Idea Flows, Economic Growth, and Trade*

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Abstract

We study a simple model of the world economy where the engine of growth is the flows of ideas. Ideas are assumed to diffuse randomly across pair of goods, where the processes used in the production of the lower cost good is adopted. We assume that producers in a country can learn from all sellers of goods to this country. We analyze how trade in goods, and impediments to it, affect this diffusion. We find that trade costs can have a large impact on the distribution of productivity. In addition, our theory provides a new micro-foundation for the Frechet distribution of productivity that is frequently used in quantitative trade models.

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1 Introduction

The free-trade institutions established after World War II initiated an era of income convergence in the economies of Europe, North America, and East Asia. Figure 1 illustrates this convergence for some large economies. Comparative studies by Ben-David (1993), Sachs and Warner (1995), Wacziarg and Welch (2008) and others have provided more detailed description and analysis. But the basic theories of international trade predict *level effects* of trade liberalization, not growth effects, so the evidence on convergence has remained at the level of measurements in search of new theory. In this paper, we provide a theoretical description of a process that is capable of generating growth and income convergence among economies engaged in trade. The new element is a focus on the relation of trade to the diffusion of ideas. In the model we develop, both domestic and international trade involve exchanges of production-related ideas, and the nature and effects of these exchanges depend on the parties involved. Changes in trade patterns give producers in one country access to new idea sources in other countries, in turn affecting technological change and growth.

The theory of trade that we use here is a static model taken directly from Alvarez and Lucas (2007), and closely related to Eaton and Kortum (2002). There is no capital: The only dynamics come from changes in technology. Our model of technology diffusion involves stochastic meetings of managers who exchange ideas. We use a version of the Kortum (1997) model of a technology frontier defined by a probability distribution, as developed in Alvarez et al. (2008).

In our model, people contribute to production in two distinct ways. First, everyone is endowed with a unit of labor time which he supplies inelastically to some production process. Then there is a second function that we call *technology management*. Every country produces a continuum of goods, and the production of each requires labor in the ordinary sense and also a technology manager. It is the knowledge level of this manager that determines labor productivity for that particular good. In the process of technology diffusion we describe below, it is these technology managers that meet, exchange ideas, and implement the best ideas in their home countries.

We impose several simplifying assumptions on this technology management function. It does not use up any of the time endowment so no occupational choice decision is involved. The productivity knowledge of each technology manager is immediately available to other producers within his own country. It is not available to producers in other countries until it is conveyed to them by a technology manager from their own country. Any one of these assumptions could be relaxed in interesting directions, but our focus here is on understanding one particular connection between trade and growth and it seems best to keep complications to a minimum.

Outline of Sections to come.

2 Closed Economy Model

We consider a closed economy where consumers have identical preferences over a $[0,1]$ continuum of goods. We use $c(s)$ to denote the consumption of an agent of each of the $s \in [0,1]$ goods for each period t. There is no intertemporal technology to transfer goods between periods. The period t utility function is given by

$$C = \left[\int_0^1 c(s)^{1-1/\eta} ds \right]^{\eta/(\eta-1)},$$

so goods enter in a symmetrical and exchangeable way. We let $s \in [0,1]$ index both a good's type and an agent's type. Each individual manages the technology to produce a single good, and the technology to produce each good is managed by one individual. Hence agent s has three roles: she consumes, she supplies labor and she manages technology s.

Using the symmetry of the utility function we group goods by their cost z: the labor required to produced this good. The production function for good s is therefore $y(s) = (1/x(s))\ l(s)$ where $x(s)$ and $l(s)$ are the cost and the labor input that correspond to the good s. The function $F(x,t)$ gives the fraction of individuals with cost *higher* than z at time t, i.e. it is a right c.d.f., and we denote by $f(x,t) = -\partial F(x,t)/\partial x$ the density of cost x.

Using this notation we can write the time t utility as

$$C(t) = \left[\int_{\mathbb{R}_+} c(x)^{1-1/\eta}\ f(x,t)\ dz \right]^{\eta/(\eta-1)},$$

where $c(x)$ is the consumption of the good s that have cost x.

The model is written assuming perfect competition. Yet, for reasons that will be clear in the context of the technology diffusion model, we assume that there is one agent managing each technology. Alternatively, we could assume that for each good there are many agents that have access to the same technology.

3 Technology Diffusion in a Closed Economy

We model technological diffusion assuming that the agent who manages the technology for a good s can learn from other technologies used to produce good $s' \neq s$. Let $x(s)$ be the cost of the technology/good s and $x(s')$ be the cost associated with technology/good s'. We assume that the agent s learns the process associated with the production of good s', and adopts it for the production of its own good, if good s' has a lower cost, i.e. $x(s') < x(s)$. We postulate that agents have α such learning opportunities per unit of time. We refer to these learning opportunities as contacts or meeting. Each of the contacts for agent s are randomly drawn from the population of *sellers* of goods.[1] The process that we have described models the diffusion of technologies across different goods. We assume that the diffusion of technology is the same across any two goods, no matter how different they are.[2] While we refer to this process as technology *diffusion*, it also involves *innovation*, since the more advance technology used for one good has to be adapted to a different good. The effect of indirected links, randomness in our diffusion process is reminiscent to us of Diamond's (1998) or Mokyr's (2005) history of technologies.[3] Next we give a mathematical description of this process as well as few comments on what we meant to capture with it.

As was described earlier, we index producers by the cost associated with them, instead of the name of the good. We describe a the cost distribution by the its *right* cdf $F(x,t)$, we formulate a discrete time law of motion for $F(x,t)$ between t and $t+\Delta$, and derive its continuous time limit. For a given level of the cost x, the distribution of the cost in period $t+\Delta$ is related to its distribution in period t according to the following law of motion:

$$F(x,t+\Delta) = F(x,t)\ F(x,t)^{\alpha\Delta} \tag{1}$$

The first term in the right hand side give the fraction of individuals with cost higher than x at time t. The second term in the right hand side $F(x,t)^{\alpha\Delta}$ is the probability that in $\alpha\Delta$ randomly drawn meetings an agent with cost x does not learn a lower cost. Given our assumption of independent draws, the probability that none of the individuals with cost above x improve their cost is given by product of these two terms. Assuming a law of large numbers, so that we convert probabilities into fractions, we obtain equation (1).

We discuss the assumptions behind our technology diffusion and the parameters that determine its speed. The nature of technology diffusion in our model is that the technology

[1] Notice that, in the case of a closed economy of our model, all the goods are produced, and therefore, the set of sellers equal the set of individual managing the technologies to produce each of the goods, i.e., the set of potential sellers.

[2] More generally, we could consider a model with symmetric sectors, each sector with a continuum of industries, where technologies are diffused across all industries within sectors. This would correspond to a minor extension of our model, where our basic setup gives the description of the model of a single sector. A tractable model with asymmetric sectors could be handled if we assume Cobb-Douglas preferences across sectors.

[3] "new technologies and materials make it possible to generate still other new technologies by recombination ... Gutenberg's press was derived from screw presses in use for making wine and olive oil, while his ink was an oil-based improvement on existing inks...", chapter 13 of Diamond (1998).

to produce a good is improved by learning from the technologies from other existing high productivity goods. In this sense, the learning is across goods, and it consists of a process of constant diffusion from the high productivity to the lower productivity technologies. In essence, this process is a race to diffuse across all the goods the technologies with the highest possible productivity which initially were used for very few goods. The parameter α determines the speed at which the learning opportunities arrive, and hence should be one of the determinants of the long run growth rate of the economy, and the speed of convergence to a balance growth path. The assumption that there is fixed number α of learning opportunities for a person managing a technology is meant to capture the use and the constraints on the time allocation of these agents. The other dimension of our model of diffusion is that agents learn from products sold in their location. As we will show later on, given this assumption trade will play an important role, since the seller of a product will be a selected group, it will be those will lowest cost for a given good.

Now we take the limit as $\Delta \downarrow 0$ of the law of motion in equation (1) to obtain:

$$\frac{1}{F(x,t)} \frac{\partial F(x,t)}{\partial t} = \alpha \log\left(F(x,t)\right) \tag{2}$$

The number of goods with arbitrarily large productivity will play a crucial role in our theory of diffusion, since it is through contacts with other goods/firms that agents learn the best practice. Thus it will be crucial to fully characterize the behavior of the initial distribution of cost in the neighborhood of $x = 0$. In particular, the elasticity of the initial distribution of cost around zero is a central determinant of the long run distribution of cost, and the growth rate in a balance growth path. The next results shows that the elasticity of the c.d.f. $1 - F(x, 0)$ of cost at $x = 0$ measures the "concentration" of goods with arbitrarily large productivity. More precisely, the ratio of the number of products with arbitrarily low cost for a c.d.f. with lower elasticity diverges towards infinity relative to the one with higher elasticity. Moreover, when two distributions have the same elasticity in the neighborhood of $x = 0$, we can measure the relative "number" of arbitrarily productive ideas in these distributions by the ratio of the densities at zero of the transformation of the cost x that has a strictly positive and finite density in the neighborhood of 0.

