

A COURSE IN GAME THEORY: *solutions to exercises*

Burton Rosenberg

Creation:
Last Update: 18 Feb 2005

1 Nash Equilibrium: Strategic Games

N.B. I have taken to calling *strategy* what the book refers to as *action profile*. In either case, this is the vector of choices each player makes, taken from the space of all possible combinations of all possible choices.

18.2 Formulate a first price auction as a strategic game and analyze its Nash equilibria.

Player i has valuation v_i and $v_i > v_{i+1} > 0$. The set of actions for each player is a bid $b_i \in [0, \infty]$. The price paid for the item is $p = \max_i \{b_i\}$ and the player of minimum index bidding this price wins. The payoff for player i is $v_i - p$ if i wins, 0 else. A Nash equilibrium is denoted $b^* = \{b_i^*\}$.

If b^* is a Nash equilibrium, then player 1 wins. Suppose I have a strategy, that is, a vector of bids, subject to the constraint that player one did not win. I infer that this strategy is not a N.E. Let $j \neq 1$ be the winner. Since j can get 0 by bidding 0, the strategy is not a N.E. if $v_j - p < 0$.

I am therefore left to consider only strategies for which $v_1 > v_j \geq p > b_1$. These are not N.E.'s since player 1 can raise his bid to slightly above v_j .

A N.E. would be $b_1 = v_2$, some other player bidding v_2 and every other player bidding $b_i \in [0, v_2]$. By this strategy player 1 gains $v_1 - v_2$ and would not change, other bids being fixed. Any other player has a payoff of 0, and could only change that by increasing his bid above v_2 , in which case the payoff would be negative, all other bids being fixed.

18.3 Show that in a second price auction the bid v_i of any player i is a weakly dominant action. Show that there are inefficient equilibria.

Consider player j with all other bids fixed at b_i , and p the highest among these bids, $p = \max_{i \neq j} \{b_i\}$.

If $p > v_j$, if j wins the payoff is negative. So i will bid v_j , lose and have payoff 0. If $p < v_j$, j will win $v_j - p > 0$ for any bid above p , and so he can bid v_j and win this amount. If $p = v_j$ then his payoff is 0 whether he wins or loses, so he can bid v_j .

We show an inefficient equilibria. Let player i bid more than v_1 and every other player bid less than v_i . Player i wins with a strictly positive payoff and every other player has payoff 0. Player i does not change his payoff unless he changes his bid and loses, with payoff 0. So he does not change his bid. Every other player does not change their payoff unless they out bid i , in which case they pay more than v_1 and have negative payoff. So they do not change their bids. We are at a N.E..

- 18.5 Formulate the *War of Attrition* as a strategic game and show that in all N.E. one of the players concedes immediately.

A player's strategy is a concession time $t_i \geq 0$. The player with the greater concession time wins $v_i - \min(t_1, t_2)$, the other player loses $\min(t_1, t_2)$. If $t_1 = t_2$ then both players win $v_i/2 - t_1$.

I found it hard to decide the value of the loser's payoff. I eventually convinced myself by looking at the player's fortune immediately before and immediately after the game. Setting the initial fortune arbitrarily to zero, the loser must pay t and gains nothing. There is no cost to losing the object beyond the cost to stay in the game, since his initial fortune did not reflect ownership of the object.

Let $t^* = (t_1^*, t_2^*)$ be a N.E.. We show that exactly one of the t_i^* is necessarily zero.

Consider a strategy (t_1, t_2) with $t_1 \neq t_2$ and both are non-zero. The loser would do better to concede immediately. Hence this cannot be a N.E. If the t_i are equal (including both zero) then either player would do better to wait the smallest amount of additional time, in this way winning rather than having a tie, and gaining an extra $v_i/2$. Hence this cannot be a N.E.

Hence if there are N.E., it is a strategy where exactly one player concedes immediately.

If $v_1 \neq v_2$ w.l.o.g. we let $v_1 > v_2$. Else the players are symmetric. There are three cases to consider,

