A Unified Model for Robust Optimization of Linear Programs with Uncertain Parameters

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This paper presents a general model for unification of the robust counterparts of uncertain linear programs (LP). We extend the robust optimization methodology for LP, introduced by Ben-Tal and Nemirovski, to a unified model in which the uncertainty region is approximated by an appropriate norm body. We derive the robust counterpart of an LP whose parameters may fall in any of the $l_1, l_2, l_\infty$ or matrix-norm bodies. An application to portfolio selection shows how an investor’s risk can be measured by a proper norm. Then, various solutions are shown for a practical example. The numerical results indicate that choosing $l_\infty$ norm leads to higher utility for an investor when the uncertainty region is large.

Keywords: Robust optimization, linear programming, data uncertainty, portfolio selection.

1. Introduction

The data of linear programs (LPs) are uncertain to some degree in most real world applications. Traditionally, the uncertainty is ignored and some nominal values of the parameters are used. These nominal values may be contaminated by various errors or be affected by random events. Difficulties with such data are typically dealt with using sensitivity and stability analysis. These methods evaluate the validity of the optimal solution with respect to the changes in each parameter.
separately [10]. However, this could lead to unacceptable results because in practice, the uncertain parameters are usually perturbed from their nominal values simultaneously. Ben-Tal and Nemirovski have shown that small perturbations of the data may make the optimal solution infeasible. Their experiments reveal that the optimal solutions for 13 of 94 NETLIB problems with nominal parameters become severely infeasible, with only 0.01%-perturbations [5].

A proactive approach for dealing with uncertainty is the one that takes into account the probabilistic information on the problem data. Stochastic programming (SP) with recourse [11], scenario optimization [20], entropic penalty methods [2], chance constraints and the recent scenario-based penalty methods [17] are well-known in this area. In these methods, the constraints are considered to be soft and thus may be violated with some acceptable cost [3]. The three main difficulties with such approaches are:

(a) Knowing the exact distribution for the data, and thus enumerating scenarios that capture the statistical distribution is rarely satisfied in practice.
(b) The size of the resulting optimization model increases drastically as a function of the number of scenarios, which poses substantial computational challenges [7].
(c) There are indeed situations in practice when an applicable solution must be feasible for all realizations of the data, and even a small violation of the constraints could not be tolerated [3].

Dealing with hard constraints is studied by Ben-Tal and Nemirovski [6]. They have introduced Robust Optimization Methodology (ROM) to address the shortcomings of soft constraint approaches [3]. In their methodology, the uncertain parameters are assumed to be bounded in an interval or an ellipsoidal region. Considering the worst-case behavior of the parameters, the robust counterpart of an uncertain LP becomes another LP or a second-order cone programming (SOCP) model depending on the type of assumption on the uncertainty. They claim that the ellipsoidal model of uncertainty is significantly less conservative than the interval region, and thus leads to more practical solutions [5]. They have also extended their method for general convex programming [4]. There is a parallel literature on robust optimization of engineering problems [9]. A unified robust modeling approach has been suggested for Design Centering in engineering application [21]. Robust optimization has also been used for discrete optimization and network flow problems [7].

This paper complements research in this area. The main contributions are:

1. Presenting a unified definition of uncertainty region for LP parameters, which depends on the concept of pdf-norm, and extends Ben-Tal and Nemirovski’s framework to more general uncertainty regions. The concept of pdf-norm has been borrowed from engineering applications.
2. Formulating the robust counterpart models of LP’s with various uncertainty regions as LP or SOCP models.
4. Comparing the quality of the solutions obtained by the proposed model using various norm definitions for the uncertainty regions, based on the concept of an individual’s utility function.
2. Generalized Uncertainty Region

We will use MATLAB inspired notation and write \((a_i ; b_i)\) to denote the column vector obtained from the column vector \(a_i\) by appending the scalar \(b_i\).

