsilon)$ and $x_2(t,\epsilon)$ consist of the outer solution

$$(U(t,\epsilon),X_1(t,\epsilon),X_2(t,\epsilon))$$
$$\sim(\gamma,1,\epsilon\mu)k_1e^{\epsilon\mu t}+(\gamma,1,-\epsilon\mu)k_2e^{-\epsilon\mu t},$$

the initial boundary layer correction

$$(v(\tau,\epsilon),m_1(\tau,\epsilon),m_2(\tau,\epsilon))\sim\left(\frac{\beta}{\epsilon},\epsilon,-\lambda\right)k_3e^{-\lambda\tau}$$

in the stretched variable $\tau=t/\epsilon$, and the terminal boundary layer correction

$$(w(\sigma,\epsilon),n_1(\sigma,\epsilon),n_2(\sigma,\epsilon))\sim(\beta,\epsilon^2,\epsilon\lambda)k_4e^{-\lambda\sigma}$$

in the stretched variable $\sigma=(1-t)/\epsilon$. Further, the limiting outer solution

$$(X_1(t,0),X_2(t,0),P_1(t,0),P_2(t,0))=(y(0),0,0,0)$$

satisfies the system (2.4) with $\epsilon=0$ and the optimal control

$$u(t,\epsilon)=k_1\gamma e^{\epsilon\mu t}+k_2\gamma e^{-\epsilon\mu t}+\frac{k_3}{\epsilon}e^{-\lambda t/\epsilon}+\epsilon\lambda k_4e^{\lambda(t-1)/\epsilon} \tag{2.7}$$

behaves asymptotically like the impulse $-\dot{y}(0)\delta(t)+y(0)$ (again in agreement with Ho).

III. Case 0: B of Rank n, Q Positive Definite

This is the simplest case that can be studied. It is quite unusual in practice, however, since it requires the dimension r of the control vector to be no less than the dimension n of the state vector. We note that it trivially satisfies the controllability–observability assumptions of Ho [3].

Beginning quite naively, let us seek a power series solution

$$X(t,\epsilon)\sim\sum_{j=0}^{\infty}X_j(t)\epsilon^j, \qquad P(t,\epsilon)\sim\sum_{j=0}^{\infty}P_j(t)\epsilon^j \tag{3.1}$$

of the Hamiltonian system (1.7). The terms of the series (3.1) follow by successively equating coefficients of powers ϵ^j in (1.7). When $\epsilon=0$, then

$$-BR^{-1}B^TP_0=0 \quad \text{and} \quad \dot{P}_0=-QX_0-A^TP_0.$$

Since $BR^{-1}B^T$ is nonsingular, $P_0=0$. Moreover, since Q^{-1} exists, the second equation implies that $X_0=0$. Continuing analogously, we find that

$$X_j(t)=0=P_j(t) \qquad \text{for all } j\geq 0. \tag{3.2}$$

The trivial outer solution thus constructed will need an initial boundary layer correction, since $X(0,\epsilon)=0$ will generally be unequal to the prescribed initial vector $x(0)$. At $t=1$, however, there is no need for a boundary layer correction, since the trivial outer solution satisfies the terminal boundary condition $p(1)=0$. Thus, we seek an asymptotic solution to (1.7) of the form

$$x(t,\epsilon)=m(\tau,\epsilon)\sim\sum_{j=0}^{\infty}m_j(\tau)\epsilon^j,\ p(t,\epsilon)=\epsilon f(\tau,\epsilon)\sim\epsilon\sum_{j=0}^{\infty}f_j(\tau)\epsilon^j \tag{3.3}$$

throughout $0\leq t\leq 1$ where the boundary layer correction $(m(\tau,\epsilon),\epsilon f(\tau,\epsilon))$ tends to zero as the stretched variable

$$\tau=t/\epsilon \tag{3.4}$$

tends to infinity. (Choice of the expansion (3.3) could be motivated by our first example.) Note that (3.3) will tend asymptotically to the trivial outer solution as $\epsilon\to 0$ provided that $t>0$.