Proposition 1. Assume that $1 - F_i(x)$, $i = 1, 2$, are two c.d.f.s, with $1 - F_i(0) = 0$, strictly increasing at zero, with elasticities $\epsilon_i(x)$, both continuous at $x = 0$, satisfying $0 < \epsilon_1(0) \leq \epsilon_2(0) < \infty$. Let the $G_i(z) = F_i(z^{\frac{1}{\epsilon_i(0)}})$ be the right c.d.f. of the transformation of the cost

$z = x^{\epsilon_i(0)}$, $i = 1, 2$. Then

$$\lim_{x \to 0} \frac{1 - F_2(x)}{1 - F_1(x)} = \begin{cases} 0 & \text{if } \epsilon_1(0) < \epsilon_2(0), \\ \frac{\partial G_2(0)/\partial z}{\partial G_1(0)/\partial z} & \text{if } \epsilon_1(0) = \epsilon_2(0). \end{cases} \tag{3}$$

Our diffusion model implies that the cost of every technology/good converges to zero, albeit at different times. In particular, for a fixed technology/good, the time path of its cost stays constant and at random times jumps down, with its path eventually converging to zero. We are interested in studying the rate, denoted by ν, at which the cross sectional distribution of costs converges to zero. We will study the distribution of cost inflated at this rate, which it will converge to a non-degenerate distribution. We formalize this discussion by introducing the notion of a normalized cost and a balanced growth path.

We define the normalized cost of a technology/good as

$$x(t) = e^{vt}\tilde{x}(t), \tag{4}$$

where $\tilde{x}(t)$ the cost of that technology/good at time t, for some growth rate $\nu > 0$. We say $\{F(\cdot, t), t \geq 0\}$ has a balanced growth path if $\Phi(x, t) = \Phi^*(x)$ where Φ^* does *not* depend on t and

$$\Phi(x, t) = F(e^{-vt}x, t)$$

is the distribution of the cost normalized at the rate rate $\nu > 0$.

Definition 1. We say that the balanced growth path is stable if $\lim_{t \to \infty} \Phi(x, t)$ converges to a non-degenerate right c.d.f. $\Phi^*(x)$ for all $x \geq 0$. We define the speed of convergence $s(F(\cdot, 0), x, t)$ as the fraction of the difference with steady state that is close per unit of time in an interval of time of length Δ at time t as

$$s(F(\cdot, 0), x, t) = \lim_{\Delta \downarrow 0} \frac{1}{\Delta} \frac{\log \Phi(x, t + \Delta) - \log \Phi(x, t)}{\log \Phi(x, t) - \log \Phi^*(x)} \quad \text{for all } t \geq 0 \;, \tag{5}$$

Notice that s depends on t, the point where it is evaluated x, as well as the whole initial distribution $F(\cdot, 0)$.

The following proposition characterizes the asymptotic behavior of the distribution of normalized cost. In particular, it shows that on a balanced growth path the cost must grow at the rate $\alpha\theta$, and the stationary distribution of the deflated cost is Weibull. We will consider any initial c.d.f. for the cost $\bar{F}$ that, in a neighborhood of cost $x = 0$, is strictly increasing and has a continuous, strictly positive elasticity.

PROPOSITION 2. Let $F(\cdot,0)$ be the time $t=0$ right c.d.f. of cost. Assume that in a neighborhood of $x=0$ its c.d.f. is strictly increasing and has a continuous elasticity with limit $1/\theta \in (0,\infty)$ as $x \downarrow 0$. Let $\lambda \in (0,\infty)$ be the initial density of the transformed variable $z = x^{1/\theta}$. Thus

$$\frac{1}{\theta} = \lim_{x\downarrow 0} \frac{x}{1-F(x,0)} \frac{\partial(1-F(x,0))}{\partial x} \quad \text{and} \quad \lambda = \lim_{z\downarrow 0} -\theta\, z^{(\theta-1)}\, \frac{\partial F(z^\theta,0)}{\partial x}\ .$$

Assume that technology is diffused according to (2). Let the path of the right c.d.f. of the cost normalized at rate $\nu = \theta\alpha$ be denoted by $\{\Phi(x,t)\}$. Then, for $t \geq 0$, the deflated right c.d.f. of cost equals

$$\Phi(x,t) = \exp\left(e^{\alpha t}\ \log\left(F\left(e^{-\nu t}x,0\right)\right)\right) \text{ for all } x \geq 0\ ,$$

and it converges pointwise at the asymptotic rate α to a Weibull distribution with right c.d.f. given by Q^* and parameters (θ,λ):

$$\lim_{t\downarrow\infty} \Phi(x,t) = \Phi^*(x) \equiv \exp\left(\lambda x^{\frac{1}{\theta}}\right) \text{ for all } x \geq 0\ ,$$

and

$$\lim_{t\downarrow\infty} s(F(\cdot,0),x,t) = \alpha \text{ for all } F(\cdot,0) \text{ and for all } x \geq 0\ .$$

Remark. *Note that since the asymptotic distribution of the deflated cost is Weibull, the corresponding asymptotic distribution of productivity will be Frechet.*

The asymptotic distribution of cost belongs to the Weibull family of distributions, which is fully characterized by two parameters, λ and θ. The shape parameter $1/\theta$ will be given by the inverse of the elasticity at $x=0$ of the initial c.d.f. of cost $1-F(x,0)$. The scale parameter λ is given by initial density at $z=0$ of the transformation of the cost $z = x^{1/\theta}$. Taken together, these two parameters measure the concentration and number of goods that can be produced with arbitrarily productive technologies (see Proposition 1). Proposition 2 means that one can partition the set of initial distributions of cost $1-F(\cdot,0)$ into classes that have the same "shape" around zero cost, i.e. the same values of (λ,θ). There is stability in the sense that the cost in an economy with initial conditions in such a class grows at the same rate and converges to the same distribution. But there is also non-stability in the sense that economies in different classes can growth at different rates and will converge to different distributions. We note that the limit distributions Φ^* do not depend on α, which itself is the only determinant of the (asymptotic) speed of convergence.

While our notion of stationary distribution is new, as well as our proof, the result in this Proposition is very closely related to known results for extreme distributions. In particular for the maximum of an infinite sequence of iid variables with finite upper bound. In that case the conditions on the elasticity is essentially the same as the von Mises condition, and the invariant distribution is Weibull. See, for example, Theorem 3.3.12 of Embrechts et al. (2003). Yet our result is different in an important way from the standard results in extreme distributions. In our set-up we obtain *geometric* growth, while, in the language of the extreme distributions, the standard result is a *linear* norming constraint, or linear growth in term of economics. Indeed the standard set-up in the mathematical statistical literature is closer to the set-up in economic models of diffusion of technologies with an exogenous idea source, such as the one by Kortum (1997). In these type of model there is no growth assymptotically.

The economics of the asymptotic growth rate of cost $\nu = \alpha\theta$ are clear enough. A higher rate of contacts per unit of time α increases the long run growth rate of productivity since diffusion is occurring faster. A higher concentration of cost near zero, that is a higher θ, increases the long run rate of productivity growth since there are better chances to learn from the technology of arbitrarily productive goods.

We can gain some understanding of the order of magnitude of the parameters α and θ by using information of the long-run growth rate of the economy ν, and information on θ which instead can be obtained either from the dispersion of productivities, or the tail of the size distribution of firms/plants, or the magnitude of trade elasticities. We turn to the description of each of these approaches.

First, since we show above that asymptotically x is Weibull distributed, then $\log(1/x)$, the log of productivity, has standard deviation equal to $\theta\pi/\sqrt{6}$, see chapter 3.3.4 of Rinne (2008). Hence we can take measures of dispersion of (log) productivity to calibrate θ. The dispersion of (log) productivity range from $0.6 - 0.75$ when measured as the value-added per worker – see Bernard et al. (2003) Table II – and around 0.8 when measures of physical total factor productivity are obtained using data on value-added, capital and labor inputs, and assumptions about the demand elasticities – see Hsieh and Klenow (2009) Table I, dispersion of TFPQ.[4] These numbers suggest a value for θ in the range $[0.5, 0.6]$.

Second, using that productivity is asymptotically distributed Frechet, and that the tail of the Frechet behaves as that of a Pareto distribution with tail coefficient $1/\theta$, we can use data on the tail of the distribution of productivity to directly infer θ. Lacking direct information on physical productivities, we can use information on the tail of the distribution

[4]Using data for eleven products for which direct measures of physical output are available Haltiwanger et al. (2008) calculate true measures of physical total factor productivity. They find that the dispersion of (log) true physical productivity is 20% higher than that measured using just value-added information.

of employment, together with a value for the elasticity of demand.[5] The tail of the size (employment) distribution of firms is approximately equal to 1.06 – see Figure 1 in Luttmer (2007). Therefore for the values of demand elasticities typically considered in the literature, say $\eta \in [3, 10]$, (see Broda and Weinstein (2006) or Hendel and Nevo (2006)) it would imply a value for θ in the range $[0.1, 0.5]$.

Third, we will show below that in the case of a model with several symmetrical locations, θ is approximately the Armington trade elasticity, which will also give us another way to measure it. This method would suggest a value for θ in $[0.1, 0.25]$ (Alvarez and Lucas, 2007).

Once we have estimate of θ regardless of its source, together with an estimate of long term growth of output ν, we can estimate the value of α, using that $\nu = \alpha\theta$. For instance, if we take the long-run growth to be 0.02, α would be in the range $[0.03, 0.2]$. Note that with a value of $\alpha = 0.1$, the half-life to convergence will be approximately 5 years.

We offer two examples to illustrate the content of the assumption that the elasticity of $1 - F(x, 0)$ at $x = 0$ is positive and finite. In the first case the elasticity of $1 - F(x, 0)$ at cost $x = 0$ is zero.