According to ROM, we call the problem

\[
\begin{align*}
\text{max} & \quad c^T x \\
\text{subject to} & \quad (a_i ; b_i)^T x \leq 0, \quad (a_i ; b_i) \in U_i, \quad \text{for } i = 1, \ldots, m \\
& \quad x \geq 0
\end{align*}
\]

an uncertain linear optimization problem. Assume that \(\hat{x} = (\hat{x}_i ; -1)^T\) and \((a_i ; b_i)\) are not known exactly; but must lie in a given uncertainty set \(U_i\). The uncertainty in \(c\) can also be absorbed in the constraints as shown later in equation (9). We build the generalized uncertainty region for \(i\)-th constraint by the following definition:

\[
U_i(p, r) = \{ (a_i ; b_i) : (a_i ; b_i) = (a_i^0 ; b_i^0) + W^i u \mid u \in B_p(r) \},
\]

where \(W^i\) is a symmetric and positive definite matrix, \((a_i^0 ; b_i^0)\) is the nominal value for \((a_i ; b_i)\) and an \(I_p\)-norm body is defined as:

\[
B_p(r) = \{ u \mid N_p(u) \leq r \}
\]

in which \(N_p(u) : \mathbb{R}^n \rightarrow \mathbb{R}^+\) is a vector norm function defined as:

\[
N_p(u) = \| u \|_p = \left( \sum_{j=1}^n |u_j|^p \right)^{1/p} \quad (I_p\text{-norm})
\]

The concept of norm body as defined above enables us to generalize the uncertainty region. This method applies to those joint statistical distributions whose level sets can be well approximated by a closed convex body. A norm which is used to characterize the contour of a pdf is referred to as a pdf-norm and the convex body associated with this norm is referred to as a norm body [8]. The choice of norm is directly related to the assumption on the statistical distribution of random variables.
parameters. Our method applies only to those symmetrical distributions whose contours can be approximated by a convex norm body. Figure 1 sketches various norm bodies in two dimensions. The maximum (infinity) norm body, is a rectangular cube which is associated with a joint uniform distribution with independent random parameters. The $l_2$-norm body is a ball (circle in Figure 1) characterizing the level sets of a standard normal distribution for independent random parameters. The $l_1$-norm body is shown by a diamond in Figure 1. In general, a bigger norm leads to a larger uncertainty region.

![Figure 1: Various norm bodies in two dimensions](image)

We are especially interested in the matrix-norm body which is shown as an ellipsoid in Figure 1 and is defined as:

\[
B_C(r) = \{ u \mid N_C(u) \leq r \}, \\
N_C(u) = \| u \|_C = \sqrt{u^T C^{-1} u}, \quad \text{(matrix-norm)}
\]

where $C$ is a symmetric positive definite $n \times n$ matrix. It characterizes the contours of a joint normal distribution with zero mean and covariance matrix $C$ [21]. Next section shows how the choice of the norm determines the robust counterpart models of LPs.

### 3. A Unified Robust Model

According to the general definition of an uncertainty region, a unified robust counterpart formulation is obtained by replacing $U_i$ in the problem (1) by $U_i(p, r)$. Consider the worst-case situation in the $i$-th constraint: $(a_i ; b_i)^T x \leq 0$.

According to the definition of $U_i(p, r)$, we have
\[
\begin{align*}
\max \quad & \left( a_i^0; b_i^0 \right)^T x + u^T W^{iT} x \right. \leq 0, \\
\text{or} \quad & \left( a_i^0; b_i^0 \right)^T x + \max_u \left[ u^T w^i(x) \right] \leq 0, \tag{7}
\end{align*}
\]

Here, we define \( w^i(x) = W^{iT} x \). Since \( l_p \) norms satisfy [18],

\[
\left\| u^T w^i(x) \right\|_p \leq \left\| w^i(x) \right\|_p ,
\]

and if the parameters are assumed to be restricted in the norm body \( N_p(u) \), then

\[
\left\| u^T w^i(x) \right\|_p \leq r \left\| w^i(x) \right\|_p . \tag{8}
\]

The maximization term in (7), is overestimated by the right hand side of (8). Therefore, the corresponding robust counterpart of (1) is,

\[
\begin{align*}
\max \quad & t \\
\text{subject to} \quad & c^0^T x - r_0 \left\| w^0(x) \right\|_{p_0} \geq t , \\
& \left( a_i^0; b_i^0 \right)^T x + r_i \left\| w^i(x) \right\|_{p_i} \leq 0 , \quad \text{for } i = 1, \ldots, m \\
& x \geq 0 . 
\end{align*} \tag{9}
\]

We call (9), a Unified Robust Model (URM) for the linear program (1). Now, we continue to specializing URM for some well-known norm bodies.