Substituting (3.3) in (1.7) implies the linear system

$$\frac{dm}{d\tau}=\epsilon Am-Sf, \qquad S=BR^{-1}B^T$$

$$\frac{df}{d\tau}=-Qm-\epsilon A^Tf \tag{3.5}$$

with the initial condition

$$m(0,\epsilon)=x(0). \tag{3.6}$$

Equating coefficients of ϵ^j for each $j\geqslant 0$ yields

$$\frac{dm_j}{d\tau}=-Sf_j+Am_{j-1}$$

$$\frac{df_j}{d\tau}=-Qm_j-A^Tf_{j-1} \tag{3.7}$$

with

$$m_{-1}\equiv f_{-1}\equiv 0, \qquad m_0(0)=x(0), \qquad m_j(0)=0, \qquad j>0.$$

Thus we have

$$\frac{d^2m_0}{d\tau^2}=SQm_0, \qquad m_0(0)=x(0). \tag{3.8}$$

Introducing the positive definite matrices $S^{1/2}$ and

$$C=\sqrt{S^{1/2}QS^{1/2}}$$

we have the unique decaying solution

$$m_0(\tau)=S^{1/2}e^{-C\tau}S^{-1/2}x(0). \tag{3.9}$$

By (3.7), then,

$$f_0(\tau)=S^{-1/2}Ce^{-C\tau}S^{-1/2}x(0)=K_0m_0(\tau) \tag{3.10}$$

with $K_0\equiv S^{-1/2}CS^{-1/2}$. Continuing, we seek the decaying solution of

$$\frac{d^2m_j}{d\tau^2}=SQm_j+S^{1/2}h_{j-1}(\tau), \qquad m_j(0)=0 \tag{3.11}$$

for each $j\geqslant 1$, where

$$h_{j-1}(\tau)=S^{-1/2}A\frac{dm_{j-1}}{d\tau}+S^{1/2}A^Tf_{j-1} \tag{3.12}$$

is a successively known exponentially decaying vector. Using the appropriate Green's function we obtain the explicit solution

$$m_j(\tau)=\frac{1}{2}S^{1/2}C^{-1}\Bigg[\int_0^\infty e^{-C(\tau+s)}h_{j-1}(s)\,ds - \int_0^\infty e^{-C(\tau-s)}h_{j-1}(s)\,ds - \int_\tau^\infty e^{C(\tau-s)}h_{j-1}(s)\,ds\Bigg] \tag{3.13}$$

and

$$f_j(\tau)=-S^{-1}\left(\frac{dm_j}{d\tau}-Am_{j-1}\right). \tag{3.14}$$

All terms in the expansion (3.3) have now been determined as exponentially decaying vectors.

By (1.6) and (3.3), the optimal control is

$$u(t,\epsilon)=-\frac{1}{\epsilon}R^{-1}B^Tf(\tau,\epsilon). \tag{3.15}$$

We observe that it decays exponentially as $\tau\to\infty$, and that in the limit as $\epsilon\to 0$, we have

$$S^{-1/2}Bu(t,\epsilon)\sim-\frac{1}{\epsilon}Ce^{-Ct/\epsilon}S^{-1/2}x(0)$$

which corresponds to Ho's result that

$$u(0)=-x(0)\delta(0)$$

when $n=r$ and $B=I$. The corresponding optimal cost is

$$J^*(\epsilon)=\frac{\epsilon}{2}\int_0^\infty\big(m^T(\tau,\epsilon)Qm(\tau,\epsilon)+f^T(\tau,\epsilon)BR^{-1}B^Tf(\tau,\epsilon)\big)\,d\tau. \tag{3.16}$$

Since the integrand decays exponentially as $\tau\to\infty$, and has an asymptotic series expansion as $\epsilon\to 0$, the optimal cost tends to zero and has an asymptotic series expansion

$$J^*(\epsilon)\sim\epsilon\sum_{k=0}^{\infty}J_k^*\epsilon^k.$$

We note that this conclusion is in agreement with the statement of Kwakernaak and Sivan [4] that systems of unlimited accuracy are possible when $r\geqslant n$. We further note that the optimal cost could be evaluated as $J^*(\epsilon)=\frac{\epsilon}{2}x^T(0)K(\epsilon)x(0)$ where $\epsilon K(\epsilon)$ is the usual Riccati gain for the infinite interval regulator problem associated with (3.5). In this regard, observe that the boundary layer controllability assumption of Yackel and Kokotović [18] is implied by the definition of Case 0.