Example 1. Assume that the c.d.f. of the cost is given by $1 - F(x, 0) = x^{1+x}$ for all $x \in [0, 1]$. The elasticity of $1 - F(x, 0)$ tends to 0 as $x \to 0$. In this case the economy does not have a balanced growth path, since the growth rate will be increasing with no bound as time passes.

The second example corresponds to a case where the elasticity of $1 - F(x, 0)$ at cost $x = 0$ is infinity.

Example 2. Assume that the c.d.f of the cost is given by $1 - F(x, 0) = \exp(-1/x)$ for all $x \geq 0$. The elasticity of $1 - F(x, 0)$ tends to ∞ as $x \to 0$. In this case the economy does not have a balanced growth path with strictly positive growth, i.e. $\nu = 0$.

The previous two examples show initial distributions for which there is no balanced growth. In the first it is because the distribution of the cost is too concentrated in a neighborhood of zero, so that as more goods are exposed to these technologies the growth rate keeps increasing. In the second example it is because the distribution of the cost are not concentrated close enough to zero, and so the learning of these technologies cannot sustain constant growth. We have chosen the examples to be continuous and have strictly increasing c.d.f.'s. In turn they imitate the case of a distribution with a mass point at cost zero, and the one of a distribution for which the support starts at a strictly positive value of the cost.

[5] The CES structure implies that employment at industry/firm with cost x satisfies $l(x) \propto (1/x)^{\eta-1}$.

The following example of an initial distribution of cost will be helpful to understand selection in our model with trade among locations.

Example 3. Consider a partition of the goods into two sets of equal size. Let the initial distribution of costs in each set be described by c.d.f.'s $1 - F_i(x)$, $i = 1, 2$, with $1 - F_i(0) = 0$, strictly increasing at zero, with elasticities $\epsilon_i(x)$, both continuous at $x = 0$, satisfying $0 < \epsilon_1(0) < \epsilon_2(0) < \infty$. The path of the distribution of the normalized cost for each set converges to the same Weibull distribution with right c.d.f. given by: $\Phi_i^*(x) = \exp(\lambda\, x^{\frac{1}{\theta}}), i = 1, 2$, where $\theta = \frac{1}{\epsilon_1(0)}$, $\lambda = \bar{G}_1'(0) + \bar{G}_2'(0) = \bar{G}_1'(0)$, and $\bar{G}_i(0), i = 1, 2$ are the c.d.f.'s of the cost index $z = x^{\frac{1}{\theta}}$.

In the example above, for large t, the shape of the distribution of costs for the entire economy will be dominated by the costs of the set of goods that initially has a highest concentration of arbitrarily productive goods. As we discussed in our first Proposition, the concentration of arbitrarily productive goods is measured by the elasticity of the c.d.f. of cost at $x = 0$, $\epsilon_i(x), i = 1, 2$. This remark previews an important property of the model of diffusion of a world economy that we study in the rest of the paper. In a world economy where individuals have meetings with producers from all countries mediated by trade flows, the stationary distribution of deflated costs in each country will be determined by the distribution of the countries with the highest initial concentration of arbitrarily productive goods.

Throughout the paper we assume the following joint restriction on the elasticity of substitution across goods η and the inverse of the elasticity at $z = 0$ of initial distribution of cost, θ, which governs the tail properties of the distribution of cost.

Assumption 1. $0 < 1 + \theta(1 - \eta)$.

This assumption guarantees that aggregate consumption is finite. Intuitively, it requires that goods are not too substitutable and/or that there is not too large a concentration of arbitrarily productive goods. The use of this assumption is similar to the one in Eaton and Kortum (2002) and Alvarez and Lucas (2007), where it is used to prove that given the Wiebull distributed cost, the output of the final good is finite. In our case the cost has a Weibull distribution only asymptotically, after deflating them at rate $\nu = \alpha\theta$. Yet, the deflated distribution of cost has a distribution for each t that is stochastically higher than the limit Weibull distribution, and hence Assumption 1 is sufficient to ensure that output is indeed finite for all $t \geq 0$.

4 Static Trade Model

We consider a world economy consisting of n locations, indexed by $i = 1, ..., n$. Each location is similar to the one described in the closed economy model. Consumers in each location have identical preferences over a $[0,1]$ continuum of goods. We use $c_i(s)$ to denote the consumption of an agent in country i of each of the $s \in [0,1]$ goods for each period t. There is no intertemporal technology to transfer goods between periods. The period t utility function is given by

$$C_i = \left[\int_0^1 c_i(s)^{1-1/\eta} ds\right]^{\eta/(\eta-1)},$$

We let $s \in [0,1]$ index both a good's type and an agent's type. We let $x_i(s)$ the cost of producing good s in location i, so that the production function for good s in location i is $y(s) = l/x_i(s)$ where l is the labor input.

Using the symmetry of the utility function we group goods by the profile $\mathbf{x} = (x_1, ..., x_n)$ of cost across the n locations, where x_i is the labor required to produced this good in location i. The function $F_i(x,t)$ gives the fraction of individuals in location i that manage a technology with cost index *higher* than x at t, i.e., $F_i(x,t)$ is the right c.d.f. of cost in location i. We denote the density of cost by $f_i(x,t) = -\partial F_i(x,t)/\partial x$. We assume that production costs are independently distributed across locations. The right c.d.f. of the distribution of the vector of cost across all locations is given by $F(\mathbf{x},t) = \prod_{i=1}^n F_i(x_i,t)$ and its density by $f(\mathbf{x},t) = \prod_{i=1}^n f_i(x_i,t)$.

Using this notation we can write the time t utility as

$$C_i(t) = \left[\int_{\mathbb{R}_+^n} c_i(\mathbf{x})^{1-1/\eta} \; f(\mathbf{x},t) d\mathbf{x}\right]^{\eta/(\eta-1)},$$

where $c_i(\mathbf{x})$ is the consumption in location i of the good s that has cost profile $\mathbf{x}$.

We use $w_i(t)$ for the time t wages in location i in units of time t numeraire. We assume that due to iceberg shipping costs, when 1 good is sent from location k a fraction κ_{ik} of the good arrives to location i. The costs κ_{ik} are the same for all goods, and κ_{ii} are normalized to one. Each good $\mathbf{x} = (x_1, ..., x_n)$ is available in location i at the following unit prices

$$\frac{x_1 w_1(t)}{\kappa_{i1}}, ..., \frac{x_n w_n(t)}{\kappa_{in}}$$

which reflect both production costs and transportation costs.

We let $p_{i,t}(\mathbf{x})$ be the prices paid for good $\mathbf{x}$ in location i at time t, which satisfy

$$p_i(\mathbf{x},t) = \min_j \left[\frac{w_j(t)}{\kappa_{ij}} x_j\right] \tag{6}$$

since agent in location i buy the good from the cheapest location. Like wages, prices are expressed in units of time t numeraire. Given prices $p_i(\mathbf{x},t)$, the ideal price index at t is the minimum cost of providing one unit of aggregate consumption $C_i(t)$:

$$p_i(t) = \left[\int_{\mathbb{R}^n_+} p_i(\mathbf{x},t)^{1-\eta} \ f(\mathbf{x},t) d\mathbf{x}\right]^{1/(1-\eta)} . \tag{7}$$

We turn to the description of the equilibrium of the trade model for given $F(\mathbf{x},t)$. We assume that there is no borrowing and lending, so that we impose trade balance in each period. Because of the static nature of the equilibrium we omit the index t in its description.

Define the sets $\mathbf{B}_{ij} \subset \mathbb{R}^n_+$ by

$$\mathbf{B}_{ij} = \left\{\mathbf{x} \in \mathbb{R}^n_+ : p_i(\mathbf{x}) = \frac{w_j}{\kappa_{ij}} x_j\right\}.$$

That is, $\mathbf{B}_{ij}$ is the set of goods that people in i want to buy from producers in j. For each i, $\cup_j \mathbf{B}_{ij} = \mathbb{R}^n_+$. Note that the sets $\mathbf{B}_{ij}$ are all cones, defined equivalently by

$$\mathbf{B}_{ij} = \left\{\mathbf{x} \in \mathbb{R}^n_+ : \frac{x_j w_j}{\kappa_{ij}} \leq \frac{x_k w_k}{\kappa_{ik}} \quad \text{for all } k\right\}.$$

The price index p_i must be calculated country by country. We have

$$\begin{aligned} p_i^{1-\eta} &= \int p_i(\mathbf{x})^{1-\eta} f(\mathbf{x}) d\mathbf{x} = \sum_{j=1}^n \int_{\mathbf{B}_{ij}} \left(\frac{x_j w_j}{\kappa_{ij}}\right)^{1-\eta} f(\mathbf{x}) d\mathbf{x} \\ &= \sum_{j=1}^n \left(\frac{w_j}{\kappa_{ij}}\right)^{1-\eta} \int_0^\infty x_j^{1-\eta} f_j(x_j) \prod_{k\neq j} F_k(a_{ijk} \ x_j) \ dx_j \ . \end{aligned}$$

where

$$a_{ijk} = \frac{w_j \kappa_{ik}}{w_k \kappa_{ij}}. \tag{8}$$

Consumption of good $\mathbf{x}$ in location i equals

$$c_i(\mathbf{x}) = \left(\frac{p_i}{p_i(\mathbf{x})}\right)^\eta C_i = \left(\frac{p_i}{p_i(\mathbf{x})}\right)^\eta \frac{w_i L_i}{p_i}.$$

where the first equality follows from individual maximization and the second replace the budget constraint $p_i C_i = w_i L_i$.

