INTERSECTION-NUMBER OPERATORS FOR CURVES ON DISCS II

Stephen P. Humphries

Abstract. Let the braid group B_n act as (isotopy classes of) diffeomorphisms of an n-punctured disc D_n. Then there is an action of B_n on a polynomial algebra $R = \mathbb{C}[a_1, \dots, a_N]$ and a way of representing simple closed curves on D_n as elements of R. Fix $k \in 2\mathbb{N}$. Using this approach we show that the image in Aut(R) of each Dehn twist τ about a simple closed γ in D_n satisfies a kind of characteristic equation when its action is restricted to the image in R of the set of curves γ having geometric intersection number k with γ.

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§1. Introduction.

Let D_n be the disc with n punctures $\pi_1, \dots, \pi_n$ and let $\mathcal{C}_m$ denote the set of isotopy classes of oriented simple closed curves on D_n which surround $m \geq 2$ of the punctures. Let $\mathcal{C} = \cup_{m=2}^{n} \mathcal{C}_m$. Then the braid group B_n acts as (isotopy classes of) diffeomorphisms of D_n [Bi, Ch.1]. In fact B_n acts transitively on $\mathcal{C}_m$ for all $2 \leq m \leq n$. The group B_n has standard generators $\sigma_1, \dots, \sigma_{n-1}$ and presentation

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \text{ for } i = 1, \dots, n-2$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \qquad \text{for } |i-j| > 1.$$

Here the generator σ_i acts as a half-twist [Bi] on D_n interchanging π_i and π_{i+1} and has a representative diffeomorphism which is supported in a tubular neighbourhood of the arc a_i (see Figure 1). For $1 \leq i < j \leq n$ let γ_{ij} be a simple closed curve isotopic to the boundary of a tubular neighbourhood of $a_i \cup a_{i+1} \cup \cdots \cup a_{j-1}$. Given any $c, d \in \mathcal{C}$ we let $\iota(c, d)$ denote the *geometric intersection-number* of c and d. This is the minimum number of points of $c' \cap d'$, where c' and d' are any simple closed curves isotopic to c and d. Note that $\iota(c, d)$ is always even since D_n is planar.

Let R be a commutative ring with identity. In a previous paper [H2, §3] we have shown that by representing the free group $\pi_1(D_n)$ using transvections (see below) and looking at certain traces we obtain an injective map $\phi : \mathcal{C} \to R_n$, where

$$R_n = R[a_{12}, a_{13}, \dots, a_{1n}, a_{21}, a_{23}, \dots, a_{2n}, \dots, a_{n1}, \dots, a_{n\,n-1}]$$

is a polynomial ring in commuting indeterminates a_{ij}, $1 \leq i \neq j \leq n$. It will be convenient for us to put $a_{ii} = 0$ for $i = 1, \dots, n$. In this situation we obtain an action of B_n on the ring R_n i.e. we have a homomorphism

$$\psi_n : B_n \to Aut(R_n);$$

the kernel of ψ_n is the centre of B_n [H1]. We note that to each curve $\gamma \in \mathcal{C}_m$ there is a $1/m$ twist τ_γ whose mth power is the Dehn twist [Bi] about the curve γ. In particular for $1 \leq k < m \leq n$ we have

$$\tau_{\gamma_{km}} = \sigma_{m-1} \dots \sigma_{k+1}\sigma_k.$$

In general if $\gamma \in \mathcal{C}_m$, then there is $\alpha \in B_n$ such that $\alpha(\gamma_{1m}) = \gamma$; in this situation we have $\tau_\gamma = \alpha\tau_{\gamma_{1m}}\alpha^{-1}$. In this paper we prove:

Theorem 1.1. *Let $n, m, r \in \mathbb{Z}, n \geq m \geq 2, r \geq 0$. Then there exist polynomials $\mathfrak{B}_{mr}(x) \in R_n[x]$ such that for all $\gamma \in \mathcal{C}$ with $\iota(\gamma_{1m}, \gamma) = 2r$ we have*

$$\mathfrak{B}_{mr}(\psi_n(\tau^m_{\gamma_{1m}}))\phi(\gamma) = 0.$$

Further, for $\gamma \in \mathcal{C}$ there is an integer $2r$ such that $\mathfrak{B}_{mr}(\psi_n(\tau^m_{\gamma_{1m}}))\phi(\gamma) = 0$ and the minimal such $2r$ is equal to $\iota(\gamma_{1m}, \gamma)$.

In part I of this paper [H4] we proved a slightly different version of this result, but only in the case $m = 2$. The polynomials obtained were different from those that we obtain in Theorem 1.1 for $m = 2$. The difference is best described by saying that the polynomials obtained in I for $m = 2$ are like the minimal polynomials for the action of $\tau^2_{\gamma_{12}}$, while those obtained in Theorem 1.1 are like the characteristic equation.

The polynomials $\mathfrak{B}_{mr}$ can be determined algorithmically (see Examples 5.5). Since B_n acts transitively on $\mathcal{C}_m, m \leq n$ and $\gamma_{1m} \in \mathcal{C}_m$ Theorem 1.1 gives and algorithm for calculating the intersection numbers $\iota(\gamma, \gamma')$ for any simple closed curves γ, γ'. Existing algorithms for calculating intersection number include those of Reinhart [R], Zieschang [Z1, Z2], Chillingworth [C1, C2], Birman and Series [BS], Cohen and Lustig [CL] and Tan [T].

We now give some more information about these polynomials $\mathfrak{B}_{mr}$. For fixed $m \geq 2$ we let $\Pi_m = T_1 T_2 \dots T_m$ where the T_i are certain $m \times m$ matrices (transvections):

$$T_i = \begin{pmatrix} 1 & 0 & \dots & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \dots & \vdots & \vdots \\ a_{i1} & a_{i2} & \dots & 1 & \dots & a_{im-1} & a_{im} \\ \vdots & \vdots & \dots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & \dots & 0 & 1 \end{pmatrix}$$

where the non-zero off-diagonal entries occur in the ith row. Here a *transvection* [A] is a matrix $M = I_m + A$ where I_m is the $m \times m$ identity matrix, $det(M) = 1$ and $A^2 = 0$. In particular, conjugates of transvections are transvections.

Now in [H2, Theorem 2.8] we have shown that for all $2 \leq m \leq n$ the characteristic polynomial

$$\chi_m(x) = \sum_{i=0}^{m} c_{mi} x^i$$

of the matrix Π_m has coefficients c_{mi} which are invariant under the action of B_m. We note that the c_{mi} are polynomials of degree m for $1 \leq i < m$ [H2; Theorem 2.8]. If $m < 6$, then they generate the ring of invariants for the action of B_m on a certain subring of R_m [H3]. Note that we have $c_{mm} = 1$ and $c_{m0} = m$.