- (a) $t < t_2$. Neither $(t, 0)$ nor $(0, t)$ are N.E. The zero player can play v_2 and increase is fortune from 0 to $v_i - t \geq v_2 - t > 0$.
- (b) $t_2 \leq t < t_1$. This case does not exist if $t_1 = t_2$. If $t_1 \neq t_2$ then $(t, 0)$ is a N.E. and $(0, t)$ is not. If $(t, 0)$ is played the first player gets his maximum payoff v_1 and this can only be changed by his playing 0, in which case his fortune decreases to $v_1/2$. The zero player can only decrease his payoff by waiting longer and not winning, or can tie or win by waiting t or more. In these cases his fortune is strictly negative. Hence $(t, 0)$ is a N.E. However (v_1, t) is preferred to $(0, t)$ by the first player, who would increase is fortune from 0 to $v_1 - t > 0$. So $(0, t)$ is not a N.E.
- (c) $t \geq t_1$. Both $(t, 0)$ and $(0, t)$ are N.E. In each case, the non-zero player can only decrease is payoff from v_i to $v_i/2$ by conceding immediately along with the zero player. The zero player can only decrease his payoff from 0 by waiting longer and not winning, or getting $t_i/2 - t \leq t_1/2 - t < t_1 - t \leq 0$ in a tie, or getting $t_i - t \leq t_1 - t < 0$ in order to win.

- 19.1 Formulate the *Location Game* as a strategic game and find all N.E. for $n = 2$ and show that there is no N.E. for $n = 3$.

We consider the case that the distribution of voters $f(x)$ is the uniform distribution. But changing the scale I believe we can reduce to this case, when $f(x) > 0$ for all x . I would have to justify this carefully. Some day.

Each player picks a location $x_i \in [0, 1]$, $i = 1, \dots, n$, or the special value \perp , which means he does not play. If $k > 0$ players choose location $x \neq \perp$ each scores $(b - a)/k$ where (a, b) is the interval in $[0, 1]$ of points closer to x than to any other distinct x_i . All players choosing \perp receive outcome *don't-play*. Among the remaining players, if there is a unique highest score that player receives outcome *win*. If the highest score is not unique, all players sharing this score receive outcome *tie*. All other players receive outcome *lose*. The preference relation for each player is,

$$\text{lose} < \text{don't-play} < \text{tie} < \text{win}$$

For $n = 2$ the only N.E. is $x_1 = x_2 = 1/2$ and the players tie. Given a strategy (x_1, x_2) any player can change his action to match that of the opponent and tie. Therefore the N.E. cannot include players that lose or don't-play. Since this is symmetric, the players must tie. Hence the only possible N.E. are of the form (x', x') with $x' \in [0, 1]$ or $(1/2 - \delta, 1/2 + \delta)$ for $\delta \in [0, 1/2]$.

Furthermore, if $x' \neq 1/2$ or $\delta \neq 0$ then one player can move his action to $1/2$ and win, assuming the other play remains fixed. Hence the only possible N.E. is $(1/2, 1/2)$. Since either player will lose against the opponent choosing $1/2$ unless he chooses $1/2$ but will tie otherwise, this is a N.E.

For $n = 3$ no N.E. exists. We consider each possible strategy (x_1, x_2, x_3) and show it is not a N.E.

If all x_i are \perp then any player can play any $x \neq \perp$ and improve their payoff by winning. If exactly two x_i are \perp then one of them can match the third player's x and improve to a tie.

We consider strategies in which exactly one x_i is \perp . Considerations of the 2 player game eliminate all strategies where one player is out unless the two players who are in tie (in fact, by both playing $1/2$). This last case, however, gives the out player an improvement by changing from \perp to the common value and thus improving his payoff to a tie.

In summary, if a strategy includes any choice of \perp is not a N.E. We now eliminate strategies in which all players play. Since a player can elect not to play if his in fact loses, we eliminate all strategies except where all players play and they tie.

If all or two players choose the same \bar{x} then there is an x' which wins. If $\bar{x} \neq 1/2$ then $x = 1/2$ is a winning choice. If $\bar{x} = 1/2$ then $x = 1/2 - \delta$ for some small δ is a winning choice. If all values are distinct the moving the leftmost value slightly rightwards, or the rightmost value slightly leftwards, would give a win for that player. Therefore in all case where the three players tie, some player has an improvement, so these are not N.E.'s.

20.2 Kakutani's fixed point theorem states: If X is a compact convex subset of \mathbb{R}^n , $f : X \rightarrow \mathcal{P}(X)$ a closed function such that $f(x)$ is non-empty and convex for all x , then f has a fixed point.

Show the necessity of the conditions: X is compact; X is convex; $f(x)$ is convex for all x ; f is closed.

If we take the non-compact $X = \mathbb{R}$ and $f(x) = [x + 2, x + 3]$ then there can be no fixed point. However \mathbb{R} is a convex subset of \mathbb{R} , $f(x)$ is non-empty and convex for all x , and f is closed. We prove that f is closed. Let $x_i \rightarrow x$, $y_i \rightarrow y$ and $y_i \in f(x_i)$. Suppose $y \notin f(x)$, w.l.o.g. $y < x + 2$. Then for all large enough i , $y_i < x + 2 - \epsilon$, for some $\epsilon > 0$. For all large enough i , $x_i > x - \epsilon$. Hence for all large enough i , $y_i \notin f(x_i)$, contradicting our assumption.