To setup the world-wide static equilibrium in t we find the derived demand of labor of location i

$$\sum_{j=1}^{n} \int_{\mathbf{B}_{ji}} c_j(\mathbf{x}) \frac{x_i}{\kappa_{ji}} f(\mathbf{x}) d\mathbf{x} = \sum_{j=1}^{n} \int_0^\infty \left(\frac{p_j}{p_j(\mathbf{x})} \right)^\eta \frac{w_j L_j}{p_j} \frac{x_i}{\kappa_{ji}} f_i(x_i) \prod_{k \neq i} F_k(a_{jik}\ x_i)\ dx_i$$
$$= \sum_{j=1}^{n} \int_0^\infty \left(\frac{p_j}{w_i} \right)^\eta \frac{w_j L_j}{p_j} \frac{x_i^{1-\eta}}{\kappa_{ji}^{1-\eta}} f_i(x_i) \prod_{k \neq i} F_k(a_{jik}\ x_i)\ dx_i\ .$$

We are now ready to define a time t equilibrium given $F(\cdot, t)$.

Definition 2. An static trade equilibrium with trade balance for cost distributions $F_k(\cdot)$ is given by a wage $w \in \Delta^{n-1}$ such that the excess labor demand are all zero, $Z_i(w) = 0$ for all i where

$$Z_i(w) = \sum_{j=1}^{n} \int_0^\infty \left(\frac{p_j(w)}{w_i} \right)^\eta \frac{w_j L_j}{p_j(w)} \frac{x_i^{1-\eta}}{\kappa_{ji}^{1-\eta}} f_i(x_i) \prod_{k \neq i} F_k(a_{jik}(w) x_i)\ dx_i - L_i\ ,$$

and where $p_j(w)$ and $a_{jik}(w)$ are defined as functions of w using equations (6)-(7)-(8).

The excess demand system in Z_i when each of the F_i are Frechet distributions is a special case of the one studied in Alvarez and Lucas (2007), where it is established that Z satisfied the gross substitute property.[6] In this paper the distributions F_i will be endogenous, as a result of technological diffusion, and not necessarily exponential.

We mention some special cases of this system.

1. The case of frictionless trade, i.e. $\kappa_{ij} = 1$ for all i, j. In this case $p_i = p$ and $p_i(\mathbf{x}) = p(\mathbf{z})$ for all locations $i = 1, 2..., n$ with

$$\begin{aligned} p(\mathbf{x}) &= \min_j \{ w_j\ x_j \} \\ p &= \left[\sum_{j=1}^{n} (w_j)^{1-\eta} \int_0^\infty x_j^{\theta(1-\eta)} f_j(x_j) \prod_{k \neq j} F_k \left(\frac{w_j}{w_k} x_j \right)\ dx_j \right]^{1/(1-\eta)}, \\ Z_i(w) &= \sum_{j=1}^{n} \int_0^\infty \left(\frac{w_i}{p} \right)^{-\eta} \frac{w_j}{p} L_j\ x_i^{1-\eta} f_i(x_i) \prod_{k \neq i} F_k \left(\frac{w_i}{w_k} x_i \right)\ dx_i - L_i \end{aligned}$$

[6]The feature that makes it a special case is the lack of tariffs, intermediate good and non-tradeable goods.

We conjecture that in this case Z satisfies the gross substitute property, and hence the equilibrium wages is unique, and easily solvable.

2. The case of identical locations. In this case the size of all the locations is the same, $L_i = L$, the trade cost are symmetric $\kappa_{ij} = \kappa$ for all $i \neq j$, and the distribution of cost for potential producer are the same $F_i(x,t) = F_j(x,t)$ for all i, j and $x \geq 0$. In this case the equilibrium wages $w_i = 1/n$.

3. The case of two locations. By Walras law we can just consider one equation in on unknown:

$$Z_1(w) = \sum_{j=1}^{2} \int_0^\infty \left(\frac{p_j(w)}{w_1}\right)^\eta \frac{w_j L_j}{p_j(w)} \frac{x_1^{1-\eta}}{\kappa_{j1}^{1-\eta}} f_1(x_1) F_2(a_{j12}(w) x_1) \, dx_1 - L_1 \ ,$$

In this case one can normalize $w_2 = 1$ and establish conditions under which $Z_1(w_1)$ is decreasing. We know that there is a unique solution to this equation in this case, based on the seminal paper by Dornbusch et al. (1977). We conjecture that the analysis on that paper also implies that Z_1 is strictly increasing.

5 Technology Diffusion Model: World Economy

We model technological diffusion assuming that the agent that manages the technology for a good s in location i can learn from other technologies used to produce good s' sold in location i. As in the model of a closed economy, the agent that manages the technology for a good s can learn from the technologies used to produced other goods $s' \neq s$. Note the added element relative to the model of one location of Section 3: there are now technology managers in different locations for the exactly the same good, but only some of them will be producing, namely those that will be the cheapest provider to some location. This is an important difference with the closed economy. In the case of multiple locations, the set of sellers in each of them is a selected group: those that are the low cost providers. The effect of selection is as if the manager of a technology contacts simultaneously the potential producers of each good s in all the locations. In this sense, learning takes place for a good s, across all locations $i = 1, ..., n$. In addition, the managers of all the technologies learn, whether or not they are producing.

In our set-up the flow of goods is accompanied by the flow of ideas, and thus trade across locations will have a direct effect on productivity, beyond the standard gains due to comparative advantage. Diamond (1998) gives a vivid discussion of historical instances of technology diffusion across space.[7] Additionally, there is a recent literature that evaluates the extent to which the technology of the producers of a location is embedded in imports from that location and effectively transferred to the purchaser location, such as Coe and Helpman (1995), Coe et al. (1997), Schiff and Wang (2008) and Acharya and Keller (2009). At the theoretical level, the idea that trade can be the conduct for technology transfers is also modeled, albeit in a different way and for a small open economy, in Helpman and Grossman (1991).

The details of our formal set up are analogous to the ones describe in Section 3, with the exception that now meetings are with sellers to a locations instead of producers in each location. As before, we index potential producers in each location by their x, instead of the name of the good s. We now consider the agent managing the technology/good s in location i with cost index $x_i(s)$, and a seller to location i from location j of good s', who has cost index $x_j(s')$. We assume that the agent s learns the process associated with the production of good s', and adopts it for the production of its own good, if good s' has a lower cost, i.e. if the costs are such that $x_i(s') < x_j(s)$. Each of the contacts for agent s are randomly drawn from the population of *sellers* of goods sold in location i.

Generalizing the model of a closed economy, we describe a discrete time law of motion for the distribution of cost $F(\mathbf{x}, t)$ between t and $t + \Delta$, and derive it continuous time limit. Assuming that $F(\mathbf{x}, t) = \prod_{i=1}^{n} F_i(x_i, t)$ we will obtain that $F(\mathbf{x}, t + \Delta) = \prod_{i=1}^{n} F_i(x_i, t + \Delta)$. We fix location i and consider a potential producer with cost x. The distribution of cost in location i, in period $t + \Delta$, is related to the distribution of cost in all locations, $i = 1, ..., n$, in period t, by the following law of motion:

$$F_i(x, t + \Delta) = F_i(x, t) \; P^i(x, t)^{\alpha\Delta} \tag{9}$$

where P^i denotes the fraction of sellers in location i with cost higher than z at time t. Hence $P^i(x, t)$ is the probability that in one randomly drawn meeting an agent with cost index x in location i does not learn a lower cost index. This fraction is given by:

$$P^i(x, t) = \sum_{j=1}^{n} \int_{\mathbf{D}_{ij}(x,t)} f(\tilde{\mathbf{x}}, t) d\tilde{\mathbf{x}} = \sum_{j=1}^{n} \int_{x}^{\infty} f_j(\tilde{x}_j) \prod_{k \neq j} F_k(a_{ijk}(t) \; \tilde{x}_j, t) \; d\tilde{x}_j,$$

[7]Chapter 13 "Necessity's mother, the evolution of technology" the author states "For all societies except the few past ones that were completely isolated, much or most new technologies is not invented locally but is instead borrowed from other societies" and "When societies do adopt a new technology from the society that invented it, the diffusion may occur in many different contexts. They include peaceful trade (as in the spread of transistors from the Unites States to Japan in 1954)...". See also chapter 12 "Blueprints and borrowed letters, the evolution of writings" for a thorough discussion of the diffusion of writing across space.

where $a_{ijk}(t) = \frac{w_j(t)}{w_k(t)} \frac{\kappa_{ik}}{\kappa_{ij}}$ and

$$\mathbf{D}_{ij}(x,t) = \left\{ \tilde{\mathbf{x}} \in \mathbb{R}^n_+ : \frac{w_j(t)}{\kappa_{ij}} \tilde{x}_j \leq \frac{w_k(t)}{\kappa_{ik}} \tilde{x}_k \quad \text{for all } k \quad \text{and } \tilde{x}_j \geq x \right\} .$$

The set $\mathbf{D}_{ij}(x,t)$ has the $\tilde{\mathbf{x}} = (\tilde{x}_1, ..., \tilde{x}_n)$ that satisfy two conditions: the first is that the products associated with $\tilde{x}_j$ are sold in i by a producer from location j, and the second is that the cost of production in location j is high enough so that the potential producer in location i with cost parameter x does not learn from this meeting.