Theorem 1.2. *Let $n \geq m \geq 2$ and $r \geq 0$. Then the coefficients of the polynomials $\mathfrak{B}_{mr}(x)$ are polynomials in $\mathbb{Q}[c_{m1}, c_{m2}, \dots, c_{m-1m}]$.*

These polynomials have the property that $\mathfrak{B}_{mr}$ divides $\mathfrak{B}_{ms}$ whenever $r \leq s$. Also, if α is a root of $\mathfrak{B}_{mr}$ and β is a root of $\mathfrak{B}_{ms}$, then $\alpha\beta$ is a root of $\mathfrak{B}_{m(r+s)}$.

The representation of B_n in $Aut(R_n)$ will be described in detail later, but should be thought of in the following way. Let $F_n =< x_1, \dots, x_n >$ denote the free group of rank n, which we identify with the fundamental group of D_n. The Magnus expansion M of F_n [Ma, MKS] is defined as follows: Let $\mathcal{P}_n$ be the algebra of formal power series in non-commutative variables $X_1, \dots, X_n$ over $\mathbb{C}$. Then M is the homomorphism $M : F_n \to \mathcal{P}_n$ given on generators by

$$M(x_i) = 1 + X_i, \quad M(x_i^{-1}) = 1 - X_i + X_i^2 - X_i^3 + \dots.$$

Then M is injective, has connections with Fox's free differential calculus and is used to define interesting representations of the braid groups [Bi]. We obtain our representation of B_n in $Aut(R_n)$ by looking at the situation where in $\mathcal{P}_n$ we have the extra relations $X_i^2 = 0$ and $X_iX_j = X_jX_i$ for all $i, j = 1, \dots, n$. This is accomplished concretely by representing the free group F_n using transvections. The standard action of B_n on F_n [Bi] then gives rise to an action of B_n on R_n which is a homomorphic image of $\mathcal{P}_n$. This is all explained in more detail in §§2, 4. A quotient of this algebra was used by Milnor [Mi1, Mi2] to study links.

For $i, j, k, \dots, r, s \in \{1, 2, \dots, n\}$ let $c_{ijk\dots rs}$ denote the *cycle* $a_{ij}a_{jk}\dots a_{rs}a_{si} \in R_n$. Then the cycles generate a subalgebra of R_n denoted Y_n. A cycle $c_{ijk\dots rs}$ will be called *simple* if $i, j, k, ..., r, s$ are all distinct. The ring Y_n is generated by the (finite number of) simple cycles. We also show

Theorem 1.3. *For $m \geq 2$ and all $\alpha \in Y_n$ there is $r \in \mathbb{Z}$ such that*

$$\mathfrak{B}_{mr}(\psi_n(\tau_{\gamma_{1m}}^m))(\alpha) = 0.$$

In what follows we will usually write $\mathfrak{B}_{mr}(\tau_{\gamma_{1m}}^m)$ instead of $\mathfrak{B}_{mr}(\psi_n(\tau_{\gamma_{1m}}^m))$.

§2 Preliminary results on the action of B_n on R_n

In this section we construct the representation $\psi_n : B_n \to Aut(R_n)$. In §4 we will give a more explicit description of the action.

Note that the action of B_n on D_n fixes the boundary of D_n and so there is an induced action of B_n on the fundamental group of D_n (where we choose a base point

p on the boundary of D_n). This fundamental group is the free group F_n of rank n. We choose a standard set of generators $x_1, \ldots, x_n$ for F_n, where x_i is a simple closed curve enclosing the ith puncture π_i and such that $x_1 x_2 \ldots x_n$ is parallel to the boundary of D_n. See Figure 1.

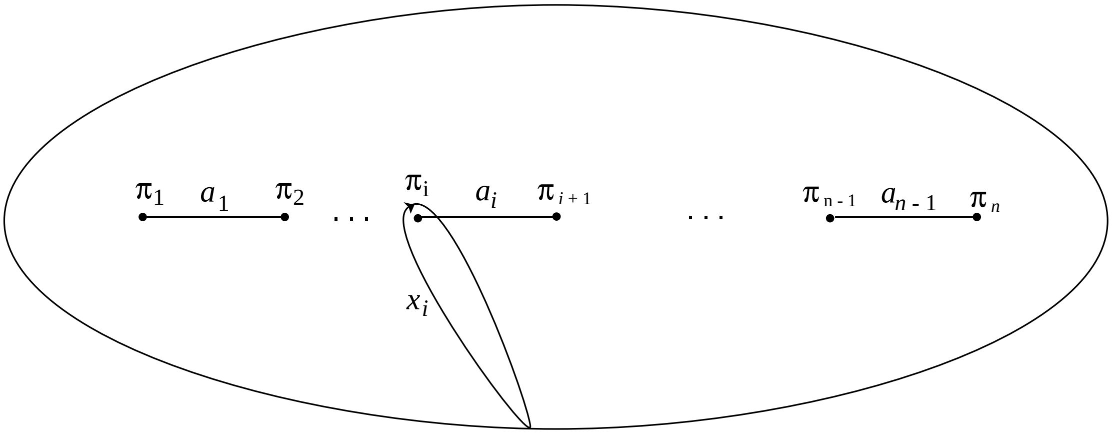

Figure 1

In Figure 1 we have shown arcs a_i and the generators x_i. The action of B_n on the generators x_i is as follows: let $1 \leq j < n$; then

$$\sigma_j(x_i) = x_i \quad \text{if} \quad i \neq j, j+1, \quad \sigma_j(x_j) = x_j x_{j+1} x_j^{-1}, \quad \sigma_j(x_{j+1}) = x_j.$$

We now find that a particularly convenient representation of the free group F_n is given by the transvections T_i. That $< T_1, \ldots, T_n >$ is a free group of rank n is shown in [H2]. This allows us to identify x_i and T_i for $i = 1, \ldots, n$ and so to identify F_n and $< T_1, \ldots, T_n >$.

Now, from the above, B_n acts by automorphisms on F_n in such a way that for $\alpha \in B_n$ the matrix $\alpha(T_i)$ is a conjugate of some T_j, $1 \leq j \leq n$ i.e. $\alpha(T_i)$ is also a transvection. Further, if $c \in \mathcal{C}$, then c represents a conjugacy class in F_n and so its *trace* is well-defined (the trace of the corresponding product of transvections in $F_n =< T_1, \ldots, T_n >$). In fact one easily sees that $trace(c) \in Y_n$ [H1]. Then a map $\phi = \phi_n : \mathcal{C} \to R_n$ is defined uniquely by

$$\phi_n(c) = trace(c) - n. \tag{2.1}$$

Thus ϕ_n can be thought of as being defined on certain conjugacy classes of elements of F_n (namely those representing simple closed curves). The map ϕ can be extended to all of F_n, by the requirement that for $s \in F_n$ we have $\phi(s) = trace(s) - n$. From [H1] (see also §4) we have

Lemma 2.1. *If $w = T_{i_1}^{k_1} T_{i_2}^{k_2} \ldots T_{i_r}^{k_r} \in < T_1, \ldots, T_n >$ where $r > 1, k_u \neq 0, i_u \neq i_{u+1}$ for $u = 1, \ldots, r-1$ and $i_r \neq i_1$, then $\phi(w)$ is a polynomial in Y_n of degree r with a unique monomial of highest degree.*

Now for $m \geq n$ and $s \in F_n$ we may also consider s as an element of F_m under the natural inclusion of F_n in F_m. In this case we note that $\phi(s)$ has the same value whether we consider s as an element of F_n or F_m.