### 3.1. Robust Optimization Model Corresponding to \( l_1 \)-norm

We choose \( l_1 \)-norm, in constraints of model (9). Therefore, the second constraint in (9) can be written as:

\[
\left( a_i^0; b_i^0 \right)^T x + r_i \sum_{k=1}^h w_k^i(x) \leq 0 , \tag{10}
\]

where \( w_k^i(x) = \sum_{j=1}^n w_k^i j x_j \) and \( w_k^i j \) are the elements of \( W^i \) for \( k = 1, \ldots, h \) and \( j = 1, \ldots, n \).

Since we want to make the constraint linear, we define two auxiliary variables as below:
The above constraint is equivalent to:

\[
\begin{align*}
& w_k^j(x) = y_k^+ - y_k^- , \\
& y_k^+ \geq 0 \text{ and } y_k^- \geq 0 , \\
& y_k^+ \cdot y_k^- = 0 .
\end{align*}
\]

The last condition will be satisfied automatically by any basic feasible solution (e.g., an optimal solution delivered by the simplex algorithm). Therefore, by extension, the above concept for the other constraints in (9) yields the following formulation:

\[
\begin{align*}
& \text{max} \quad t \\
& \text{subject to} \quad c^{0T} x - n_0 \sum_{k=1}^{h_0} (y_k^0 + y_k^-) \geq t , \\
& \quad (a_i^0; b_i^0)^T x + r_i \sum_{k=1}^{h_i} (y_k^+ + y_k^-) \leq 0 , \quad \text{for } i = 1, \ldots, m , \\
& \quad y_k^+, y_k^- \geq 0 , \quad \text{for } k = 1, \ldots, h_i \text{ and } i = 0,1,\ldots,m , \\
& \quad x \geq 0 .
\end{align*}
\]

This is a new robust formulation for uncertain linear programs (1) associated with \( l_1 \)-norm, which we call diamond robust model. This is a linear programming problem and is expected to be used by an aggressive decision maker because its parameters are assumed to fall in it and it has the smallest region among the others.

We can use another formulation for the \( l_1 \)-norm description of model (9) which leads to an equivalent to interval robust optimization introduced in ROM [5]. Using the triangular inequality, we can replace the upper bound terms in constraint (10) as below:

\[
\begin{align*}
|w_k^j(x)| & = |y_k^+ + y_k^-| , \\
& \quad |y_k^+ + y_k^-| \leq \sum_{j=1}^{n+1} |w_k^j| |x_j| , \\
& \quad |w_k^j| |x_j| \leq (a_i^0; b_i^0)^T x + r_i \sum_{k=1}^{h_i} \sum_{j=1}^{n+1} |w_k^j| |x_j| .
\end{align*}
\]

where \( x_{n+1} = -1 \), then

\[
|w_k^j(x)| \leq (a_i^0; b_i^0)^T x + r_i \sum_{k=1}^{h_i} \sum_{j=1}^{n+1} |w_k^j| |x_j| .
\]

(12)
Following the worst-case approach, the second constraint of (9) can be written as:

\[
(a_i^0; b_i^0)^T x + r_i \sum_{k=1}^{h_0} \sum_{j=1}^{n+1} |w^i_{kj}| x_j \leq 0.
\]  

This is an extreme upper bound for this constraint which makes the model so conservative. By extending the above treatment to the other constraints in (9), the model becomes

\[
\text{max } t
\]

subject to

\[
c^T x - r_0 \sum_{k=1}^{h_0} \sum_{j=1}^{n} |w^0_{kj}| x_j \geq t, \\
(a_0^0; b_0^0)^T x + r_i \sum_{k=1}^{h_i} \sum_{j=1}^{n+1} |w^i_{kj}| x_j \leq 0, \text{ for } i = 1, \ldots, m
\]

\[
x \geq 0.
\]

This is the same as the interval robust optimization problem in [5].

3.2. Robust Optimization Model Corresponding to $l_2$-norm

Choosing $l_2$-norm, the second constraints in (9) can be written as:

\[
(a_i^0; b_i^0)^T x + r_i \sqrt{\sum_{k=1}^{h_i} \left(w^i_{k}(x)\right)^2} \leq 0, \quad i = 1, 2, \ldots, m.
\]  

The model becomes

\[
\text{max } t
\]

subject to

\[
c^T x - r_0 \sqrt{\sum_{k=1}^{h_0} \left(w^0_{k}(x)\right)^2} \geq t, \\
(a_0^0; b_0^0)^T x + r_i \sqrt{\sum_{k=1}^{h_i} \left(w^i_{k}(x)\right)^2} \leq 0, \quad i = 1, \ldots, m
\]

\[
x \geq 0.
\]