When $\epsilon=0$ the Hamiltonian (1.4) is extremized by taking

$$\frac{\partial H}{\partial u}=B^Tp=0 \tag{3.18}$$

and a singular problem results since $\partial^2H/\partial u^2=0$. For B of rank n, (3.18) implies that $p=0$, while the costate equation (1.5) implies that $x=0$ since Q is nonsingular. Thus the trivial outer solution for Case 0 corresponds to a singular arc. Example 1 simply illustrates the asymptotic solution of Case 0.

IV. CASE 1: B OF RANK $r \leqslant n$, B^TQB POSITIVE DEFINITE

The $n \times r$ matrix B of rank r is equivalent to the matrix

$$\tilde{B} = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

where I is the $r \times r$ identity matrix, i.e., there are nonsingular matrices M and N such that

$$\tilde{B} = MBN.$$

Further, if we introduce new variables

$$\tilde{x} = Mx, \qquad \tilde{u} = N^{-1}u, \tag{4.1}$$

problem (1.1)–(1.3) would be transformed into a new problem of the same form with

$$\tilde{A} = MAM^{-1}, \qquad \tilde{B} = MBN,$$

$$\tilde{Q} = (M^T)^{-1}QM^{-1}, \qquad \tilde{R} = N^TRN.$$

Symmetry and positive definiteness are also preserved. The required matrices M and N can be determined by a process similar to Gaussian elimination. The constructive technique for accomplishing this is discussed in Wilkinson [21] and elsewhere.

Let us now assume that the change of variables (4.1) has already been made and take

$$B = \begin{bmatrix} 0 \\ I \end{bmatrix}. \tag{4.2}$$

We note that this is also assumed in Ho's study of singular problems, while a presentation avoiding this transformation is given by Jameson [19]. Let us also partition A, Q, x, and p in the corresponding manner, i.e.,

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix},$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \text{and} \quad p = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

(omitting the upper blocks if $r = n$, corresponding to Case 0). Note that

$$Q_{21} = Q_{12}^T \quad \text{and} \quad B^TQB = Q_{22}$$

is positive definite. We rewrite the control relation (1.6) and the two-point problem (1.7) as

$$u = -\frac{1}{\epsilon^2}R^{-1}p_2 \tag{4.3}$$

and

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2, & x_1(0) \text{ prescribed} \\ \epsilon^2\dot{x}_2 &= \epsilon^2A_{21}x_1 + \epsilon^2A_{22}x_2 - R^{-1}p_2, & x_2(0) \text{ prescribed} \\ \dot{p}_1 &= -Q_{11}x_1 - Q_{12}x_2 - A_{11}^Tp_1 - A_{21}^Tp_2, & p_1(1) = 0 \\ \dot{p}_2 &= -Q_{12}^Tx_1 - Q_{22}x_2 - A_{12}^Tp_1 - A_{22}^Tp_2, & p_2(1) = 0. \end{aligned} \tag{4.4}$$

Since the system (4.4) is singularly perturbed, we expect its asymptotic solution to consist of an outer solution plus boundary layer corrections at both endpoints (cf., the two examples). Specifically, let us seek an asymptotic solution to (4.4) of the form

$$\begin{aligned} x_1(t,\epsilon) &= X_1(t,\epsilon) + \epsilon m_1(\tau,\epsilon) + \epsilon^2 n_1(\sigma,\epsilon) \\ x_2(t,\epsilon) &= X_2(t,\epsilon) + m_2(\tau,\epsilon) + \epsilon n_2(\sigma,\epsilon) \\ p_1(t,\epsilon) &= P_1(t,\epsilon) + \epsilon f_1(\tau,\epsilon) + \epsilon^2 g_1(\sigma,\epsilon) \\ p_2(t,\epsilon) &= \epsilon^2 P_2(t,\epsilon) + \epsilon f_2(\tau,\epsilon) + \epsilon^2 g_2(\sigma,\epsilon) \end{aligned} \tag{4.5}$$

where the m_i's and f_i's$\to 0$ as the (left) stretched variable $\tau = t/\epsilon$ tends to infinity, and the n_i's and g_i's$\to 0$ as the (right) stretched variable $\sigma = (1-t)/\epsilon$ tends to infinity. (We note that the ϵ multipliers are inserted in (4.5) to simplify the calculations. They could have been omitted and zero terms would result after much effort, since the solution and its asymptotic representation in the form (4.5) are unique.) Within (0,1), then, the asymptotic solution will be given by the outer solution

$$(X_1(t,\epsilon), X_2(t,\epsilon), P_1(t,\epsilon), \epsilon^2 P_2(t,\epsilon)). \tag{4.6}$$