Now for all i, j we have $trace(T_iT_j) = a_{ij}a_{ji} + n$ and in general if $A, B \in F_n$, then

$$trace(AT_iA^{-1}BT_jB^{-1}) = b_{ij}b_{ji} + n$$

where $b_{ij} \in R_n$ (see [H1] and §4). It is also easy to see that there is a natural choice so that

$$b_{ij} = a_{ij} + \text{terms of higher degree.}$$

Now for $\alpha \in B_n$ the image $\alpha(T_i)$ is a conjugate AT_jA^{-1} for some $A \in F_n$ and $1 \leq j \leq n$. Here the action of α on the a_{ij} is defined by

$$trace(\alpha(T_i)\alpha(T_j)) = \alpha(a_{ij})\alpha(a_{ji}) + n,$$

(see §4 for more details) so that it has the following naturality property (with respect to the action of B_n on F_n): for all $w \in < T_1, \ldots, T_n >$ we have

$$\phi(\alpha(w)) = \alpha(\phi(w)).$$

For example the action of σ_i is given by

$$\begin{aligned} &\sigma_i(a_{i\,i+1}) = a_{i+1\,i}, \quad \sigma_i(a_{i+1\,i}) = a_{i\,i+1}, \\ &\sigma_i(a_{h\,i}) = a_{h\,i+1} + a_{hi}a_{i\,i+1}, \quad \sigma_i(a_{h\,i+1}) = a_{h\,i}, \\ &\sigma_i(a_{i\,h}) = a_{i+1\,h} - a_{i+1\,i}a_{ih}, \quad \sigma_i(a_{i+1\,h}) = a_{i\,h}, \end{aligned} \tag{2.2}$$

where $1 \leq h \leq n$ and $h \neq i, i+1$.

It follows from [H1,Theorem 2.5 and Theorem 6.2] that the kernel of the action of B_n on R_n is the centre of B_n and that if B_n and R_n are thought of as sub-objects of B_{n+1} and R_{n+1} (respectively), then the action of B_n on R_{n+1} is faithful. The proof is essentially and application of Lemma 2.1. We note as in [H2] that there is a natural ring involution $*$ on R_n which commutes with the action of B_n, so that for $\alpha \in B_n$ we have

$$\alpha(w)^* = \alpha(w^*) \tag{2.3}$$

for all $w \in R_n$. This involution is determined by its action on the generators a_{ij} which is as follows:

$$a_{ij}^* = -a_{ji}.$$

This involution has the following property:

$$trace(A^{-1}) = trace(A)^*,$$

for all $A \in F_n$. Thus for $c \in \mathcal{C}$ we have $\phi(c^{-1}) = \phi(c)^*$, where c^{-1} is the curve c with its orientation reversed. We also have $b_{ji} = -b_{ij}^*$, for b_{ij}, b_{ji} as in (2.1).

§3 Preliminaries on symmetric functions

We will need the following results on symmetric functions; these results can all be found in [M].

Let $a_1, \dots, a_n$ be algebraically independent indeterminates. For $(\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ we let a^α denote the monomial

$$a_1^{\alpha_1} a_2^{\alpha_2} \dots a_n^{\alpha_n}.$$

Recall that a partition is a sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ of non-negative integers such that $\lambda_1 \geq \lambda_2 \geq \dots$. The number r is called the *length* of the partition and the *weight* of the partition λ is the sum $|\lambda| = \lambda_1 + \lambda_2 + \dots + \lambda_r$. The polynomial

$$m_\lambda(a_1, a_2, \dots, a_r) = \sum a^\alpha,$$

where the sum is taken over all distinct permutations α of $\lambda = (\lambda_1, \dots, \lambda_r)$. The m_λ form a basis for the ring of symmetric polynomials in the variables $a_1, \dots, a_r$.

The *elementary symmetric functions* e_i are defined as follows: for $r \geq 0$ let

$$e_r = \sum_{i_1 < i_2 < \dots < i_r} a_{i_1} a_{i_2} \dots a_{i_r} = m_{(1^r)}.$$

For a partition λ we let

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \dots e_{\lambda_r}.$$

Note that the degree of m_λ or e_λ is $|\lambda|$. The elementary symmetric functions satisfy:

$$\prod_{i=1}^{r} (1 - t a_i) = \sum_{i=0}^{r} e_i (-t)^i.$$

From [M, p.35] we get

Lemma 3.1. *For all $r \geq 0$ we have:*

$$\begin{aligned}\prod_{i,j=1}^{r} (1 + a_i b_j) &= \sum_\lambda e_\lambda(a_1, \dots, a_r) m_\lambda(b_1, \dots, b_r) \\ &= \sum_\lambda m_\lambda(a_1, \dots, a_r) e_\lambda(b_1, \dots, b_r).\end{aligned}$$

Here the sum in the right hand side is over all partitions of weight at most r. □

We note that since the μ_λ and the e_λ are both algebraically independent generators for the ring of symmetric polynomials, that it is possible to express the μ_λ as linear combinations of the e_λ with rational coefficients, and vice versa.