This is an SOCP model and is the same as the ellipsoidal robust optimization model introduced in [5].
3.3. Robust Optimization Model Corresponding to Infinity-norm

Taking $l_{\infty}$-norm, the second constraints in (9) can be written as:

$$(a_i^0; b_i^0)^T x + r_i \max_k \left| w_k^i (x) \right| \leq 0, \quad i = 1, 2, ..., m. \quad (17)$$

Moreover, let us assume that $\max_k \left| w_k^i (x) \right| = \tau_i$, then we can substitute the above constraints, with the followings:

$$(a_i^0; b_i^0)^T x + r_i \tau_i \leq 0, \quad i = 1, 2, ..., m, \quad \left| w_k^i (x) \right| \leq \tau_i \quad i = 1, 2, ..., m. \quad (18)$$

Then, the optimization problem becomes

$$\max \quad t$$

subject to

$$c^0^T x - r_0 \tau_0 \geq t,$$

$$(a_i^0; b_i^0)^T x + r_i \tau_i \leq 0, \quad \text{for } i = 1, ..., m, \quad (19)$$

$$-\tau_i \leq \sum_{j=1}^{n} w_k^j x_j \leq \tau_i, \quad \text{for } k = 1, ..., h_i; \quad \text{and } \quad i = 0, ..., m,$$

$$x \geq 0.$$

This is a new robust formulation for uncertain linear programs (1), which we call rectangular robust model. It leads to some conservative solutions among others.

3.4. Robust Optimization Model Corresponding to Matrix-norm

We now assume that the random parameters are correlated with one another. This dependency is formulated by choosing the matrix-norm in which $C$ is the covariance matrix. Let us consider a multivariate Gaussian distribution with parameters $(\mu, C)$ as

$$f(z; \mu, C) = \left(2\pi e \det C\right)^{-1/2} \exp \left(-1/2(z-\mu)^T C^{-1} (z-\mu)\right), \quad (20)$$

where $z$ is the vector of random parameters for the $i$-th constraint in (1). Its level sets are concentric ellipsoids defined by
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\[ E(\mu, C, r) = \{ z \in \mathbb{R}^{n+1} \mid (z - \mu)^T C^{-1} (z - \mu) \leq r^2 \} \].

(21)

This is the same as the matrix-norm body with radius \( r \). In order to show the equivalence of the matrix-norm with \( l_2 \)-norm definition of the uncertainty region, one can define \( z = (a_i ; b_i) \), \( \mu = (a_i^0 ; b_i^0) \) for each \( i \) and use the following affine transformation,

\[
(a_i ; b_i) = L'u + (a_i^0 ; b_i^0),
\]

(22)

where \( L' \) is a lower triangular matrix obtained from Cholesky factorization of \( C = L' L \). The ellipsoid \( E \) can be expressed as:

\[
U_i(2, r) = \{(a_i ; b_i) : (a_i ; b_i) = (a_i^0 ; b_i^0) + L'u \mid u \in B_2(r)\}.
\]

(23)

Here \( L' \) plays the same role as \( W^i \) does in a general uncertainty region. The main difference is that \( L \) can be a lower triangular matrix in this case, whereas in \( l_2 \)-norm formulation \( W \) is a diagonal matrix. Therefore, the constraints in model (9) can be written as

\[
(a_i^0 ; b_i^0)^T x + r_i \left\| L'i x \right\|_2 \leq 0.
\]

(24)

and the model becomes

\[
\begin{align*}
\max_t \\
\text{subject to} \\
(c^0 x - r_0 \left\| L'i x \right\|_2 \geq t, \\
(a_i^0 ; b_i^0)^T x + r_i \left\| L'i x \right\|_2 \leq 0, \quad \text{for } i = 1, \ldots, m \\
x \geq 0.
\end{align*}
\]

(25)

This is another SOCP problem and can be solved by an efficient interior-point algorithm [19].

We conclude that URM not only covers the robust models introduced in ROM [5] but also includes some new models for dealing with uncertainty in LP problems.