Now we take the limit as $\Delta \downarrow 0$ of the law of motion in (9) to obtain:

$$\frac{1}{F_i(x,t)} \frac{\partial F_i(x,t)}{\partial t} = \alpha \log \left(\sum_{j=1}^{n} \int_x^\infty f_j(\tilde{x}_j, t) \prod_{k \neq j} F_k(a_{ijk}(t) \ \tilde{x}_j, t) \ d\tilde{x}_j \right) \tag{10}$$

for all $x \geq 0$ and all $i = 1, ..., n$.

We are ready to define an equilibrium with technology diffusion.

Definition 3. An equilibrium with technology diffusion given the initial distributions of cost $\{F_i(\cdot, 0)\}_{i=1,..,n}$ is a path of wages and cost distributions $\{w_i(t), F_i(\cdot, t)\}$ for all $t \geq 0$ and $i = 1, ..., n$ so that:

1. $\{w_i(t)\}_{i=1,..,n}$ is an static trade equilibrium given $\{F_i(\cdot, t)\}_{i=1,..,n}$ for all $t \geq 0$
2. $\{F_i(\cdot, t)\}_{i=1,..,n}$ satisfies the law of motion of technology diffusion described in equation (10) for all $t \geq 0$ where $a_{ijt}(t)$, defined in equation (8), are given by the equilibrium wages, and given the initial condition $\{F_i(\cdot, 0)\}_{i=1,..,n}$.

The definition of an equilibrium with technology diffusion is a fixed point on paths for wages and cost distributions. One the one hand, the static trade model requires the world-wide distribution of costs. On the other hand, the diffusion of technologies depends on the equilibrium of the trade model, which determines which producer sell in location i, and hence from whom the potential producers of location i can learn from.

Our process of technology diffusion is one where through time cost are getting smaller for each goods. To analyze its long run behavior we define a type of balanced growth path, where we study the distribution of the relative values of costs. In particular, we say $\{F_i(\cdot, t), t \geq 0, i = 1, ..., n\}$ has a balanced growth path if $\Phi_i(x,t) = \Phi_i^*(x)$ and

$$\Phi_i(x,t) = F_i(e^{-vt}x, t), \ \ i = 1, ...n,$$

is the distribution of the cost normalized at the rate $\nu > 0$, as defined in equation (4). In a balanced growth the wage and output in all location grow at the rate ν. We define the density of normalized cost by $\phi_i(x,t) = -\partial\Phi_i(x,t)/\partial x$.

Using these definitions, replacing them into equation (10) and changing variables we obtain

$$\frac{1}{\Phi_i(x,t)}\left[\frac{\partial\Phi_i(x,t)}{\partial t} - \nu x\phi_i(x,t)\right] = \alpha \log\left(\sum_{j=1}^{n}\int_x^\infty \phi_j(\tilde{x}_j,t)\prod_{k\neq j}\Phi_k(a_{ijk}(t)\ \tilde{x}_j,t)\ d\tilde{x}_j\right). \quad (11)$$

We state our assumption on the initial distribution of the cost x_i in each location:

Assumption 2. Let $\epsilon_i(x,0)$ be the elasticity of the c.d.f. $1-F_i(x,0)$ for the cost x_i at time $t=0$. The elasticities are all continuous at $x_i = 0$, satisfying $0 < \epsilon_1(0,0) \leq ... \leq \epsilon_n(0,0)$, and $\epsilon_1(0,0) < \infty$.

The following proposition characterizes the dynamics of the left tails of the distribution of cost of the n locations.

Proposition 3. Let $F_i(\cdot,0)$ be the time $t = 0$ right c.d.f. of cost in locations $i = 1,...,n$, satisfying assumption 2. Let $\theta \equiv 1/\epsilon_1(0,0)$, where $\epsilon_1(0,0) \leq \epsilon_i(0,0)$, all i, and let $\lambda_i \equiv \partial G_i(0,0)/\partial z$ be the initial density of the transformed variable $z = x^{1/\theta}$ evaluated at $z = 0$. Assume that technology is diffused according to equation (10) with $0 < \kappa_{ij} \leq 1$, all i,j. Assume that there exist a balance growth path and that the corresponding invariant distributions has positive and finite elasticity at zero. Then, the growth rate in a balance growth path is

$$\nu = n\alpha\theta, \quad (12)$$

and for all $t > 0$ and all i the elasticity of $1 - F_i(x,t)$ at $x = 0$, $\epsilon_i(0,t)$, satisfies

$$\epsilon_i(0,t) = 1/\theta, \quad (13)$$

and for all $t \geq 0$ and all i

$$\lim_{x\to 0}\frac{1-\Phi_i(x,t)}{1-\Phi_1(x,t)} = \frac{\partial G_i(0,t)/\partial z}{\partial G_1(0,t)/\partial z} = \frac{\lambda_i e^{-n\alpha t} + \left(\frac{1}{n}\sum_{j=1}^{n}\lambda_j\right)(1-e^{-n\alpha t})}{\lambda_1 e^{-n\alpha t} + \left(\frac{1}{n}\sum_{j=1}^{n}\lambda_j\right)(1-e^{-n\alpha t})}. \quad (14)$$

After the first instance for which diffusion takes places through trade, the distribution of cost in each location shares the same concentration of very low costs, i.e., equation (13) holds. In particular, the concentration of low costs in each location equals that of the location with the highest initial concentration, i.e., the lowest elasticity of the initial c.d.f. around $x = 0$, given by $1/\theta$. While locations share the same concentration of low costs for all $t > 0$, the relative number of these costs in each location differ along the transition to a stationary equilibrium. As discussed in Proposition 1, for distributions with the same concentration of low costs, the relative number of these costs is measured by the ratio of the density of the transformation of the cost $z = x^{1/\theta}$ at $z = 0$. Equation (14) gives an expression for the evolution of the ratio of the densities of the transformed cost at $z = 0$ for each location relative to the location with the highest initial number of low costs. This expression follows from the solution of a linear system of o.d.e. that describes the evolution of $\partial G_i(0, t)/\partial x$. Overtime, the relative number of low costs in each location converges monotonically to 1. In words, since the distributions immediately have the same concentration of ideas, and asymptotically they have the same number of low cost ideas, then the left tail of the distribution of cost of all locations are equalized.

Notice that the characterization of the left tail of the distribution of cost is independent of the magnitude of trade costs and the details of the static trade equilibrium. Intuitively, the diffusion of arbitrarily productive ideas is not affected by proportional trade barriers or by the relative cost of labor in different locations, as trade patterns for these goods are going to be determined by purely technological considerations.

An important corollary of Proposition 3 is that calibrations of the shape parameter θ based on the property of the right tail of the distribution of productivities are going to be robust to trade distortions.

Next we turn to some special cases of the set-up.

5.1 The case of two locations

In this case the static trade model can be analyzed and/or solve using different strategies, as explained above. Equation (11) specializes to:

$$\frac{1}{\Phi_1(x,t)}\left[\frac{\partial \Phi_1(x,t)}{\partial t} - 2\alpha x \phi_1(x,t)\right] = \tag{15}$$
$$= \alpha \log\left(\int_x^\infty \phi_1(\tilde{x}_1,t)\Phi_2\left(\frac{w_1\kappa_{12}}{w_2}\tilde{x}_1,t\right)d\tilde{x}_1 + \int_x^\infty \phi_2(\tilde{x}_2,t)\Phi_1\left(\frac{w_2}{w_1\kappa_{12}}\tilde{x}_2,t\right)d\tilde{x}_2\right)$$

and

$$\frac{1}{\Phi_2(x,t)}\left[\frac{\partial \Phi_2(x,t)}{\partial t} - 2\alpha x\phi_2(x,t)\right] =$$
$$= \alpha \log\left(\int_x^\infty \phi_1(\tilde{x}_1,t)\Phi_2\left(\frac{w_1}{w_2\kappa_{21}}\tilde{x}_1,t\right)d\tilde{x}_1 + \int_x^\infty \phi_2(\tilde{x}_2,t)\Phi_1\left(\frac{w_2\kappa_{21}}{w_1}\tilde{x}_2,t\right)dx_2\right)$$

We consider the alternative representation of the first equation:

$$\frac{1}{\Phi_1(x,t)}\left[\frac{\partial \Phi_1(x,t)}{\partial t} - 2\alpha x\phi_1(x,t)\right] = \tag{16}$$
$$= \alpha \log\left(\Phi_1(x,t)\Phi_2(x,t) + \int_x^{\frac{w_2}{w_1\kappa_{12}}x}\int_{\frac{w_1\kappa_{12}}{w_2}x_1}^{x} \phi_1(\tilde{x}_1,t)\phi_2(\tilde{x}_2,t)d\tilde{x}_2\; d\tilde{x}_1\right)$$

for the case that $\kappa_{12} < \frac{w_2}{w_1}$, or in the complementary case:

$$\frac{1}{\Phi_1(x,t)}\left[\frac{\partial \Phi_1(x,t)}{\partial t} - 2\alpha x\phi_1(x,t)\right] =$$
$$= \alpha \log\left(\Phi_1(x,t)\Phi_2(x,t) + \int_{\frac{w_2}{w_1\kappa_{12}}x}^{x}\int_x^{\frac{w_1\kappa_{12}}{w_2}\tilde{x}_1} \phi_1(\tilde{x}_1,t)\phi_2(\tilde{x}_2,t)d\tilde{x}_2 d\tilde{x}_1\right)$$

This representation shows that whenever $\kappa_{12} \neq \frac{w_2}{w_1}$ the diffusion is slower. The case where $\kappa_{12} = \frac{w_2}{w_1}$ has the interpretation of a situation where potential producers from location 1 has two meetings, one with a randomly drawn potential producers from location 1 and the other with a randomly drawn producer from location 2.