We have the following easy result:

Lemma 3.2. *Suppose that p is a polynomial in $\mathbb{C}[a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_r]$ of degree m in $a_1, a_2, \dots, a_n$. Suppose that p is invariant under the natural action of S_n on $a_1, a_2, \dots, a_n$. Then p is a polynomial in $\mathbb{C}[e_\lambda(a_1, a_2, \dots, a_n), b_1, b_2, \dots, b_r]$ where the degree of each of the $e_\lambda(a_1, a_2, \dots, a_n)$ that we need is at most m.* □

§4 Action of B_n on R_n continued

In this section we describe in greater detail the action of B_n on R_n as outlined in §2 so as to be able to give explicit formulae for the action of $\tau^m_{\gamma_{1m}}$. In general [A] a transvection in $SL_n(Q)$ (for a commutative ring Q with identity) can be defined as a pair $T = (\phi, d)$ where $d \in Q^n$ and ϕ is an element of the dual space of Q^n satisfying $\phi(d) = 0$. The action is given by

$$T(x) = x + \phi(x)d \quad \text{for all} \quad x \in Q^n.$$

Then we have [H1, Lemma 2.1]

Lemma 4.1. *Let $T = (\phi, d)$ and $U = (\psi, e)$ be two transvections. Then for all $\lambda \in \mathbb{Z}$ we have*

$$U^\lambda T U^{-\lambda} = (\phi - \lambda\phi(e)\psi, U^\lambda(d)). \quad \square$$

Let $\{T_1 = (\phi_1, d_1), \ldots, T_n = (\phi_n, d_n)\}$ be a fixed set of transvections in $SL_n(R_n)$ where $\phi_i(d_j) = a_{ij}$ for all $1 \le i \ne j \le n$. For any set of transvections $T = \{T_1 = (\phi_1, e_1), \ldots, T_n = (\phi_n, e_n)\}$ we let $M(T)$ denote the $n \times n$ matrix $(\phi_i(e_j))$ and we call $M(T)$ the *M-matrix* of the set of transvections T.

Any monomial that can be written in the form $a_{j_1j_2}a_{j_2j_3}\ldots a_{j_{r-1}j_r}$ will be called a j_1j_r-word. Note that by (2.2) if $\alpha \in B_n$ and $1 \le i \ne j \le n$, then $\alpha(a_{ij})$ is a sum of rs-words, where $\alpha(T_i)$ is a conjugate of T_r and $\alpha(T_j)$ is a conjugate of T_s. Let $\alpha \in B_n$ where $\alpha(T_i) = w_iT_jw_i^{-1}$ in freely reduced form for $i = 1, ..., n$ and where $w_i = w_i(T_1, \ldots, T_n)$. Then for $i = 1, \ldots, n$ we have $w_iT_iw_i^{-1} = (\psi_i, f_i)$ for some ψ_i, f_i determined by Lemma 4.1, which shows that

$$\psi_i = q_1\phi_1 + \cdots + q_n\phi_n \quad \text{and} \quad f_i = p_1d_1 + \cdots + p_nd_n,$$

where $p_1, \ldots, p_n, q_1, \ldots, q_n \in R_n$. We define the action of B_n on R_n by

$$\alpha(a_{ij}) = \psi_i(f_j).$$

This agrees with the previous definition. Thus the M-matrix $M(\alpha(T_1), \ldots, \alpha(T_n))$ is $\alpha(M(T))$.

From Lemma 2.3 of [H1] we have:

Lemma 4.2. *Let $1 \le i, j \le n$, $\alpha \in B_n$ where $\alpha(T_i) = C_1T_kC_1^{-1}, \alpha(T_j) = C_2T_pC_2^{-1}$, with $C_1, C_2 \in < T_1, \ldots, T_n >$ and let $C = C_1^{-1}C_2 = T^{q_1}_{j_1}\ldots T^{q_r}_{j_r}$ be freely reduced with $j_r \ne p,\ j_1 \ne k, q_s \ne 0$ for $s = 1, \ldots, r$ and $j_s \ne j_{s+1}$, for $s = 1, \ldots, r-1$. Then*

$$\alpha(a_{ij}) = \sum_{h=1}^{n} A_h a_{hp}$$

where A_h is equal to the sum of all the products of the form

$$q_{r_1}q_{r_2}\ldots q_{r_m}a_{kj_{r_1}}a_{j_{r_1}j_{r_2}}\ldots a_{j_{r_{m-1}}j_{r_m}}$$

where $1 \leq r_1 < r_2 < \cdots < r_m \leq r$ *and* $j_{r_m} = h$*. If* $p \neq j_r$*, then the summand of* $\alpha(a_{ij})$ *of highest degree is unique and is equal to*

$$\pm q_1 q_2 \ldots q_r a_{kj_1} a_{j_1 j_2} \ldots a_{j_{r-1} j_r} a_{j_r p}. \quad \square$$

For example if $\alpha(T_1) = T_3 T_2^{-1} T_1 T_2 T_3^{-1}$ and $\alpha(T_2) = T_2^{-1} T_3 T_2$, then we would have $C = T_2 T_3^{-1} T_2^{-1}$ and

$$\alpha(a_{12}) = a_{13} + a_{13} a_{32} a_{23} + a_{12} a_{23} a_{32} a_{23}.$$

For each $m \geq 2$ we let $\tau_m = \tau_{\gamma_{1m}} = \sigma_{m-1} \ldots \sigma_2 \sigma_1$, the $1/m$ twist about the curve $\gamma_{1,m}$. Then we have:

Lemma 4.3. *(i) For all* $1 \leq i \leq m$ *we have*

$$\tau_m^m(x_i) = (x_1 x_2 \ldots x_m) x_i (x_1 x_2 \ldots x_m)^{-1};$$

(ii) For all $1 \leq i \neq j \leq m$ *we have* $\tau_m^m(a_{ij}) = a_{ij}$*;*
(iii) For $1 \leq i \leq m < r \leq n$ *we have*

$$\tau_m^m(a_{ri}) = a_{ri} + \sum_{s=1}^{m} \sum_{1 \leq j_1 < j_2 < \cdots < j_s \leq m} a_{rj_1} a_{j_1 j_2} a_{j_2 j_3} \ldots a_{j_{s-1} j_s} a_{j_s i}.$$

(iv) For $1 \leq i \leq m < r \leq n$ *we have*

$$\tau_m^m(a_{ir}) = a_{ir} + \sum_{s=1}^{m} \sum_{1 \leq j_1 < j_2 < \cdots < j_s \leq m} (-1)^s a_{ij_s} a_{j_s j_{s-1}} \ldots a_{j_3 j_2} a_{j_2 j_1} a_{j_1 r}.$$

(v) For all $m < u \neq v \leq n$ *we have* $\tau_m(a_{uv}) = a_{uv}$*.*

Proof. (i) follows from the action of B_n given in §2. The rest all follow from (i) and Lemma 4.2. $\square$

For $y > m$ let v_y be the vector $(a_{y1}, a_{y2}, \ldots, a_{ym})$. Let $w_y = (a_{1y}, a_{2y}, \ldots, a_{my}) = -v_y^*$. Then we have :

Lemma 4.4. *For all* $2 \leq m \leq n$ *there is a* $m \times m$ *matrix* $M_m \in SL(m, R_m)$ *such that*

$$\tau_m^{hm}(v_y) = v_y M_m^h,$$

for all $h \in \mathbb{Z}$ *and all* $n \geq y > m$*. In fact* $M_m = T_1 T_2 \ldots T_m$*.*
We also have $\tau_m^{hm}(w_y) = w_y (M_m^*)^h$ *for all* $h \in \mathbb{Z}$*.*

Proof. Fix such a y. Consider the matrix $\Pi_m = T_1 T_2 \ldots T_m$. Its (i,k) entry is

$$\sum_{s=0}^{m} \sum_{i \leq j_1 < j_2 < \cdots < j_s \leq m} a_{ij_1} a_{j_1 j_2} \ldots a_{j_{s-1} j_s} a_{j_s k}.$$

But by Lemma 4.3 this is exactly equal to the coefficient of a_{yi} in $\tau_m^m(a_{yk})$, as required for $h = 1$. But the entries of Π_m are invariant under the action of τ_m^m since these entries are in R_m (use Lemma 4.3 again). So we see that $\tau_m^{hm}(v_y) = v_y M_m^h$, for all $h \in \mathbb{Z}$.

