QUARTIC TRIGONOMETRIC BÉZIER CURVE WITH A SHAPE PARAMETER

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ABSTRACT

Analogous to the cubic Bézier curve, a quartic trigonometric Bézier curve with a shape parameter is presented in this paper. Each curve segment is generated by four consecutive control points. The shape of the curve can be adjusted by altering the values of shape parameters while the control polygon is kept unchanged. These curves are closer to the control polygon than the cubic Bézier curves, for all values of shape parameter. With the increase of the shape parameter, the curve approaches to the control polygon.

KEYWORDS: Trigonometric Bézier Basis Function, Trigonometric Bézier Curve, Shape Parameter

1. INTRODUCTION

The treatment of curves and surfaces in Computer Aided Geometric Designing (CAGD) and Computer Graphics (CG) requires developing the proper equations and algorithms for both computation and programming purposes. Curves describing engineering objects are generally smooth and well behaved. Products such as car bodies, ship hulls, airplane fuse lane and wings, propeller blades, shoe insoles and bottles are a few examples that require free form curves and surfaces. Therefore to fulfill these requirements, parametric representation of curves and surfaces is widely used in the field of CAGD and CG. The type of input data and its influence on the control of the resulting curve determine the use and effectiveness of curve in design. Designers need a curve representation that is directly related to the control points and is flexible enough to bend, twist or change the curve shape by changing one or more control points. For these reasons, Bézier curve and surface plays a significant role in the field of CAGD and CG. However, they have many shortcomings due to polynomial forms. In particular, they cannot represent exactly transcendental curves such as the helix and the cycloid etc. To overcome the shortcomings, many bases are presented using trigonometric functions or the blending of polynomial and trigonometric functions in [1-6].

Some of these existing methods have no shape parameters; hence the shape of the curves or surfaces cannot be modified when once their control points are determined. Many authors have studied different kinds of spline for curve and surface with shape parameters through incorporating parameters into the classical basis functions, where the parameters can adjust the shape of the curves and surfaces without changing the control points [7, 8, 9].

Recently, many papers investigate the trigonometric Bézier-like polynomial, trigonometric spline and their applications. Nikolis et al. [10] discussed the applications of trigonometric spline in dynamic systems. Dyllong, Su et al. [11, 12] modified a trajectory for robot manipulators using trigonometric spline. Neamtu et al. [13] presented some methods for designing the cam profile with trigonometric spline in CNC. Many kinds of methods based on trigonometric polynomials were also established for free form curves and surfaces modeling [14, 15, 16, 17, 18, 19]. In recent years, several new trigonometric splines have been studied in the literature; see [20, 21, 22, 23, 24]. Xi-An Han et al. [25] presented the cubic trigonometric Bézier curve with two shape parameters. However, up to now, quartic trigonometric
curves like those of Bézier type have not been studied. In this paper the quartic trigonometric Bézier curve with a shape parameter is presented.

The paper is organized as follows. In section 2, quartic trigonometric Bézier basis functions with a shape parameter are established and the properties of the basis functions are shown. In section 3, quartic trigonometric Bézier curves are given and some properties are discussed. By using shape parameter, shape control of the curves is studied. In section 4, the approximability of the quartic trigonometric Bézier curves and cubic Bézier curves corresponding to their control polygon are shown.

2. QUARTIC TRIGONOMETRIC BÉZIER BASIS FUNCTIONS

Firstly, the definition of quartic trigonometric Bézier basis functions is given as follows.

2.1 The Construction of the Basis Functions

Definition 2.1: For an arbitrarily selected real values of $\lambda$, where $\lambda \in [-1, 1]$, the following four functions of $t$ ($t \in [0,1]$) are defined as quartic trigonometric Bézier basis functions with a shape parameter $\lambda$:

\[
\begin{align*}
    b_0(t) &= \frac{1}{4}(1 - \sin \frac{\pi}{2} t)^2 (1 - \lambda \sin \frac{\pi}{2} t)^2 \\
    b_1(t) &= \frac{1}{4}(1 - \frac{1}{2}(1 - \cos \frac{\pi}{2} t)^2 (1 - \lambda \cos \frac{\pi}{2} t)^2) \\
    b_2(t) &= \frac{1}{4}(1 - \frac{1}{2}(1 - \sin \frac{\pi}{2} t)^2 (1 - \lambda \sin \frac{\pi}{2} t)^2) \\
    b_3(t) &= \frac{1}{4}(1 - \cos \frac{\pi}{2} t)^2 (1 - \lambda \cos \frac{\pi}{2} t)^2
\end{align*}
\] (2.1)

For $\lambda = 0$, the basis functions are quadratic trigonometric polynomials. For $\lambda \neq 0$, the basis functions are quartic trigonometric polynomials.