If we take the non-convex $X = [0, 1] \cup [2, 3]$ and $f(x) = [2, 3]$ for $x \in [0, 1]$ and $f(x) = [0, 1]$ for $x \in [2, 3]$ then there can be no fixed point. However X is compact, $f(x)$ is non-empty and

convex for all x and f is closed. We prove that f is closed. W.l.o.g. assume $x_i \rightarrow x \in [0, 1]$. The sequence x_i must eventually remain in $[0, 1]$ hence $f(x_i)$ will eventually be constantly $[2, 3]$. So the sequence y_i will eventually remain in $[2, 3]$ hence converge to an element in $[2, 3]$ since it is closed.

Let $X = [0, 1]$, compact and convex, and,

$$f(x) = \begin{cases} [4, 5] & x \in [0, 2] \\ [0, 1] \cup [4, 5] & x \in [2, 3] \\ [0, 1] & x \in (3, 5] \end{cases}$$

The map $f(x)$ is not always convex, and there is no fixed point. We show f is closed. Let $x_i \rightarrow x$. If x is well inside the constant regions of f then eventually $f(x_i)$ stabilizes. Hence y_i is eventually a sequence in a fixed closed set and its limit is in that set. This leaves the cases $x_i \rightarrow 2$ and $x_i \rightarrow 3$. The y_i are a convergent sequence in the closed set $[0, 1] \cup [4, 5]$ and hence has a limit in this set. Since $f(2) = f(3) = [0, 1] \cup [4, 5]$ the limit of y_i will be in the limit of x_i .

If we take $X = [0, 5]$ and $f(x) = (1/2, 1]$ if $x \in [0, 1/2]$ and $f(x) = [0, 1/2]$ if $x \in (1/2, 1]$, there is no fixed point. However X is compact, convex, $f(x)$ is non-empty and convex for all x . We show that f is not closed. Let $x_i = 1/2$ and $y_i = 1/2 + 1/i$. Both converge to $1/2$ and $y_i \in f(x_i)$. However $1/2 \notin f(1/2)$.

- 20.4 Show that in a symmetric game there is a N.E. of the form (a_1^*, a_1^*) . Give an example of a finite symmetric game that has only asymmetric equilibria.

Not done. My guess is to apply Kakutani's theorem to the space $\{(a, a) \mid (a, a) \in A \times A\}$. The necessary qualities of convexity and so forth should carry forward into this subspace, where symmetry might be required to make sure the set $B(a)$ always touches this subspace (and so is not empty).

2 Nash Equilibrium: Competitive Games

- 24.1 Let G be a strictly competitive (zero-sum) game with N.E.

- Show that if some of 1's payoffs are increased in such a way that the resulting game G' is strictly competitive than G' has no N.E. in which player 1 is worse off than she was in a N.E. of G .
- Show that the game that results if 1 is prohibited from using one of her actions does not have a N.E. of higher payoff than a N.E. in G .
- Construct (necessarily) non-strictly competitive games where the above properties do not hold.

(a) In a competitive game 1's N.E. x^* is a maximinimizer. If $u'(x, y) \geq u(x, y)$, then $\min_{y \in A_2} u'(x, y) \geq \min_{y \in A_2} u(x, y)$ for each x , hence if a maximum on the L.H.S. exists it is at least as big as the maximum over the R.H.S.

(b) If 1 has his set of actions modified $A'_1 \subset A_1$, maximizing over A_1 includes more points, so can only be larger.

(c) In some strategic games, increasing 1's payoff in certain situations might decrease his payoff in equilibrium. Consider the two games,

	CA	CB		CA	CB
RA	4,2	2,1	RA	4,2	2,1
RB	1,1	1,2	RB	5,1	3,2

The first game has unique N.E. (4, 2). The second has N.E. (3, 2). Even though we have increased the payoff for 1 in certain places to arrive at the second game, the equilibrium payoff for 1 is larger in the first game.

(c bis) In some strategic games, restricting 1's play, might increase his payoff. Consider the game:

	CA	CB
RA	3,3	1,1
RB	4,1	2,2

The full game N.E. is (2, 2). If the row player cannot play RB, then the column player would play CA, and the equilibrium (3, 3) results.