1) The Outer Solution: We suppose the outer solution has an asymptotic expansion with

$$(X_1(t,\epsilon), X_2(t,\epsilon), P_1(t,\epsilon), P_2(t,\epsilon)) \sim \sum_{j=0}^{\infty} (X_{1j}, X_{2j}, P_{1j}, P_{2j})\epsilon^j. \tag{4.7}$$

Since the outer solution must satisfy the system (4.4), we will obtain the terms of its expansion (4.7) by successively equating coefficients of like powers of ϵ in (4.4). It follows that

$$\begin{aligned} \dot{X}_{10} &= A_{11}X_{10} + A_{12}X_{20} \\ \dot{X}_{20} &= A_{21}X_{10} + A_{22}X_{20} - R^{-1}P_{20} \\ \dot{P}_{10} &= -Q_{11}X_{10} - Q_{12}X_{20} - A_{11}^TP_{10} \\ 0 &= -Q_{12}^TX_{10} - Q_{22}X_{20} - A_{12}^TP_{10}. \end{aligned} \tag{4.8}$$

Thus

$$P_{20}=R\left(A_{21}X_{10}+A_{22}X_{20}-\dot{X}_{20}\right),$$
$$X_{20}=-Q_{22}^{-1}\left(Q_{12}^TX_{10}+A_{12}^TP_{10}\right)$$

and the system (4.8) reduces to the $2(n-r)$-dimensional linear system

$$\dot{X}_{10}=\left(A_{11}-A_{12}Q_{22}^{-1}Q_{12}^T\right)X_{10}-A_{12}Q_{22}^{-1}A_{12}^TP_{10}$$
$$\dot{P}_{10}=-\left(Q_{11}-Q_{12}Q_{22}^{-1}Q_{12}^T\right)X_{10}-\left(A_{11}^T-Q_{12}Q_{22}^{-1}A_{12}^T\right)P_{10}. \tag{4.9}$$

Applying the boundary conditions for x_1 and p_1 with $\epsilon=0$, (4.5) implies

$$X_{10}(0)=x_1(0) \quad \text{and} \quad P_{10}(1)=0. \tag{4.10}$$

The linear Hamiltonian system (4.9)–(4.10) has a unique solution since the matrices $A_{12}Q_{22}^{-1}A_{12}^T$ and $Q_{11}-Q_{12}Q_{22}^{-1}Q_{12}^T$ are semidefinite (cf., Bucy [20]). Thus, the leading terms of the outer expansion are uniquely determined. Note that the reduced problem (4.9)–(4.10) can be interpreted as the canonical equations for a state regulator problem of state dimension $n-r$, corresponding to Ho's results for the singular arc.

Equating higher order coefficients results in the linear system

$$\dot{X}_{1j}=A_{11}X_{1j}+A_{12}X_{2j},$$
$$\dot{X}_{2j}=A_{21}X_{1j}+A_{22}X_{2j}-R^{-1}P_{2j},$$
$$\dot{P}_{1j}=-Q_{11}X_{1j}-Q_{12}X_{2j}-A_{11}^TP_{1j}-A_{21}^TP_{2,j-2},$$
$$\dot{P}_{2,j-2}=-Q_{12}^TX_{1j}-Q_{22}X_{2j}-A_{12}^TP_{1j}-A_{22}^TP_{2,j-2}.$$

Thus, we can solve for P_{2j} and X_{2j} and there remains the lower dimensional system

$$\dot{X}_{1j}=\left(A_{11}-A_{12}Q_{22}^{-1}Q_{12}^T\right)X_{1j}-A_{12}Q_{22}^{-1}A_{12}^TP_{1j}-A_{12}Q_{22}^{-1}\left(A_{22}^TP_{2,j-2}+\dot{P}_{2,j-2}\right)$$
$$\dot{P}_{1j}=-\left(Q_{11}-Q_{12}Q_{22}^{-1}Q_{12}^T\right)X_{1j}-\left(A_{11}^T-Q_{12}Q_{22}^{-1}A_{12}^T\right)P_{1j}+Q_{12}Q_{22}^{-1}\left(A_{22}^TP_{2,j-2}+\dot{P}_{2,j-2}\right)-A_{21}^TP_{2,j-2}. \tag{4.11}$$