2.2 The Properties of the Basis Functions

Theorem 2.1: The basis functions (2.1) have the following properties:

- **Non-Negativity**: $b_i(t) \geq 0$ for $i = 0,1,2,3$.
- **Partition of Unity**: $b_i(t) \geq 0$ for $i = 0,1,2,3$.
- **Symmetry**: $b_i(t; \lambda) = b_{3-i}(1 - t; \lambda)$, for $i = 0,1,2,3$.
- **Monotonicity**: For a given parameter $t$, as the shape parameter $\lambda$ increases, $b_0(t)$ and $b_3(t)$ decreases and as the shape parameter $\lambda$ decreases, $b_1(t)$ and $b_2(t)$ increases.

Proof

For $t \in [0,1]$ and $\lambda \in [-1,1]$, then

$0 \leq (1 - \sin \frac{\pi}{2} t)^2 \leq 1$,

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$0 \leq (1 - \lambda \sin \frac{\pi}{2} t)^2 \leq 1$,

$0 \leq (1 - \lambda \cos \frac{\pi}{2} t)^2 \leq 1$
It is obvious that $b_i(t) \geq 0$ for $i = 0, 1, 2, 3$.

$$\sum_{i=0}^{3} b_i(t) = \frac{1}{4} (1 - \sin \frac{\pi}{2} t)^2 (1 - \lambda \sin \frac{\pi}{2} t)^2 + \frac{1}{2} (1 - \frac{1}{2} (1 - \cos \frac{\pi}{2} t)^2 (1 - \lambda \cos \frac{\pi}{2} t)^2) + \frac{1}{2} (1 - \frac{1}{2} (1 - \sin \frac{\pi}{2} t)^2 (1 - \lambda \sin \frac{\pi}{2} t)^2) + \frac{1}{4} (1 - \cos \frac{\pi}{2} t)^2 (1 - \lambda \cos \frac{\pi}{2} t)^2 = 1.$$

The remaining cases follow obviously.

The curves of the quartic trigonometric Bézier basis functions for $\lambda = 1$, $\lambda = 0$ and $\lambda = -1$ are shown in Figure 1, 2 and 3 respectively.

![Figure 1: Quartic Trigonometric Bézier Basis Functions for $\lambda = 1$](image1)

![Figure 2: Quartic Trigonometric Bézier Basis Functions for $\lambda = 0$](image2)

![Figure 3: Quartic Trigonometric Bézier Basis Functions for $\lambda = -1$](image3)

3. QUARTIC TRIGONOMETRIC BÉZIER CURVE

3.1 The Construction of the Quartic Trigonometric Bézier Curve

**Definition 3.1:** Given points $P_i$ ($i = 0, 1, 2, 3$) in $R^2$ or $R^3$.

Then

$$r(t) = \sum_{i=0}^{3} b_i(t) P_i, \quad t \in [0,1], \lambda \in [-1,1]$$

is called a quartic trigonometric Bézier curve with a shape parameter.
From the definition of the basis function some properties of the quartic trigonometric Bézier curve can be obtained as follows:

**Theorem 3.1:** The quartic trigonometric Bézier curves (3.1) have the following properties:

- **Terminal Properties**
  
  \[ r(0) = \frac{1}{4}[P_0 + 2P_1 + P_2], \]
  
  \[ r(1) = \frac{1}{4}[P_3 + 2P_2 + P_1], \]
  
  \[ r'(0) = (1 + \lambda) \frac{\pi}{6}(P_2 - P_0), \]
  
  \[ r'(1) = (1 + \lambda) \frac{\pi}{6}(P_3 - P_1) \] (3.2)