To see the equivalence compare the derivative w.r.t x of the following two expressions. The first one is the term inside of the logarithm of the right hand side of equation (15). The second is the term inside of the logarithm of the right hand side of equation (16). For the second term it is useful to first perform the inner integral.

5.2 The case of n identical locations

As we explain above in this case by hypothesis $L_i = L$, $F_i(x,t) = F(x,t)$, $\kappa_{ij} = \kappa$, and the resulting wages are equal. The equation describing the evolution of their identical distribution, equation (11), specializes to:

$$\frac{1}{\Phi(x,t)}\left[\frac{\partial\Phi(x,t)}{\partial t}-n\alpha x\phi(x,t)\right]= \tag{17}$$
$$=\alpha\log\left(\int_x^\infty \phi(\tilde{x},t)\left(\left[\Phi\left(\kappa\tilde{x},t\right)\right]^{n-1}+(n-1)\left[\Phi(\tilde{x},t)\right]^{n-2}\Phi\left(\frac{\tilde{x}}{\kappa},t\right)\right)\,d\tilde{x}\right)$$

In this case, a larger trade cost κ implies a slower diffusion of technology, and hence a long run distribution of relative cost that is stochastically higher. The sensitivity of the distribution Φ w.r.t. to κ depends on the level of κ. For instance, for κ around the case of costless trade $\kappa = 1$ trade cost has only second order effect on Φ. In the case of costless trade the stationary distribution of relative cost is Weibull with parameters given by elasticity of the initial c.d.f. at $x = 0$, and the $\phi(0)$. The case of n identical locations with no trade cost corresponds to the close economy case studied above (see Alvarez et al. (2008), where we also derived that the distribution converged to exponential.)

We first establish that the right hand side of equation (17) is decreasing in κ, so that for higher trade cost the stationary distribution of technology is smaller in the first order stochastically dominated sense. For the purpose of technology diffusion, this case can be analyze as the case of two non-identical locations, but with equal wages. This is because trade costs induce a larger fraction of meeting with high cost domestic producers, and fewer meetings with each of the remaining $n-1$ symmetric foreign locations. Mathematically, we can set $\Psi_1(x) = \Psi(x)$ and $\Psi_2(x) = [\Psi(x)]^{n-1}$. Replacing this specification into the two locations equation (15) we can derive the expression in equation (17).

The effect of κ around the case of costless trade can be seen by realizing that the derivative of the term inside the logarithm in the right hand side of equation equation (17) vanishes for $\kappa = 1$.

5.3 Numerical Examples

In this section we present preliminary numerical examples to illustrate the effect of trade costs on the stationary distribution of productivity, and the transitional dynamics of the distribution of productivity following a decline in trade costs. We consider two cases: (1) the case of n symmetric locations facing trade costs $\kappa_{ij} = \kappa$, and (2) the case of 1 asymmetric and $n-1$ symmetric locations facing trade costs $\kappa_1 = \kappa_{1j} \leq \kappa_{ji} = 1$, $j = 2, ..., n$, $i \neq j$.

5.3.1 n Symmetric Locations

We first study a world economy with n symmetric locations. For each location i the distribution of the cost normalized at the rate $n\alpha\theta$ is given by equation (17). We calibrate the values of θ and α following our discussion in Section 3, and consider various alternative values for the number of locations, n.[8]

In Figure 1 we show the effect of changes in the trade cost κ on the stationary distribution of productivities, $1/x$. The left panel shows the stationary densities of productivities for four different values of $\kappa = 0.5, 0.7, 0.9, 1.0$, for a world economy with 10 locations, $n = 10$. As the trade cost increases, i.e., κ decreases, the potential producers in each location are exposed to worse ideas, $P_i(x)$ increases, and therefore, there is more mass in the left tail of the distribution of productivities. We can also see that the right tail of the distribution of productivity is unaffected, illustrating the rusult in Proposition 3. The right panel shows the average productivity as a function of the trade cost for five world economies differing in their size, as measured by the number of locations, n = 2, 5, 10, 20 and 100.

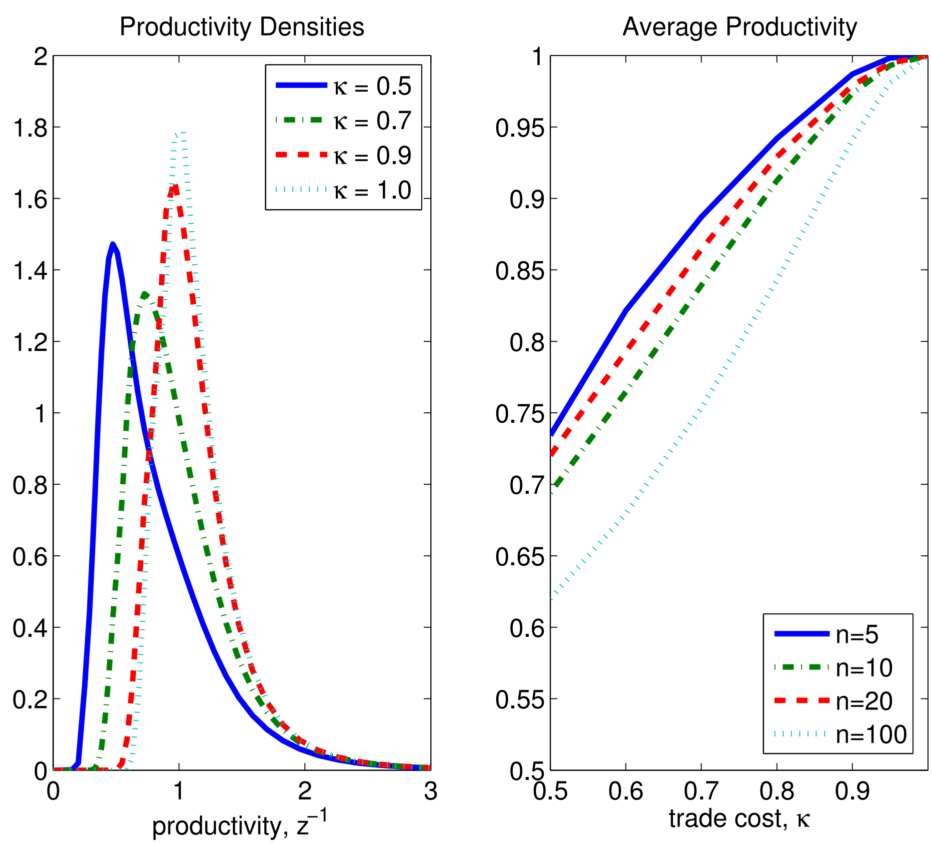

Figure 1: Trade Cost and the Stationary Distribution of Productivity, n Symmetric Locations. The left panel shows the stationary densities of productivities for different values of $\kappa = 0.5, 0.7, 0.9, 1.0$. The right panel shows the average productivity as a function of the trade cost for different values of $n = 5, 10, 20, 100$.

[8]In particular, we set $\theta = 0.2$ to be consistent with the available evidence on the right tail of the distribution of productivity, and set $\alpha = 0.02\theta n$, to match a growth rate 0.02. For each value of n we recalibrate α.

In Figure 2 we ilustrate the transitional dynamics induced by a decline in trade costs. We consider the transition of a world economy consisting of 10 locations from an initial stationary equilibrium in which trade among each these locations must incur a cost of 30%, $\kappa = 0.7$, to a situation of costless trade, $\kappa = 1$. In the left panel of Figure 2 we shows the evolution of the distribution of productivity following the decline in trade cost by plotting the density of productivity in four periods starting from the initial stationary, $t = 0$, 5, 10, and 50. In the right panel of Figure 2 we present the associated time path of average productivity.

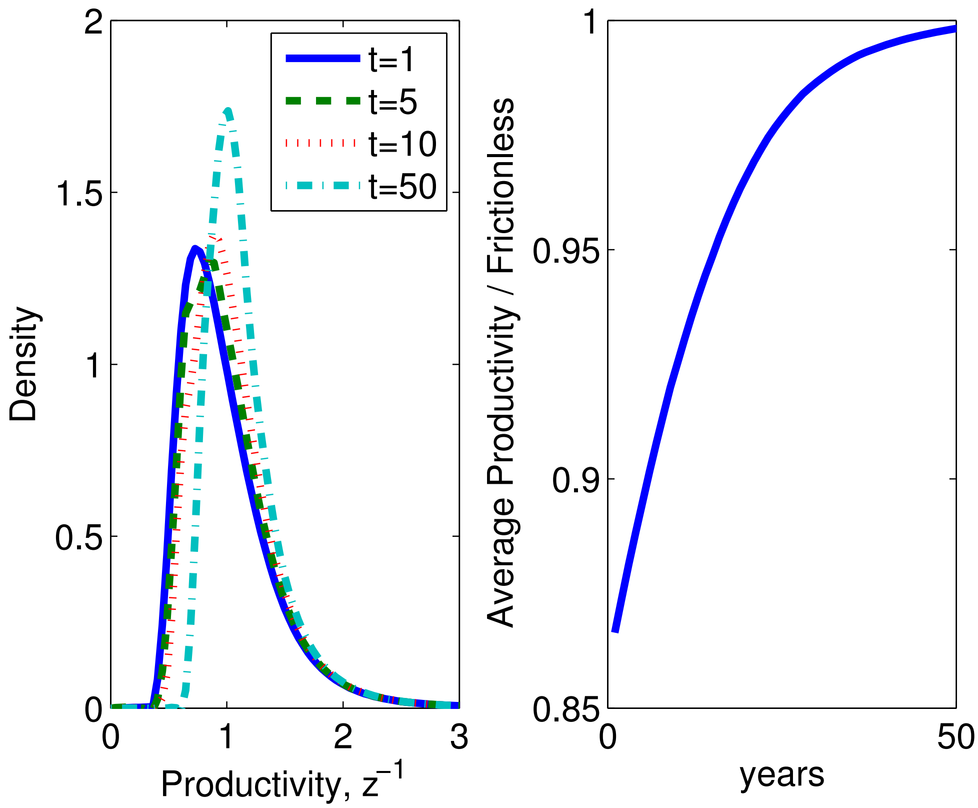

Figure 2: Transitional Dynamics, n Symmetric Locations. The left panel shows the evolution of the distribution of productivities following the elimination of trade costs, from $\kappa = 0.7$. The right panel shows the time path of average productivity for world economy with 10 locations, $n = 10$.