Likewise, from the coefficient of ϵ^j in the boundary conditions for x_1 and p_1, we have

$$X_{1j}(0)=-m_{1,j-1}(0), \quad P_{1j}(1)=-g_{1,j-2}(0) \tag{4.12}$$

since $n_1(1/\epsilon,\epsilon)$ and $f_1(1/\epsilon,\epsilon)$ are asymptotically negligible. Since (4.11)–(4.12) is a nonhomogeneous version of the problem (4.9)–(4.10), its solution is unique. Thus, the outer expansion can be uniquely generated termwise up to specification of the preceding terms in the series expansion for the boundary layer corrections $m_1(0,\epsilon)$ and $g_1(0,\epsilon)$. Note that the outer solution must generally fail to represent the solution near the endpoints since the boundary conditions for $x_2(0,\epsilon)$ and $p_2(1,\epsilon)$ have not been observed.

2) The Initial Boundary Layer Correction: By linearity, both the outer solution and the boundary layer corrections must satisfy the system (4.4). Thus the initial boundary layer correction

$$(\epsilon m_1, m_2, \epsilon f_1, \epsilon f_2) \tag{4.13}$$

must satisfy the linear system

$$\frac{dm_1}{d\tau}=\epsilon A_{11}m_1+A_{12}m_2,$$
$$\frac{dm_2}{d\tau}=\epsilon^2A_{21}m_1+\epsilon A_{22}m_2-R^{-1}f_2,$$
$$\frac{df_1}{d\tau}=-\epsilon Q_{11}m_1-Q_{12}m_2-\epsilon A_{11}^Tf_1-\epsilon A_{21}^Tf_2,$$
$$\frac{df_2}{d\tau}=-\epsilon Q_{12}^Tm_1-Q_{22}m_2-\epsilon A_{12}^Tf_1-\epsilon A_{22}^Tf_2 \tag{4.14}$$

for $\tau\geqslant 0$. When $\epsilon=0$, then

$$\frac{dm_{10}}{d\tau}=A_{12}m_{20}, \quad \frac{dm_{20}}{d\tau}=-R^{-1}f_{20},$$
$$\frac{df_{10}}{d\tau}=-Q_{12}m_{20}, \quad \frac{df_{20}}{d\tau}=-Q_{22}m_{20}. \tag{4.15}$$

Thus, we seek the decaying solution of

$$\frac{d^2m_{20}}{d\tau^2}=R^{-1}Q_{22}m_{20}.$$

Introducing the positive definite matrices $R^{1/2}$ and

$$C=\sqrt{R^{-1/2}Q_{22}R^{-1/2}}$$

we have

$$m_{20}(\tau)=R^{-1/2}e^{-C\tau}R^{1/2}m_{20}(0), \tag{4.16}$$

where the initial condition for x_2 implies

$$m_{20}(0)=x_2(0)-X_{20}(0). \tag{4.17}$$

Integrating the remaining equations of (4.15) and applying the boundary condition at infinity,

$$m_{10}(\tau)=-A_{12}Q_{22}^{-1}f_{20}(\tau)$$
$$f_{10}(\tau)=Q_{12}Q_{22}^{-1}f_{20}(\tau) \quad \text{and}$$
$$f_{20}(\tau)=Q_{22}\sqrt{R^{-1/2}Q_{22}^{-1}R^{1/2}}\, m_{20}(\tau). \tag{4.18}$$