4. Robust Portfolio Selection Problems

Suppose that there are \( n \) different assets in the market. The return of $1 invested in asset \( j \) is a random variable, which is assumed to be distributed symmetrically in its domain. The problem is
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to allocate $1 among the assets in order to get the highest possible total return on the selected portfolio. A natural model for this problem is an uncertain linear optimization problem:

$$\max_{x_j} \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} x_j = 1 ,$$

$$x_j \geq 0 , \text{ for } j = 1, \ldots, n ,$$

where $c_j$ is the uncertain return of the asset $j$. Both the nominal optimal solution and risk-neutral stochastic programming approach result in the same solution: the budget should be invested in the assets which have the maximal nominal returns. Mathematically, there is always an extreme point solution that is optimal. The extreme points of the underlying feasible region are the unit vectors in $\mathbb{R}^n$ in this case. Therefore, there always exists a unit vector which is optimal. This solution, however, is unreliable and risky. To reduce risk, one can use URM which results in the following optimization problem:

$$\max_t \quad t$$

subject to

$$c^T x - r \left\| W^T x \right\|_p \geq t ,$$

$$\sum_{j=1}^{n} x_j = 1 ,$$

$$x \geq 0 .$$

First, we assume that the vectors of random returns are independent. Therefore, we define $W$ as the matrix of variances, i.e., $W = \text{Diag}(\sigma_j)$, for $j = 1, \ldots, n$. Next, by choosing various $l_p$-norms, we build various optimization problems for modeling uncertainty in returns and getting different results.

4.1. Portfolio Selection Problem with $l_1$-norm

$$\max_t \quad t$$

subject to

$$c^0^T x - r \sum_{j=1}^{n} \sigma_j x_j \geq t ,$$

$$\sum_{j=1}^{n} x_j = 1 ,$$

$$x \geq 0 .$$
This model is the sum of standard deviations of portfolio returns and is similar to the mean-absolute deviation (MAD) model in [12, 13]. However, just like the problem (26), to find an optimal solution of (28), it suffices to choose \( j \) with maximum \( c_j^0 - r\sigma_j \) and set that \( x_j \) to 1 and all others to zero. We have included the model here for exposition purposes.

4.2. Portfolio Selection Problem with \( l_2 \)-norm

\[
\begin{align*}
\max & \quad t \\
\text{subject to} & \quad c^{0T}x - r\sqrt{\sum_{j=1}^{n} \sigma_j^2 x_j^2} \geq t, \\
& \quad \sum_{j=1}^{n} x_j = 1, \\
& \quad x \geq 0.
\end{align*}
\]

This model is an SOCP problem which is the same as the Markowitz mean-variance model in [14, 15].

4.3. Portfolio Selection Problem with \( l_\infty \)-norm

\[
\begin{align*}
\max & \quad t \\
\text{subject to} & \quad c^{0T}x - r_0\tau_0 \geq t, \\
& \quad \sigma_j x_j \leq \tau_0 \quad \text{for } j = 1, \ldots, n, \\
& \quad \sum_{j=1}^{n} x_j = 1, \\
& \quad x \geq 0.
\end{align*}
\]

This is similar to the minimax model introduced by Young in [22]. Next, we assume that the vector of random returns are correlated and define \( C \) as their covariance matrix. Choosing a matrix-norm, we build robust optimization problems corresponding to correlated returns as follows.
4.4. Portfolio Selection Problem with $l_c$-norm

$$\begin{align*}
\text{max} & \quad t \\
\text{subject to} & \quad c^{0T} x - r \left\| L^T x \right\|_2 \geq t, \\
& \quad \sum_{j=1}^n x_j = 1, \\
& \quad x \geq 0,
\end{align*}$$

(31)

where $C = L^T L$. This is also an SOCP problem that is another presentation of mean-variance model.

The $l_c$ and $l_2$-norm model are essentially a regeneration of Markowitz portfolio model [14, 15] as defined in (27).

The choice of $p$ in $l_p$ norms depends on the utility function of the investor.

In the next section, we compare the numerical solutions obtained by choosing various norm selections, according to the problems (28) to (31).

6. Example

Let the possible investments be eight risky assets, taken from [23]. Table 1 in the appendix shows historical information on the assets and their means, standard deviations and covariance matrix. First, let us assume that the returns are independent. We solve $l_1$, $l_2$ and $l_\infty$ models corresponding to (28), (29) and (30). The results of solving robust models are reported respectively in Tables 2, 3, 4 and 5 in the appendix. We then compare various efficient frontiers corresponding to the results of mean-variance model with those of $l_p$-models. We do in the same way for $l_c$ (matrix-norm) which considers the correlation among the returns.