Now $v_y^* = -w_y$ and if we apply the automorphism * to the equation $\tau_m^{hm}(v_y) = v_y M_m^h$, and use (2.3), then we obtain $w_y (M_m^*)^h = \tau_m^{hm}(w_y)$ as required. $\square$

Lemma 4.5. *The matrix $\Pi_n = T_1T_2\dots T_n$ has n distinct roots.*

Proof. Let T_i' be the $n\times n$ matrix T_i with all the indeterminates a_{ij} replaced by an indeterminate b_i. We prove by induction that there are positive real numbers b_i such that for all $m\geq 2$ the matrix $T_1'T_2'\dots T_m'$ has $n-m$ eigenvalues equal to 1, the rest being distinct and not equal to 1. The case $m=2$ is easy (take $b_1=b_2=1$). Now assume that $T_1'T_2'\dots T_m', 2\leq m<n$ satisfies the inductive hypothesis for some positive real choice of $b_1,\dots,b_m$. Consider $P=T_1'T_2'\dots T_m'T_{m+1}'$. Since $b_1,\dots,b_m>0$, the matrix $T_1'T_2'\dots T_m'$ has all strictly positive entries in its first m rows. Thus the trace of P is a non-constant function of b_{m+1}. Thus the characteristic polynomial of P is a function which depends continuously and non-constantly on b_{m+1}. When $b_{m+1}=0$, this polynomial has m distinct roots not equal to 1, and so we can choose b_{m+1} small enough so that P has $m+1$ distinct eigenvalues not equal to 1. (In fact if one takes $b_1=1$ and $b_i=1/(i-1)$ for $n\geq i\geq 2$, then the eigenvalues are $\frac{n-2}{n-1},\frac{n-3}{n-2},\dots,\frac{2}{1}$ together with the two positive real and distinct roots of the quadratic $z^2-(2n-1)z+n-1$). □

Lemma 4.6. *Suppose that the matrix $T_1T_2\dots T_n$ has characteristic polynomial $\chi_n(x)$. Then the matrix $(T_1T_2\dots T_n)^*$ has characteristic polynomial*

$$\chi_n(x)^* = (-x)^n\chi(1/x).$$

The eigenvalues of $(T_1T_2\dots T_n)^*$ are the inverses of the eigenvalues of $T_1T_2\dots T_n$.

Proof. The first claim is an immediate consequence of [H2; Corollary 2.7], which says that the same result is true for all matrices in $< T_1,\dots,T_n >$. The second claim follows from the first. □

Now suppose that μ is a monomial in R_n. Then we can write $\mu=\mu_1\mu_2\mu_3\mu_4$ where each μ_i is a monomial with

$$\mu_1\in R_m,\ \ \mu_2\in R[a_{ik}|1\leq i\leq m<k],\ \ \mu_3\in R[a_{ki}|1\leq i\leq m<k]$$
$$\text{and}\ \ \mu_4\in R[a_{ik}|m<i,k\leq n]. \tag{4.1}$$

Now τ_m^m fixes μ_1 and μ_4 and its action on the generators a_{ik} occurring in μ_2 and μ_3 is given by Lemma 4.3. Further, we note that if $\mu\in Y_n$, then μ_2 and μ_3 have the same degrees.

§5 Kronecker products

We now wish to recall some elementary facts about Kronecker products of matrices. The basic reference is [HJ, §4.2]. Let $A=(a_{ij})$ be an $n\times n$ matrix and $B=(b_{ij})$ be an $m\times m$ matrix. Then the *Kronecker* (or *tensor*) product $A\otimes B$ is the $nm\times nm$ block matrix

$$A\otimes B=\begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B\\ a_{21}B & a_{22}B & \dots & a_{2n}B\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1}B & a_{n2}B & \dots & a_{nn}B\end{pmatrix}.$$

This can be interpreted as follows: Suppose that A and B are the matrices of linear transformations $\alpha: V\to V$ and $\beta: W\to W$ acting on vector spaces V and W with respect to bases u_i and v_i (respectively). Then the action of $\alpha\otimes\beta$ on $V\otimes W$ with respect to the basis $u_i\otimes v_j$ is given by $A\otimes B$.

Lemma 5.1. *[HJ; p. 245] If the eigenvalues of A are $a_1, \dots, a_n$ and the eigenvalues of B are $b_1, \dots, b_m$ (both counting multiplicities), then the eigenvalues of $A \otimes B$ are $a_i b_j$ for all $i = 1, \dots, n$, $j = 1, \dots, m$ (counting multiplicities).* □

We apply this to the situation where $y > m$ is fixed and τ_m^m is acting on the subring $R[a_{iy}, a_{yj} | 1 \le i, j \le m]$ of R_n. Then by Lemma 4.3 τ_m^m fixes each element of R_m and there are matrices M_m, M_m^* such that $\tau_m^m(v_y) = v_y M_m$ and $\tau_m^m(w_y) = w_y M_m^*$. But then, by the above remarks, the action of τ_m^m on products like $a_{iy}a_{yj}$ is completely determined by the Kronecker product $N_m = M_m \otimes M_m^*$. In fact, since the matrix M_m does not depend on y the Kronecker product N_m also determines the action of τ_m^m on products like $a_{iy}a_{zj}$ for all $n \le y, z > m$. Thus we have

Lemma 5.2. *Let $\chi_N(t)$ denote the characteristic polynomial of N_m. If we have a monomial $\mu = \mu_1\mu_2\mu_3\mu_4 \in Y_n$ as in (4.1), where μ_2 and μ_3 have degree 1, then*

$$\chi_N(\tau_m^m)(\mu) = 0.$$

Proof. If $\mu = \mu_1\mu_2\mu_3\mu_4$, then τ_m^m fixes each of μ_1 and μ_4; thus $\chi_N(\tau_m^m)(\mu) = \mu_1\mu_4\chi_N(\tau_m^m)(\mu_2\mu_3)$, the latter being 0 by the above discussion. □