Thus the leading terms are uniquely obtained. In particular, we have found the value $m_{10}(0)$ needed to specify the second-order terms of the outer expansion. We note that the positivity of Q_{22} implies the boundary layer controllability required by Yackel and Kokotović [18]. For the higher order terms we find that

$$\frac{dm_{1j}}{d\tau}=A_{12}m_{2j}+A_{11}m_{1,j-1},$$

$$\frac{dm_{2j}}{d\tau}=-R^{-1}f_{2j}+A_{21}m_{1,j-2}+A_{22}m_{2,j-1},$$

$$\frac{df_{1j}}{d\tau}=-Q_{12}m_{2j}-Q_{11}m_{1,j-1}-A_{11}^Tf_{1,j-1}-A_{21}^Tf_{2,j-1},$$

$$\frac{df_{2j}}{d\tau}=-Q_{22}m_{2j}-Q_{12}^Tm_{1,j-1}-A_{12}^Tf_{1,j-1}-A_{22}^Tf_{2,j-1},\tag{4.19}$$

while the initial condition for x_2 yields

$$m_{2j}(0)=-X_{2j}(0).\tag{4.20}$$

Since $m_{2j}\to 0$ as $\tau\to\infty$, we have

$$m_{2j}(\tau)=\frac{1}{2}R^{-1/2}C^{-1}\cdot\left\{e^{-C\tau}\left(\int_0^\infty e^{-Cs}h_{j-1}(s)\,ds-2CR^{1/2}m_{2j}(0)\right)-\int_0^\tau e^{C(\tau-s)}h_{j-1}(s)\,ds-\int_\tau^\infty e^{C(\tau-s)}h_{j-1}(s)\,ds\right\}\tag{4.21}$$

where

$$h_{j-1}(\tau)=R^{-1/2}(Q_{12}^Tm_{1,j-1}+A_{12}^Tf_{1,j-1}+A_{22}^Tf_{2,j-1})+R^{1/2}\left(A_{21}\frac{dm_{1,j-2}}{d\tau}+A_{22}\frac{dm_{2,j-1}}{d\tau}\right).$$

Decaying solutions for m_{1j}, f_{1j}, and f_{2j} are then obtained by integration. Thus the initial boundary layer correction is uniquely determined termwise. We note that the value $m_{1j}(0)$ needed for the outer expansion is also successively found.

3) The Terminal Boundary Layer Correction: Since the terminal boundary layer correction

$$(\epsilon^2n_1,\epsilon n_2,\epsilon^2g_1,\epsilon^2g_2)\tag{4.22}$$

satisfies (4.4),

$$\frac{dn_1}{d\sigma}=-\epsilon A_{11}n_1-A_{12}n_2$$

$$\frac{dn_2}{d\sigma}=-\epsilon^2A_{21}n_1-\epsilon A_{22}n_2+R^{-1}g_2$$

$$\frac{dg_1}{d\sigma}=\epsilon Q_{11}n_1+Q_{12}n_2+\epsilon A_{11}^Tg_1+\epsilon A_{21}^Tg_2$$

$$\frac{dg_2}{d\sigma}=\epsilon Q_{12}^Tn_1+Q_{22}n_2+\epsilon A_{12}^Tg_1+\epsilon A_{22}^Tg_2\tag{4.23}$$

for $\sigma\geqslant 0$. From the terminal condition for p_2, we have

$$g_2(0,\epsilon)=-P_2(1,\epsilon).\tag{4.24}$$

Thus, when $\epsilon=0$

$$\frac{dn_{10}}{d\sigma}=-A_{12}n_{20},\quad \frac{dn_{20}}{d\sigma}=R^{-1}g_{20},\quad \frac{dg_{10}}{d\sigma}=Q_{12}n_{20},$$

$$\frac{dg_{20}}{d\sigma}=Q_{22}n_{20},\quad\text{and}\quad g_{20}(0)=-P_{20}(1),\tag{4.25}$$

so

$$g_{20}(\sigma)=R^{1/2}e^{-C\sigma}R^{-1/2}g_{20}(0)\tag{4.26}$$

and the decaying vectors n_{10}, n_{20}, and g_{10} follow by integration. The initial value $g_{10}(0)$ needed for the outer expansion is then obtained. Higher order terms follow analogously. Thus, the complete asymptotic representation (4.5) has been uniquely obtained termwise. We note that in the special case when $n=r$, the outer solution and the terminal boundary layer correction will both be trivial asymptotically, as we previously found for Case 0.