- **Symmetry**
  P_0, P_1, P_2, P_3 and P_0, P_2, P_1, P_0 define the same quartic trigonometric Bézier curve in different parameterizations, i.e.,
  \[ r(t; \lambda; P_0, P_1, P_2, P_3) = r(1 - t; \lambda; P_3, P_2, P_1, P_0), t \in [0,1], \lambda \in [-1,1] \] (3.4)

- **Geometric Invariance**
  The shape of a quartic trigonometric Bézier curve is independent of the choice of coordinates, i.e. (3.1) satisfies the following two equations:
  \[ r(t; \lambda; P_0 + q, P_1 + q, P_2 + q, P_3 + q) = r(t; \lambda; P_0, P_1, P_2, P_3) + q, \]
  \[ r(t; \lambda; P_0 * T, P_1 * T, P_2 * T, P_3 * T) = r(t; \lambda; P_0, P_1, P_2, P_3) * T, \] (3.5)
  \[ t \in [0,1], \lambda \in [-1,1] \]
  where q is arbitrary vector in \( R^2 \) or \( R^3 \), and T is an arbitrary \( d \times d \) matrix, \( d = 2 \) or 3.

- **Convex Hull Property**
  The entire quartic trigonometric Bézier curve segment lies inside its control polygon spanned by \( P_0, P_1, P_2, P_3 \).

### 3.2 Shape Control of the Quartic Trigonometric Bézier Curve

For \( t \in [0,1] \), we rewrite (3.1) as follows:

\[ r(t) = \sum_{i=0}^{3} P_i c_i(t) + \frac{1}{4} \lambda^2 \sin \frac{\pi}{2} t (1 - \sin \frac{\pi}{2} t)^2 (P_0 - P_2) + \frac{1}{4} \lambda^2 \cos \frac{\pi}{2} t (1 - \cos \frac{\pi}{2} t)^2 (P_3 - P_1) + \]

\[ \lambda \sin \frac{\pi}{2} (1 - \sin \frac{\pi}{2} t)^2 \frac{(P_0 + P_2)}{2} + \lambda \cos \frac{\pi}{2} t (1 - \cos \frac{\pi}{2} t)^2 \frac{(P_3 + P_1)}{2} \] (3.6)

Where

\[ c_0(t) = \frac{1}{4} (1 - \sin \frac{\pi}{2} t)^2, \]

\[ c_1(t) = \frac{1}{2} (1 - \frac{1}{2} (1 - \cos \frac{\pi}{2} t)^2), \]

\[ c_2(t) = \frac{1}{2} (1 - \frac{1}{2} (1 - \sin \frac{\pi}{2} t)^2), \]
Obviously, shape parameter $\lambda$ affects curve on the control edge $P_0 - P_2$, $(P_3 - P_1)$, $(P_0 + P_2)/2$ and $(P_1 + P_3)/2$. Therefore as the shape parameter $\lambda$ increases, the quartic trigonometric Bézier curve approximates the control polygon. Figure 4 shows some computed examples with different values of shape parameter $\lambda$.

![Figure 4: Quartic Trigonometric Bézier Curves with Different Values of Shape Parameter $\lambda$](image)

These curves are generated by setting $\lambda=1$ in (a), $\lambda=0$ in (b), $\lambda=-0.5$ in (c) and $\lambda=-1$ in (d).

### 4. APPROXIMABILITY

Control polygon provides an important tool in geometric modeling. It is an advantage if the curve being modeled tends to preserve the shape of its control polygon. Now we show the relations of the quartic trigonometric Bézier curves and cubic Bézier curves corresponding to their control polygon.