5.3.2 1 Asymmetric, $n - 1$ Symmetric Locations

In the previous exercises we illustrate the effect of symmetric trade barriers. We now explore the impact of unilateral trade barriers by considering a world economy consisting of n locations, where $n - 1$ locations are able to import at no cost from any other location, while the first location faces an iceberg cost $\kappa_1 \leq 1$ when importing from any of the other location.

These trade costs are asymmetric in that the cost of exporting to location 1 is different from the cost importing. We focus on the case of a world economy with 10 locations. Alternatively, we can interpret this world economy as consistent of two locations of asymmetric size.

In Figure 3 we show the density of the distribution of productivity in the first location, left panel, and the other $n-1$ locations, right panel, for different levels of the trade cost incurred by the first location. The effects in the distribution of productivities of the first locations of unilateral trade costs are similar to the effects of symmetric trade costs, see Figure 1, although somewhat smaller in magnitude.

In the right panel of Figure 3 we can observe the effect on the distribution of productivities in the $n-1$ symmetric locations. Even though trade in these locations is not directly distorted by trade barriers, in general equilibrium their wages decline, and they consume more of their own goods that are produced with relatively unproductive technologies. Potential producers in these locations end up having relatively worst meetings, implying that their stationary distribution have more mass on the left tail. Nevertheless, the impact on the distribution of productivity of the other $n-1$ locations is quantitatively very small.

A Proofs

Proof. (of Proposition 1). We first consider the case $0 < \epsilon_1(0) < \epsilon_2(0) < \infty$. If $\epsilon_i(x)$ is the elasticity of $1 - F_i(x)$, we can write:

$$1 - F_i(x) = (1 - F_i(\bar{x})) \exp\left(\int_x^{\bar{x}} \frac{\epsilon_i(t)}{t} dt\right)$$

for some $\bar{x} > x$. Hence,

$$\frac{1 - F_2(x)}{1 - F_1(x)} = \frac{1 - F_2(\bar{x})}{1 - F_1(\bar{x})} \exp\left(\int_x^{\bar{x}} \frac{\epsilon_2(t) - \epsilon_1(t)}{t} dt\right).$$

If $\epsilon_i(x)$, $i = 1, 2$ are continuous at $x = 0$, satisfying $0 < \epsilon_1(0) < \epsilon_2(0) < \infty$, and considering a small $\bar{x}$ we get

$$\frac{1 - F_2(x)}{1 - F_1(x)} \leq \frac{1 - F_2(\bar{x})}{1 - F_1(\bar{x})} \exp\left(\int_x^{\bar{x}} \frac{\bar{\epsilon}}{t} dt\right) = \frac{1 - F_2(\bar{x})}{1 - F_1(\bar{x})}\, \bar{x}^{\bar{\epsilon}}\, x^{-\bar{\epsilon}},$$

for some $\epsilon_2(0) - \epsilon_1(0) \leq \bar{\epsilon} < \infty$. Letting x go to zero we obtain the desired result. If $0 < \epsilon_1(0) = \epsilon_2(0) < \infty$, the desired result is a direct application of L'Hopital rule and the definition of $G_i(z)$, $G_i(z) = F_i(z^{\frac{1}{\epsilon_i(0)}})$.

Proof. (of Proposition 2). The proof of the proposition follows from Lemma 4-Lemma 6.

Lemma 4, which follows from Proposition 1 and l'Hopital rule, is useful for the interpretation of cost x and transformed cost $z = x^{\frac{1}{\theta}}$.

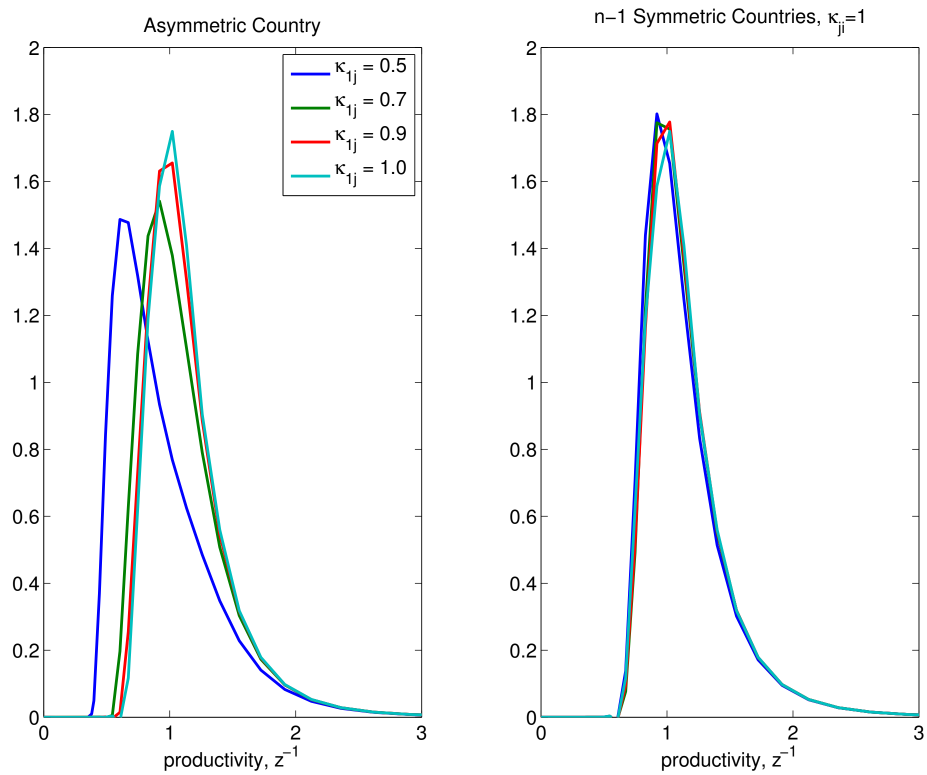

Figure 3: Unilateral Trade Cost and the Stationary Distribution of Productivity, 1 Asymmetric, $n-1$ Symmetric Locations. The left panel shows the stationary density of productivity in the asymmetric location, for different levels of the iceberg cost of importing to this location. The right panel shows the stationary density of productivity in the $n-1$ symmetric locations, for different levels of the iceberg cost of importing to the asymmetric location.

Lemma 4. Let $\epsilon(x)$ be the elasticity of the c.d.f. $1 - F(x)$ of the cost x and $f(x)$ be its density. Then

$$\lim_{x \downarrow 0} f(x) = \begin{cases} \infty & \text{if } \lim_{x\downarrow 0} \epsilon(x) < 1 \ , \\ f(0) \in (0, \infty) & \text{if } \lim_{x\downarrow 0} \epsilon(x) = 1 \ , \\ 0 & \text{if } \lim_{x\downarrow 0} \epsilon(x) > 1 \ . \end{cases} \tag{A-1}$$

Consider a c.d.f. for the initial cost that has a finite positive elasticity at $x = 0$. For any such distribution of cost, we can always choose an associated cost index $z = x^{1/\theta}$ that has a strictly positive and finite density at $z = 0$. This amounts to a convenient change of variables as formalize in the following Lemma, whose proof is immediate.

Lemma 5. Let $\epsilon(x)$ be the elasticity of the c.d.f. $1 - F(x)$ for the cost x, and let $\theta = \lim_{x\downarrow 0} \frac{1}{\epsilon(x)}$. Then the cost index $z = x^{\frac{1}{\theta}}$ has c.d.f $1 - F(z)$ with elasticity one at $z = 0$, and hence it has a strictly positive and finite density at zero.

We now study the evolution of the distribution of cost index z described by equation

$$\frac{1}{G(z,t)} \frac{\partial G(z,t)}{\partial t} = \alpha \log \left(G(z,t)\right) \ , \tag{A-2}$$

the analog of the law of motion for the cost derived in equation (2). Without loss of generality, we choose a cost index that has a strictly positive and finite density at $z = 0$, i.e., following Lemma 4 and Lemma 5, we choose a cost index z such that $x = z^{\theta}$, where $\theta = \lim_{x\downarrow 0} \frac{1}{\epsilon(x)}$. Integrating equation (A-2) we obtain the following simple expression for the evolution the path of right c.d.f. for the cost index $G(z,t)$:

$$\log G(z,t) = e^{\alpha t} \log G(z,0) \ .$$

Lemma 6 characterizes the asymptotic behavior of the distribution of the deflated cost index. In particular, it shows that in a balance growth path the cost index must grow at the rate α, and the stationary distribution of the relative cost index is exponential.