4) The Optimal Control and the Optimal cost: The control relation (4.3) implies that the optimal control $u(t,\epsilon)$ will be asymptotically represented in the form

$$u(t,\epsilon)=-\frac{R^{-1}}{\epsilon}f_2(\tau,\epsilon)-R^{-1}P_2(t,\epsilon)-R^{-1}g_2(\sigma,\epsilon)\tag{4.27}$$

throughout $0\leqslant t\leqslant 1$. Thus the optimal control tends to the singular arc

$$U_0(t)\equiv -R^{-1}P_{20}(t)$$

within $(0,1)$. Using the familiar Riccati approach (cf., [1]), it is easy to see that this implies the limiting control law

$$U_0(t)=\gamma(t)X_{10}(t)\tag{4.28}$$

where

$$\gamma(t)=\dot{\Gamma}-A_{21}-A_{22}\Gamma+\Gamma(A_{11}+A_{12}\Gamma)$$

for $X_{20}(t)=\Gamma(t)X_{10}(t)$. Likewise, the control behaves like the impulse function

$$\frac{1}{\epsilon}R^{-1/2}Ce^{-Ct/\epsilon}R^{1/2}(\Gamma(0)X_{10}(0)-x_2(0))\tag{4.29}$$

near $t=0$ while the derivative of the control will be impulse-like at both endpoints (cf., Ho [3]).

The cost function (1.3) and (4.27) imply that the optimal cost is given by

$$J^*(\epsilon)=\frac{1}{2}\int_0^1 L(t,\epsilon)\,dt$$

where

$$L(t,\epsilon)=x_1^T(t,\epsilon)Q_{11}x_1(t,\epsilon)+2x_1^T(t,\epsilon)Q_{12}x_2(t,\epsilon)+x_2^T(t,\epsilon)Q_{22}x_2(t,\epsilon)+\frac{1}{\epsilon^2}p_2^T(t,\epsilon)R^{-1}p_2(t,\epsilon).$$

Also (4.5) implies that L will have the form

$$L(t,\epsilon)=L_1(t,\epsilon)+L_2(\tau,\epsilon)+\epsilon L_3(\sigma,\epsilon)$$

where the L_i's have asymptotic series expansions, and L_2 and L_3 decay exponentially as τ (or σ) tends to infinity. In defining L_2 and L_3, products of functions of t and τ (or σ) are written as functions of τ (or σ) by setting $t=\epsilon\tau$ (or $t=1-\epsilon\sigma$), while products of functions of τ and functions of σ are omitted since they are asymptotically negligible throughout $0\leqslant t\leqslant 1$. Thus the optimal cost is given by

$$J^*(\epsilon)=\frac{1}{2}\int_0^1 L_1(t,\epsilon)\,dt+\frac{\epsilon}{2}\int_0^\infty L_2(\tau,\epsilon)\,d\tau+\frac{\epsilon^2}{2}\int_0^\infty L_3(\sigma,\epsilon)\,d\sigma. \quad (4.30)$$

It will have a power series expansion

$$J^*(\epsilon)\sim\sum_{k=0}^{\infty}J_k^*\epsilon^k \quad (4.31)$$

as $\epsilon\to 0$. The leading term J_0^* is unaffected by the control, but corresponds to the cost for the limiting trajectory (X_{10},X_{20}) within $0<t<1$. Note that $J_0^*=0$ ("unlimited accuracy" as $\epsilon\to 0$) if $X_{10}(t)\equiv 0\equiv X_{20}(t)$ (or $X_{10}\equiv 0\equiv P_{10}$). This will follow if $x_1(0)=0$.

We observe also that the limiting solution within $0<t<1$ corresponds to a singular arc determined by the dynamic system of order $n-r$ for X_{10}. Recall that singular arcs are determined by the condition

$$\frac{\partial H}{\partial u}=B^Tp=p_2=0 \quad (4.32)$$

where the Hamiltonian H is given by (1.4) with $\epsilon=0$. The Hamilton–Jacobi equations $\dot{x}=\partial H/\partial p$, $\dot{p}=-\partial H/\partial x$ then imply the linear system

$$\begin{aligned}\dot{x}_1&=A_{11}x_1+A_{12}x_2,\\ \dot{x}_2&=A_{21}x_1+A_{22}x_2+u\\ \dot{p}_1&=-Q_{11}x_1-Q_{12}x_2-A_{11}^Tp_1\\ 0&=-Q_{12}^Tx_1-Q_{22}x_2-A_{12}^Tp_1.\end{aligned}$$