Now the action of τ_m^m on an arbitrary monomial in Y_n is given by

Proposition 5.3. *The action of τ_m^m on any monomial $\mu = \mu_1\mu_2\mu_3\mu_4 \in Y_n$ where μ_2 and μ_3 have degree r is determined by the r-fold Kronecker product $N_m^{\otimes r}$. If $\chi_{N^{\otimes r}}(t)$ denotes the characteristic polynomial of $N_m^{\otimes r}$, then*

$$\chi_{N^{\otimes r}}(\tau_m^m)(\mu) = 0. \quad \square$$

Corollary 5.4. *If $\alpha \in Y_n$, then there is $r \in \mathbb{Z}$ such that*

$$\chi_{N^{\otimes r}}(\tau_m^m)(\alpha) = 0. \quad \square$$

This proves Theorem 1.3, since we can let

$$\mathfrak{B}_{m0}(x) = x - 1 \quad \text{and} \quad \mathfrak{B}_{mr}(x) = \chi_{N^{\otimes r}}(x) \quad \text{for} \quad r \ge 1.$$

We note that $\iota(\gamma_{1m}, \gamma) = 0$ if and only if $\tau_m^m(\gamma) = \gamma$ if and only if $\mathfrak{B}_{m0}(\tau_m^m)(\phi(\gamma)) = 0$.

Now by Lemma 4.6 the eigenvalues of Π_m and Π_m^* are inverses of each other. Thus by Lemma 5.1 the matrix $N_m = M_m \otimes M_m^*$ has 1 as an eigen-value with multiplicity at least $m \ge 2$. Thus also from Lemma 5.1 we see that any root of $\chi_{N^{\otimes r}}$ is also a root of $\chi_{N^{\otimes s}}$ whenever $0 \le r < s$. It also easily follows that if α is a root of $\chi_{N^{\otimes r}}$ and β is a root of $\chi_{N^{\otimes s}}$, then $\alpha\beta$ is a root of $\chi_{N^{\otimes r+s}}$. This is the proof of the second part of Theorem 1.2.

Examples 5.5. (i) If $m = 2$, then we let $I = -2 - a_{12}a_{21}$ so that $x^2 + Ix + 1$ is the characteristic polynomial of M_2. It has roots $a, 1/a$ say, since $det(M_2) = 1$. Then by Lemma 4.6 M_2^* also has roots $a, 1/a$ and so $\chi_N(x)$ has roots $1, 1, a^2, 1/a^2$ (by Lemma 5.1). The roots of $\chi_{N^{\otimes r}}(x)$ are thus all of the 4^r products $a_1a_2 \dots a_r$ for all possible $a_i = 1, 1, a^2, 1/a^2$. To find $\chi_{N^{\otimes r}}(x)$ one multiplies out

$$\prod_{a_1,\dots,a_r=1,1,a^2,1/a^2} (x - a_1)(x - a_2)\dots(x - a_r)$$

and finds that this is a polynomial in x and I.

To be more specific we let $s_0(x) = x - 1$ and for $i \geq 1$ we let

$$s_i = (x - a^{2i})(x - 1/a^{2i}).$$

Thus for example, we have

$$\begin{aligned}
s_1(x) &= x^2 + (2 - I^2)x + 1;\\
s_2(x) &= x^2 + (-2 + 4I^2 - I^4)x + 1;\\
s_3(x) &= x^2 + (2 - 9I^2 + 6I^4 - I^6)x + 1;\\
s_4(x) &= x^2 + (-2 + 16I^2 - 20I^4 + 8I^6 - I^8)x + 1.
\end{aligned}$$

The s_i are quadratic polynomials in x related to the polynomials $\mathfrak{A}_i$ defined in [H4]. We have

$$\mathfrak{B}_{2\,0}(x) = s_0(x), \quad \mathfrak{B}_{2\,1}(x) = s_0(x)^2 s_1(x).$$

By the above we see that $\mathfrak{B}_{2\,m+1}(x) = \chi_{N^{\otimes m+1}}(x)$ will factor into a product of the form

$$\mathfrak{B}_{2m}(x)^2 C_{m1}(x) C_{m2}(x)$$

where $C_{m1}(x)$ (respectively $C_{2m}(x)$) is obtained from $\mathfrak{B}_{2m}(x)$ by replacing each root α of $\mathfrak{B}_{2m}(x)$ by αa^2 (respectively α/a^2). Thus each $\mathfrak{B}_{2m}(x)$ is a product of powers of the $s_i(x)$. This has the effect of replacing $s_0(x)$ by $s_1(x)$, $s_1(x)$ by $s_0(x)^2 s_2(x)$, $s_2(x)$ by $s_1(x)s_3(x)$, $s_3(x)$ by $s_2(x)s_4(x)$ etc. We obtain

$$\begin{aligned}
\mathfrak{B}_{21}(x) &= s_0(x)^2 s_1(x);\\
\mathfrak{B}_{22}(x) &= s_0(x)^6 s_1(x)^4 s_2(x);\\
\mathfrak{B}_{23}(x) &= s_0(x)^{20} s_1(x)^{15} s_2(x)^6 s_3(x); \;\; etc.
\end{aligned}$$

Using the fact that $\mathfrak{B}_{2\,m+1}(x) = \mathfrak{B}_{2m}(x)^2 C_{m1}(x) C_{m2}(x)$ for C_{1m}, C_{2m} as described above one easily sees that the general result is

$$\mathfrak{B}_{2n}(x) = \prod_{i=0}^{n} s_i^{\binom{2n}{n-i}}.$$

(ii) For the case $m = 3$ we see that if M_3 has roots a, b, c (which are distinct by Lemma 4.5), then $abc = 1$ since $det(M_3) = 1$. Further, M_3^* has roots $1/a, 1/b, 1/c$

and so $N_3 = M_3 \otimes M_3^*$ has roots $1, 1, 1, \frac{a}{b}, \frac{a}{c}, \frac{b}{a}, \frac{b}{c}, \frac{c}{a}, \frac{c}{b}$ (by Lemma 5.1). This allows one to find that