**Theorem 4.1**: Suppose $P_0, P_1, P_2$ and $P_3$ are not collinear; the relationship between quartic trigonometric Bézier curve $r(t)$ (3.1) and the cubic Bézier curve $B(t) = \sum_{i=0}^{3} P_i \binom{3}{i} (1-t)^{3-i}t^i$ with the same control points $P_i$ ($i = 0, 1, 2, 3$) are as follows:

- $r(0) = \frac{1}{4}[P_0 + 2P_1 + P_2], B(0) = P_0$
- $r(1) = \frac{1}{4}[P_1 + 2P_2 + P_3], B(1) = P_3$
- $r(\frac{1}{2}) - p^* = \frac{1}{2}(\sqrt{2} - 1)^2(\sqrt{2} - \lambda)^2 \left[ B\left(\frac{1}{2}\right) - p^* \right]$

where $p^* = \frac{1}{2}(P_1 + P_2)$

**Proof**

According to (3.2), we have

- $r(0) = \frac{1}{4}[P_0 + 2P_1 + P_2]$

and

- $r(1) = \frac{1}{4}[P_1 + 2P_2 + P_3]$
Performing simple computation, we have

\[ B(0) = P_0 \quad \text{and} \quad B(1) = P_3 \]

\[ B \left( \frac{1}{2} \right) - P^* = \frac{1}{8}(P_0 - P_1 - P_2 + P_3) \quad (4.2) \]

and according to (4.2), we have

\[ r \left( \frac{1}{2} \right) - P^* = \frac{1}{16}(\sqrt{2} - 1)^2(\sqrt{2} - \lambda)^2(P_0 - P_1 - P_2 + P_3) \]

\[ = \frac{1}{2}(\sqrt{2} - 1)^2(\sqrt{2} - \lambda)^2 \left( B \left( \frac{1}{2} \right) - P^* \right) \]

Then (4.1) holds.

From Figure 5, we can see that the quartic trigonometric Bézier curve (red lines for \( \lambda = 1, \lambda = 0 \) and \( \lambda = -1 \) respectively) is closer to the control polygon than the cubic Bézier curve (blue lines) for all values of \( \lambda \in [-1,1] \).

![Figure 5: The Relationship between Quartic Trigonometric Bézier Curves and the Cubic Bézier Curves](image)

**5. COMPOSITE QUARTIC TRIGONOMETRIC BÉZIER CURVES**

The condition of \( C^1 \) continuity between two quartic trigonometric Bézier curves is discussed as follows:

Let a quartic trigonometric Bézier curve \( r(t; \lambda) \) with control points \( P = (P_0, P_1, P_2, P_3) \), \( P_i \in R^2 \) or \( R^3 \) be given as (3.1) and a second curve \( r^*(t; \lambda^*) \) with control points \( P^* = (P'_0, P'_1, P'_2, P'_3) \), \( P_i \in R^2 \) or \( R^3 \) by

\[ r^*(t) = \sum_{i=0}^{3} b_i(t)P'_i, \quad t \in [0,1], \lambda^* \in [-1,1] \]

Clearly, for the composite curve to be \( C^1 \) continuous, it is necessary and sufficient that

\[ r(1) = r^*(0) \quad (5.1) \]

\[ r'(1) = r^*(0) \quad (5.2) \]

For ordinary cubic Bézier curves it is well known that the condition of continuity is as follows:

**Lemma 5.1:** Given two segments of cubic Bézier curves with control points \( P \) and \( P^* \), they are defined as

\[ B(t) = \sum_{i=0}^{3} P_i B_i^0(t), t \in [0,1], \]

and

\[ B^*(t) = \sum_{i=0}^{3} P'_i B_i^0(t), t \in [0,1], \]
Then the necessary and sufficient condition of continuity is

- For $C^0$ continuity,
  \[ P_3 = P_0; \]
- For $C^1$ continuity,
  \[ P_3 = P_0^* = \frac{1}{2}(P_2 + P_1^*); \]

According to the terminal properties of quartic trigonometric Bézier curves, the theorem below shows the condition of continuity of the composite quartic trigonometric Bézier curves.

Theorem 5.1: Given two segments of quartic trigonometric Bézier curves with the control points $P$ and $P^*$, then the necessary and sufficient condition of continuity is

- For $C^0$ continuity,
  \[ P_1 + 2P_2 + P_3 = Q_0 + 2Q_1 + Q_2; \]
- For $C^1$ continuity,
  \[ (1 + \lambda)(P_3 - P_1) = (1 + \lambda')(Q_2 - Q_0) \]
  When $\lambda = \lambda'$, straightforward computation gives that $P_1 + P_2 = Q_0 + Q_1$.

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