Since Q_{22}^{-1} exists, along a singular arc we have

$$x_2=Q_{22}^{-1}Q_{12}^Tx_1-Q_{22}^{-1}A_{12}^Tp_1, \qquad u=\dot{x}_2-A_{21}x_1-A_{22}x_2 \quad (4.33)$$

and

$$\dot{x}_1=(A_{11}-A_{12}Q_{22}^{-1}Q_{12}^T)x_1-A_{12}Q_{22}^{-1}A_{12}^Tp_1,$$

$$\dot{p}_1=-(Q_{11}-Q_{12}Q_{22}^{-1}Q_{12}^T)x_1-(A_{11}^T-Q_{12}Q_{22}^{-1}A_{12}^T)p_1. \quad (4.34)$$

If we solve (4.34) subject to the boundary conditions for $x_1(0)$ and $p_1(1)$, the resulting singular arc (x_1,x_2,u) coincides with the limiting optimal solution $(X_{10},X_{20},-R^{-1}P_{20})$ within $(0,1)$. Note that an initial impulse must be expected since $X_{20}(0)$ is generally unequal to the prescribed value $x_2(0)$. An interesting alternative interpretation of the singular arc is provided by the limiting state regulator problem

$$\dot{X}_{10}=A_{11}X_{10}+A_{12}X_{20}, \qquad X_{10}(0) \text{ prescribed}$$

with

$$J_0=\frac{1}{2}\int_0^1[X_{10}^TQ_{11}X_{10}+2X_{10}^TQ_{12}X_{20}+X_{20}^TQ_{22}X_{20}]\,dt \quad (4.35)$$

which acts like a standard dynamical system of state dimension $n-r$ with the r-dimensional vector X_{20} acting as the control. This is the viewpoint of Ho [3]. It can also be achieved by a formal time-scale separation, as in Kelley [22] and elsewhere.

V. Conclusions

Summarizing our formal results and applying known theory for singular perturbation problems (e.g., Harris [12]), we have the following Theorem.

Theorem: Consider the state regulator problem (1.1)–(1.3). For every positive integer $N>0$, the optimal control u, the corresponding trajectory x, and the optimal cost J^* will have uniform asymptotic approximations as follows:

Case 0: If B has rank n and Q is positive definite,

$$u(t,\epsilon)=\frac{1}{\epsilon}v\left(\frac{t}{\epsilon},\epsilon\right)=\frac{1}{\epsilon}\sum_{k=0}^{N}v_k\left(\frac{t}{\epsilon}\right)\epsilon^k+0(\epsilon^N)$$

$$x(t,\epsilon)=m\left(\frac{t}{\epsilon},\epsilon\right)=\sum_{k=0}^{N}m_k\left(\frac{t}{\epsilon}\right)\epsilon^k+0(\epsilon^{N+1})$$

and

$$J^*(\epsilon)=\epsilon\sum_{k=0}^{N}J_k^*\epsilon^k+0(\epsilon^{N+2}).$$

Case 1: If B^TQB is positive definite

$$u(t,\epsilon)=\frac{1}{\epsilon}v\left(\frac{t}{\epsilon},\epsilon\right)+U(t,\epsilon)+w\left(\frac{1-t}{\epsilon},\epsilon\right)$$

$$=\frac{1}{\epsilon}\sum_{k=0}^{N}\left(v_k\left(\frac{t}{\epsilon}\right)+\epsilon U_k(t)+\epsilon w_k\left(\frac{1-t}{\epsilon}\right)\right)\epsilon^k+0(\epsilon^N)$$

$$x(t,\epsilon) = m\left(\frac{t}{\epsilon},\epsilon\right) + X(t,\epsilon) + \epsilon n\left(\frac{1-t}{\epsilon},\epsilon\right)$$

$$= m_0\left(\frac{t}{\epsilon}\right) + X_0(t) + \sum_{k=1}^{N}\left(m_k\left(\frac{t}{\epsilon}\right) + X_k(t) + n_{k-1}\left(\frac{1-t}{\epsilon}\right)\right)\epsilon^k + 0(\epsilon^{N+1})$$

and

$$J^*(\epsilon) = \sum_{k=0}^{N} J_k^* \epsilon^k + 0(\epsilon^{N+1}).$$

In both cases, the functions of $\tau = t/\epsilon$ *tend to zero as* τ *tends to infinity. Likewise, the functions of* $\sigma = 1 - t/\epsilon$ *tend to zero as* σ *tends to infinity.*

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