Figures 2 and 3 show the efficient frontiers corresponding to $l_p$-norm models (dotted and dashed lines) with mean-variance efficient frontier (solid line). It shows that $l_1$, $l_\infty$ and $l_c$ efficient frontiers have similar behaviors to that of the mean-variance model and can also be taken as suitable metrics for modeling risk [16].
Table 1: Historical Data

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A Unified Model for Robust Optimization of Linear…, Payam Hanafizadeh, Abbas Seifi

Table 2: The result of $l_1$ - norm model

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Figure 2: Comparison of various efficient frontiers corresponding to norm body with MATLAB mean-variance efficient frontier (independent).
Now, an important question is that what kind of norm is appropriate for modeling risk. We use an individual’s utility function as a criterion for choosing a suitable norm. The utility function used in [1] is expressed here for illustration:

$$u(t) = 1 - e^{-kt},$$

where $t$ is the total return of an investment and $k > 0$ is called a risk aversion constant. By this assumption, the total return of the investment is a normal random variable and the expected utility is given by

$$\bar{u}(t) = 1 - e^{(-k\bar{t} + 0.5k^2\sigma^2)},$$

where $\bar{t}$ is the mean of total return and $\sigma$ is the total standard deviation. Table 6 shows the expected value of the utility function for the solutions with low and high degree of norms, where $l_1, l_2$ and $l_\infty$ is used respectively. The results show that when the uncertainty is small (at radius 0.0884 and 0.1768), the expected utility for $l_1$ and $l_2$ are bigger than that of $l_\infty$ and when the uncertainty is large (at radius from 0.7071 to 2.8284), the expected utility for $l_\infty$ becomes higher. It is also depicted in Figure 4.
**Table 3:** The result of $l_2$- norm model

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A Unified Model for Robust Optimization of Linear..., Payam Hanafizadeh, Abbas Seifi
Table 4: The result of $l_\infty$-norm model

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Table 5: The result of the $l_\infty$-norm model

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Table 6: Expected values of the utility function corresponding to $l_1$, $l_2$ and $l_\infty$.

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Figure 4: The plot of expected value of the utility function corresponding to $l_1$, $l_2$ and $l_\infty$ norms

6. Conclusion

In this paper, a general uncertainty region is constructed using the concept of pdf-norms. We assume that LP parameters could fall in any of $l_1$, $l_2$, $l_\infty$ or matrix-norm bodies. A unified robust model (URM) has been presented using this general definition of uncertainty region for LP parameters. Deterministic robust counterparts for each problem with its special norm definition have also been derived. Portfolio selection problem was modeled by the proposed unified model and a numerical example is solved. The model provides suitable metrics for dealing with risk. The numerical results show that choosing higher bigger norms leads to higher utility for an investor when the uncertainty region is large.
A Unified Model for Robust Optimization of Linear…, Payam Hanafizadeh, Abbas Seifi

REFERENCE


A Unified Model for Robust Optimization of Linear…, Payam Hanafizadeh, Abbas Seifi


Vanderbei, R.J. Linear Optimization. Lecture 18, convex optimization, expected utility optimization, Department of Operation research and financial engineering, Princeton University, NJ

**Appendix**

\[
\text{Cov} = \begin{pmatrix}
0.0009 & -0.0001 & 0.0001 & 0.0001 & -0.0003 & 0.0003 & -0.0013 & 0.0008 \\
-0.0001 & 0.0232 & 0.0113 & 0.0106 & 0.0118 & 0.0115 & 0.0110 & -0.0141 \\
0.0001 & 0.0113 & 0.0283 & 0.0297 & 0.0329 & 0.0075 & 0.0219 & -0.0185 \\
0.0001 & 0.0106 & 0.0297 & 0.0319 & 0.0371 & 0.0071 & 0.0231 & -0.0166 \\
-0.0003 & 0.0118 & 0.0329 & 0.0371 & 0.0500 & 0.0076 & 0.0245 & -0.0164 \\
0.0003 & 0.0115 & 0.0075 & 0.0071 & 0.0076 & 0.0065 & 0.0044 & -0.0115 \\
-0.0013 & 0.0110 & 0.0219 & 0.0231 & 0.0245 & 0.0044 & 0.0554 & -0.0140 \\
0.0008 & -0.0141 & -0.0185 & -0.0166 & -0.0164 & -0.0115 & -0.0140 & 0.1271
\end{pmatrix}
\]