Lemma 6. Let $1 - G$ be the c.d.f corresponding to the initial distribution of cost index, and assume that $0 < \lambda \equiv g(0) < \infty$, where $g(z)$ is the density of the cost index, $g(z) = -\partial G(z)/\partial z$. Then, the path of the distribution of deflated cost index with right c.d.f. $\{\Psi(z,t)\}$ converges to a exponential distribution with right c.d.f. given by: $\Psi^*(z) = \exp(\lambda \ z)$ if and only if $\nu = \alpha$.

Proof. (of Lemma 6). Using the definition of the deflated right c.d.f. of the cost index $\{\Phi(x,t)\}$ and the notion of a balance growth path

$$\log\left[\Phi^*(x)\right] \quad = \quad \lim_{t\to\infty} \log \Phi(x,t) = lim_{t\to\infty} e^{\alpha t} \log\left[G(e^{-vt}x, 0)\right]$$

using L'Hopital rule

$$\log\left[\Phi^*(x)\right] \;=\; lim_{t\to\infty}\frac{\nu}{\alpha}e^{(\alpha-\nu)t}x\frac{G_1(e^{-\nu t}x,0)}{G(e^{-\nu t}x,0)}$$

provided that $\nu = \alpha$ the stationary distribution converges to a non-degenerated limit

$$\log\left[\Phi^*(x)\right] \;=\; G_1(0,0)\ x = -\lambda\ x \ \text{ for all } x > 0,$$

thus the stationary distribution of the deflated cost index Φ^* is a non-degenerate exponential with parameter λ.

Lemma 7. Assume $\nu = \alpha$. The asymptotic speed of convergence of $\{\Phi(x,t)\}$ equals α.

Proof. (of Lemma 7). Following the definition (5), the speed of convergence of the distribution of the deflated cost index x, $\Phi(x,t)$, equals

$$\begin{aligned} s(G(\cdot,0),x,t) &= \lim_{\Delta\downarrow 0}\frac{1}{\Delta}\frac{\log\Phi(x,t+\Delta)-\log\Phi(x,t)}{\log\Phi(x,t)-\log\Phi^*(x)} \\ &= \alpha\frac{e^{\alpha t}\log\left[G(e^{-\alpha t}x,0)\right]-x\frac{G_x(e^{-\alpha t}x,0)}{G(e^{-\alpha t}x,0)}}{e^{\alpha t}\log\left[G(e^{-\alpha t}x,0)\right]-x\lambda}. \end{aligned}$$

Taking the limit as $t\to\infty$,

$$\begin{aligned} \lim_{t\to\infty} s(G(\cdot,0),x,t) \;&=\; \lim_{t\to\infty}\alpha\frac{e^{\alpha t}\log\left[G(e^{-\alpha t}x,0)\right]-x\frac{G_x(e^{-\alpha t}x,0)}{G(e^{-\alpha t}x,0)}}{e^{\alpha t}\log\left[G(e^{-\alpha t}x,0)\right]-x\lambda} \\ &=\; \alpha. \end{aligned}$$

Proof. (of Proposition 3). We first study the dynamics of the distribution of the transformed cost $z = x^{1/\theta}$, with right c.d.f. $G_i(z,t) = F_i(z^\theta,t)$ and density $g_i(z,t) = -\partial G_i(z,t)/\partial z$. This transformed cost has the property that its initial density evaluated at $z = 0$ is finite for all locations $i = 1,...,n$, and strictly positive for at least location $i = 1$. Using equation (10) and the definition of $G_i(z,t)$ it is straightforward to the law of motion

$$\frac{1}{G_i(z,t)}\frac{\partial G_i(z,t)}{\partial t} = \alpha\log\left(\sum_{j=1}^{n}\int_x^\infty g_j(\tilde z_j,t)\prod_{k\neq j}G_k(a_{ijk}(t)\ \tilde z_j,t)\ d\tilde z_j\right) \tag{A-3}$$

Differentiating this equation w.r.t. z and evaluating at $z = 0$ we obtain a system of ordinary differential equations for the densities of locations $i = 1,...,n$, evaluated at $z = 0$:

$$\frac{\partial g_i(0,t)}{\partial t} = \alpha\sum_{j=1}^{n}g_j(0,t),$$

whose solution equals

$$g_i(0,t) = g_i(0,0) + n\alpha \left(\frac{1}{n} \sum_{j=1}^{n} g_j(0,t) \right) \left(e^{n\alpha t} - 1 \right). \tag{A-4}$$

The ratio of the right c.d.f. of the normalized cost in location i relative to that in location 1, in the neighborhood of $x = 0$, equals

$$\begin{aligned}
\lim_{x \to 0} \frac{1 - \Phi_i(x,t)}{1 - \Phi_1(x,t)} &= \lim_{x \to 0} \frac{1 - F_i(e^{-\nu t}x, t)}{1 - F_1(e^{-\nu t}x, t)} \\
&= \frac{g_i(0,t)}{g_1(0,t)}, \\
&= \frac{g_i(0,0)e^{-n\alpha t} + n\alpha \left(\frac{1}{n} \sum_{j=1}^{n} g_j(0,0) \right) (1 - e^{-n\alpha t})}{g_1(0,0)e^{-n\alpha t} + n\alpha \left(\frac{1}{n} \sum_{j=1}^{n} g_j(0,0) \right) (1 - e^{-n\alpha t})},
\end{aligned}$$

where the second equality follows from L'Hopital rule and the last equality uses equation (A-4). We next study the tail behavior of the distribution of the transformed cost, normalized at the rate ν, $\Psi_i(z,t) = G_i(e^{-\nu t}z, t)$, whose law of motion is the following straightforward modification of equation (11),

$$\frac{1}{\Psi_i(x,t)} \left[\frac{\partial \Psi_i(z,t)}{\partial t} - \nu z \psi_i(z,t) \right] = \alpha \log \left(\sum_{j=1}^{n} \int_{z}^{\infty} \psi_j(\tilde{z}_j, t) \prod_{k \neq j} \Psi_k(a_{ijk}(t)\ \tilde{z}_j, t)\ d\tilde{z}_j \right). \tag{A-5}$$

In turn, the behavior of $\Psi_i(z,t)$ in the neighborhood of $z = 0$ can be described by the following first order approximation of equation (A-5) around $z = 0$

$$\frac{1}{\Psi_i(z,t)} \frac{\partial \Psi_i(z,t)}{\partial t} = \left(-\alpha \sum_{j=1}^{n} \psi_j(0,t) + \nu \psi_i(0,t) \right) z.$$

Integrating this equation between $[0,t]$, we obtain

$$\Psi_i(z,t) = \exp \left[\log \Psi_i(z,0) + \left(-\alpha \sum_{j=1}^{n} \int_0^t \psi_j(0,\tilde{t}) d\tilde{t} + \nu \int_0^t \psi_i(0,\tilde{t}) d\tilde{t} \right) z \right]. \tag{A-6}$$

Using that the density of the transformed and normalized cost evaluated at $z = 0$ equals

$$\psi_i(0,t) = \psi_i(0,0)e^{-\nu t} + \left(\frac{1}{n} \sum_{j=1}^{n} \psi_j(0,0) \right) \left(e^{(n\alpha - \nu)t} - e^{-\nu t} \right),$$

we can re-write equation (A-6) as

$$\begin{aligned}
\Psi_i(z,t) &= \exp\left[\log \Psi_i(z,0) - \psi_i(0,0)z - \left(\frac{1}{n}\sum_{j=1}^{n}\psi_j(0,0)\right)\left(e^{(n\alpha-\nu)t} - e^{-\nu t}\right)z\right], \\
&\approx \exp\left[-\left(\frac{1}{n}\sum_{j=1}^{n}\psi_j(0,0)\right)\left(e^{(n\alpha-\nu)t} - e^{-\nu t}\right)z\right], \\
&= \exp\left[-\left(\frac{1}{n}\sum_{j=1}^{n}g_j(0,0)\right)\left(e^{(n\alpha-\nu)t} - e^{-\nu t}\right)z\right], \qquad \text{(A-7)}
\end{aligned}$$

where the second equality follows from $\log \Psi_i(z,0) \approx \psi_i(0,0)z$ in the neighborhood of $z=0$, and the last equality uses that $\psi_j(0,0) = g_j(0,0)$. Provided that $\nu = n\alpha$, equation (A-7) has a non-degenerate limit at $t \to \infty$

$$\lim_{t\to\infty} \Psi_i(z,t) = \exp\left[-\left(\frac{1}{n}\sum_{j=1}^{n}\psi_j(0,0)\right)z\right].$$

Doing a simple change of variables, we also obtain the evolution of the distribution of the cost, $x = z^\theta$, normalized at the rate $n\alpha\theta$, in the neighborhood of $x=0$,

$$\Phi_i(x,t) = \exp\left[-\left(\frac{1}{n}\sum_{j=1}^{n}g_j(0,0)\right)\left(1 - e^{-n\alpha\theta t}\right)x^{\frac{1}{\theta}}\right],$$

and its asymptotic distribution in the neighborhood of $x=0$,

$$\lim_{t\to\infty} \Phi_i(x,t) = \exp\left[-\left(\frac{1}{n}\sum_{j=1}^{n}\psi_j(0,0)\right)x^{\frac{1}{\theta}}\right].$$

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