$$\begin{aligned}
\mathfrak{B}_{31}(x) &= (x-1)^3[x^6 + (3 - p_1p_2)(x + x^5) + (6 - 5p_1p_2 + p_1^3 + p_2^3)(x^2 + x^4) \\
&\qquad + (7 + 2p_1^3 - 6p_1p_2 - p_1^2p_2^2 + 2p_2^3)x^3 + 1]; \\
\mathfrak{B}_{32}(x) &= (x-1)^{15}[x^3 + (3p_1p_2 - p_2^3 - 3)x^2 + (p_1^3 - 3p_1p_2 + 3)x - 1]^2 \times \\
&[x^3 + (-p_1^3 + 3p_1p_2 - 3)x^2 + (-3p_1p_2 + p_2^3 + 3)x - 1]^2 \times \\
&[x^6 + (3 - p_1p_2)(x + x^5) + (6 - 5p_1p_2 + p_1^3 + p_2^3)(x^2 + x^4) \\
&\qquad + (7 + 2p_1^3 - 6p_1p_2 - p_1^2p_2^2 + 2p_2^3)x^3 + 1]^8 \times \\
&[x^6 + (2p_1^3 - p_1^2p_2^2 - 4p_1p_2 + 2p_2^3 + 3)(x + x^5) \\
&+ (p_1^6 - 6p_1^4p_2 + 2p_1^3 + 19p_1^2p_2^2 - 6p_1p_2^4 - 20p_1p_2 + p_2^6 + 2p_2^3 + 6)(x^2 + x^4) \\
&+ (-2p_1^6 + 4p_1^5p_2^2 - p_1^4p_2^4 + 4p_1^4p_2 - 16p_1^3p_2^3 - 4p_1^3 + 4p_1^2p_2^5 + 26p_1^2p_2^2 + 4p_1p_2^4 \\
&\qquad - 24p_1p_2 - 2p_2^6 - 4p_2^3 + 7)x^3 + 1].
\end{aligned}$$

where $p_1 = a_{12}a_{21} + a_{13}a_{31} + a_{23}a_{32} - a_{13}a_{32}a_{21}$ and $p_2 = -p_1^*$. As in the case $m = 2$ one can find a (more complicated) recursion for the polynomials $\mathfrak{B}_{3n}(x)$.

(iii) For $m = 4$ similar reasoning allows one to find that

$$\begin{aligned}
\mathfrak{B}_{41}(x) =&(x-1)^4[(1 + x^{12}) + (x + x^{11})(-p_2p_3 + 4) \\
&+ (x^2 + x^{10})(-2p_1^2 + p_1p_2^2 + p_1p_3^2 - 4p_2p_3 + 10) \\
&+ (x^3 + x^9)(-p_1^2p_2p_3 - 8p_1^2 + 7p_1p_2^2 + 7p_1p_3^2 - p_2^4 - 13p_2p_3 - p_3^4 + 20) \\
&+ (x^4 + x^8)(p_1^4 - 8p_1^2p_2p_3 - 16p_1^2 + p_1p_2^3p_3 + 18p_1p_2^2 \\
&\qquad + p_1p_2p_3^3 + 18p_1p_3^2 - 3p_2^4 - p_2^2p_3^2 - 24p_2p_3 - 3p_3^4 + 31) \\
&+ (x^5 + x^7)(4p_1^4 - p_1^3p_2^2 - p_1^3p_3^2 - 19p_1^2p_2p_3 - 24p_1^2 + 5p_1p_2^3p_3 + 29p_1p_2^2 \\
&\qquad + 5p_1p_2p_3^3 + 29p_1p_3^2 - 6p_2^4 - p_2^3p_3^3 - 2p_2^2p_3^2 - 34p_2p_3 - 6p_3^4 + 40) \\
&+ x^6(6p_1^4 - 2p_1^3p_2^2 - 2p_1^3p_3^2 + p_1^2p_2^2p_3^2 - 24p_1^2p_2p_3 - 28p_1^2 + 6p_1p_2^3p_3 \\
&\quad + 34p_1p_2^2 + 6p_1p_2p_3^3 + 34p_1p_3^2 - 7p_2^4 - 2p_2^3p_3^3 - 40p_2p_3 - 7p_3^4 + 44)].
\end{aligned}$$

Here

$$\begin{aligned}
p_3 =& -a_{21}a_{14}a_{43}a_{32} + a_{13}a_{32}a_{21} + a_{21}a_{14}a_{42} + a_{31}a_{14}a_{43} + a_{32}a_{24}a_{43} - a_{12}a_{21} \\
&- a_{13}a_{31} - a_{23}a_{32} - a_{14}a_{41} - a_{24}a_{42} - a_{34}a_{43} - 4, \\
p_1 =& a_{23}a_{32}a_{14}a_{41} - a_{13}a_{32}a_{24}a_{41} - a_{23}a_{31}a_{14}a_{42} + a_{13}a_{31}a_{24}a_{42} \\
&- a_{13}a_{34}a_{42}a_{21} - a_{12}a_{24}a_{43}a_{31} + a_{12}a_{21}a_{34}a_{43} + a_{12}a_{23}a_{31} - a_{13}a_{21}a_{32} \\
&+ a_{12}a_{24}a_{41} + a_{13}a_{34}a_{41} - a_{21}a_{14}a_{42} + a_{23}a_{34}a_{42} - a_{31}a_{14}a_{43} \\
&- a_{32}a_{24}a_{43} + 2a_{12}a_{21} + 2a_{13}a_{31} + 2a_{23}a_{32} + 2a_{14}a_{41} + 2a_{24}a_{42} + 2a_{34}a_{43} + 6, \\
p_2 =& p_3^*.
\end{aligned}$$

These examples, together with other calculations made in the preparation of this paper, were accomplished using MAGMA [MA].

As we are seeing in these examples, the coefficients of $\mathfrak{B}_{mr}(x)$ are polynomials in the coefficients of $\chi_{M_m}(x)$, as stated in Theorem 1.2. That this is the case in general follows from Lemma 3.2 and Lemma 5.2 applied to $N_m = M_m \otimes M_m^*$.

§6 Geometric intersection numbers of curves on a disc

Let $\gamma, \gamma' \in \mathcal{C}$. We wish to find the geometric intersection number $\iota(\gamma, \gamma')$. Since B_n acts transitively on each $\mathcal{C}_m$ we may without loss assume that $\gamma' = \gamma_{1m}$ for some $m \geq 2$. For $i = 1, \ldots, n$ let b_i denote a vertical arc in D_n meeting the arc a_i exactly once, meeting no other $a_j, j \neq i$, and with its endpoints on the boundary of D_n. Then $D_n \setminus b_i$ has two components, one containing the arcs $a_1, \ldots, a_{i-1}$ and part of a_i. This component we will think of as the disc D_m, so that γ_{1m} is the boundary curve of $D_m \subset D_n$.

Let μ denote the monomial of greatest degree in $\phi_n(\gamma)$ and write $\mu = \mu_1\mu_2\mu_3\mu_4$ as we have done in (4.1). Write $\mu = \mu(\gamma), \mu_i = \mu_i(\gamma)$. We note that $\phi(\gamma) \in Y_n$ and so $deg(\mu_2) = deg(\mu_3)$. Then we have

Proposition 6.1. *For $\gamma \in \mathcal{C}$ we have*

$$\iota(\gamma_{1m}, \gamma) = degree(\mu_2(\gamma)\mu_3(\gamma)) = 2degree(\mu_2(\gamma)).$$

Proof. We may assume that γ meets γ_{1m} in $\iota(\gamma_{1m}, \gamma)$ points. Then, since γ_{1m} is the boundary curve of $D_m \subset D_n$, we see that γ meets b_m in exactly $\iota(\gamma_{1,m}, \gamma)$ points. Now let $w \in< T_1, \ldots, T_n >$ be a cyclically reduced word representing γ. Then we can write

$$w = w_1y_1w_2y_2 \ldots w_ky_k \quad \text{where} \quad w_i \in< T_1, \ldots, T_m >, \;\; y_i \in< T_{m+1}, \ldots, T_n > .$$

Here we may assume that w_i and y_i are all non-trivial. Now using Lemma 4.2 we see that $\phi(\gamma) = trace(w) - n$ is a sum of monomials of the form

$$\pm m_1 a_{i_1 j_1} m_1' a_{j_1' i_1'} m_2 a_{i_2 j_2} m_2' a_{j_2' i_2'} \ldots m_h a_{i_h j_h} m_h' a_{j_h' i_h'},$$

where each m_i is a monomial in R_m, each monomial m_i' is in $R[a_{uv}|m < u, v \leq n]$ and $1 \leq i_r, i_r' \leq m < j_r, j_r' \leq n$ for all $1 \leq r \leq h \leq k$. We note that $\iota(\gamma_{1m}, \gamma) = 2k$ since for every adjacent pair of words w_1y_1, $y_1w_2, \ldots, w_ky_k$, y_kw_1 in w the curve γ must cross the arc b_m and conversely, each crossing must correspond to one of these pairs of adjacent words in w. This proves the result. $\square$

Thus if $\iota(\gamma_{1m}, \gamma) = 2k$, then by the proposition 5.3 we see that $\mathfrak{B}_{mk}(\tau_m^m)(\phi(\gamma)) = 0$. This proves the first part of Theorem 1.1.

Now the above argument never used the fact that γ was simple closed curve, so what we have proved is

Proposition 6.2. *Let γ be any closed curve on the disc D_n and let*

$$w = T_{i_1}^{\pm 1} T_{i_2}^{\pm 1} \ldots T_{i_k}^{\pm 1} \in< T_1, \ldots, T_n >$$

be any cyclically reduced word representing γ. Then the geometric intersection number $\iota(\gamma_{1m}, \gamma)$ is equal to the number of $j \leq k$ such that $i_j \leq m$ and $i_{j+1} > m$ or $i_j > m$ and $i_{j+1} \leq m$ (indices taken mod k). Further, if $2r = \iota(\gamma_{1m}, \gamma)$, then $\mathfrak{B}_{mr}(\tau_m^m)(\phi(\gamma)) = 0$. $\square$

In order to conclude the proof of Theorem 1.1 it will suffice to prove

Proposition 6.3. *For all $m \geq 2$ and $r \geq 1$ we have*

$$\mathfrak{B}_{mr}(\tau_m^m)((a_{1\,m+1}a_{m+1\,1})^{r+1}) \neq 0.$$

Proof. One checks the following facts about degrees using Lemma 4.3:
(i) $deg(\tau_m^{km}(a_{1\,m+1})) = km+1$;
(ii) $deg(\tau_m^{km}(a_{1\,m+1}a_{m+1\,1})) = 2(km+1)$;
(iii) $\mathfrak{B}_{mr}(x)$ has degree m^{2r} in the variable x;
(iv) $deg((\tau_m^m)^{m^{2r}}(a_{1\,m+1}a_{m+1\,1})) = 2(m^{2r}m+1)$.

We also have
(v) The coefficient of $x^{m^{2r}-k}$ in $\mathfrak{B}_{mr}(x)$ is a polynomial in the a_{ij} of degree no greater than $2rmk$.

To see this last fact we note as in §5 that if a_i are the eigenvalues of $M_m = T_1 \dots T_m$, and if a'_i are the eigenvalues of M^*, then $a_i a'_j$ are the eigenvalues of $N_m = M_m \otimes M_m^*$ and $a_{i_1}a'_{j_1}a_{i_2}a'_{j_2}\dots a_{i_k}a'_{j_k}$ are the eigenvalues of $N_m^{\otimes k}$. Thus the coefficient of $x^{m^{2r}-k}$ in $\mathfrak{B}_{mr}(x)$ is the sum of all terms of the form $a_{i_1}a'_{j_1}a_{i_2}a'_{j_2}\dots a_{i_k}a'_{j_k}$. Such a sum is clearly invariant under the action of $S_m \times S_m$ and so is a polynomial in the coefficients of the characteristic polynomial of M and M^* (use Lemma 3.2). This polynomial has degree $2k$ (use Lemma 3.2 again) as a polynomial in these coefficients and so has degree $2mk$ as a polynomial in the a_{ij} (since, as noted in §1, the coefficients of the characteristic polynomial of M have degree m in R_n). This proves statement (v).

Continuing with the proof of Proposition 6.3 we see that if c_k is the coefficient of $x^{m^{2r}-k}$ in $\mathfrak{B}_{mr}(x)$, then

$$deg(c_k(\tau_m^m)^{m^{2r}-k}((a_{1\,m+1}a_{m+1\,1})^{r+1})) \leq 2mrk + (r+1)(2((m^{2r}-k)m+1))$$

for all $k \geq 0$ and where we have equality for $k=0$. It follows that

$$deg((\tau_m^m)^{m^{2r}}((a_{1\,m+1}a_{m+1\,1})^{r+1})) - deg(c_k(\tau_m^m)^{m^{2r}-k}((a_{1\,m+1}a_{m+1\,1})^{r+1})) \geq 2mk.$$

This proves Proposition 6.3 and concludes the proof of Theorem 1.1. □

Remark 6.4. Suppose that γ is a multicurve on D_n i.e. γ is the disjoint union of simple closed curves on D_n. Then the orbit of γ under the action of B_n contains a curve of the form $\gamma_{i_1j_1} \cup \gamma_{i_2j_2} \cup \dots \cup \gamma_{i_sj_s}$ where $i_k < j_k$ for all k and the intervals $[i_k, j_k]$ satisfy

$$[i_u, j_u] \cap [i_v, j_v] = \emptyset \quad \text{or} \quad [i_u, j_u] \subset [i_v, j_v] \quad \text{or} \quad [i_v, j_v] \subset [i_u, j_u].$$

Thus we can, without loss, assume that γ is the above multicurve.

Now if $\gamma' \in \mathcal{C}$, then we let $r_k = \iota(\gamma_{i_kj_k}, \gamma')$. From Theorem 1.1 it follows that

$$[\mathfrak{B}_{j_1-i_1+1\,r_1}(\tau_{\gamma_{i_1j_1}}) + \dots + B_{j_s-i_s+1\,r_s}(\tau_{\gamma_{i_sj_s}})](\phi(\gamma)) = 0.$$

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