3 Nash Equilibrium: Bayesian Games

N.B. I would like to explain how I understand the notation on page 26 concerning Bayesian games, in particular, statements such as “a lottery $L_i(a^*, t_i)$ over $A \times \Omega$: the probability assigned by $L_i(a^*, t_i)$ to $((a^*(j, \tau_j(\omega)))_{j \in N}, \omega)$ is the player i 's posterior belief that the state is ω when he receives the signal t_i .”

A strategy (action profile) a^* in a Bayesian game is a choice of action for every player for every possible signal the player can receive. It is a big thing in that this will be reduced to just one action per player once we determine what signal each player gets. This reduction is denoted $(a^*(j, \tau_j(\omega)))_{j \in N}$. Read: each player ($j \in N$) picks the action according to its received signal ($\tau_j(\omega)$) We can denote this by $a^*(\omega)$ as well, but this does hide the very important role of the signal functions τ_i .

The outcome of the game is still not determined by $a^*(\omega)$ since the true state of the world ω might be involved. So a complete description of the outcome is $(a^*(\omega), \omega)$. From this the consequences can be determined and a preference order on $(a^*(\omega), \omega)$ determined, different for each player.

For each ω this gives an lottery $L_i(a^*, t_i)$. In fact there is not one lottery but a lottery for each player for each of his signals t_i . Hence the notation $L_i(a^*, t_i)$ refers to the lottery played by i after he has received signal t_i . Given player i received signal t_i he now knows that the world's state is in the set $\Omega(t_i) = \{\omega \mid \tau_i(\omega) = t_i\}$. For $\omega \in \Omega(t_i)$ the lottery assigns probability $\mu_i(\omega \mid \omega \in \Omega(t_i))$, where μ_i is player i 's *subjective* probability measure (think likelihood) on Ω . The lottery is then a strategy valued random variable on the set $\Omega(t_i)$.

For a given lottery the outcome varies with choice of $\omega \in \Omega(t_i)$ for two reasons: although i 's action is fixed, because his signal is fixed, other player's might change their action; also the payoffs might depend directly on ω . The functions τ_i are public knowledge. Therefore when evaluating a lottery, looking at outcomes over all states of the world $\omega \in \Omega(t_i)$, I can correctly deduce from a^* and ω what play my adversaries will make.

27.2 Formulate Bach or Stravinsky as a Bayesian game and find the N.E.

The states of the world are $\Omega = \{BB, BS, SB, SS\}$. The signal functions are $\tau_1(XY) = X$ and $\tau_2(XY) = Y$, where $X, Y \in \{B, S\}$. The players actions are from the set $A = \{B, S\}^4$ indexed by $(1, B), (1, S), (2, B), (2, S)$. The probability distribution μ_i on Ω is modeled as a biased coin flipped twice. Note that in this model if one player is known to favor B , then so does the other player, since both use the same bias on their coins.

- (a) $BBBB, SSSS$ are N.E. The game is independent of Ω , each player plays the same no matter what signal is received, and each lottery has a unique outcome. Since strategies BB and SS cannot be modified unilaterally by either player without decreasing the player's payoff, we have a N.E.
- (b) $BBSS, SSBB$ are not N.E. Regardless of ω player 1 plays B , player 2 plays S . Given player 2 playing S , player 1 can improve is fortune by playing S instead of B . The other strategy is argued similarly.
- (c) $BBBS, BBSB, SSBS, SSSB, BSBB, SBBB, BSSS, SBSS$ are not N.E. Consider the player who plays either B or S depending on his signal. For one of the two signals he certainly faces that the players will go to separate concerts, and for that signal he would be better to play differently. For instance, $BBBS$ is not a N.E. because player 2 when given signal S will play S against player 1 playing B regardless of signal, resulting in a outcome inferior to player 2 playing B .

None of these above depend upon the probability distribution. The remaining cases depend upon whether the coin bias is large enough to overcome a player's signal.

Lemma 1 *Suppose a player's signal is X and the probability of the opponent playing X is p_X . If $p_X > 1/3$, the player has larger payoff playing X ; if $p < 1/3$, the payoff is greater playing Y ; else the player is indifferent.*

The proof is the calculation: the payoff is $2p_X$ to play the signal, $(1 - p)$ otherwise. For $p_X > 1/3$ the first is greater, for $p_X < 1/3$ the second is greater.

We call a probability distribution where the bias is in $(1/3, 2/3)$ as unbiased, if the bias is in $(0, 1/3) \cup (2/3, 1)$ it is biased, if it is either $1/3$ or $2/3$ we call it critical. Note that these values $1/3$ and $2/3$ depend on the exact values 2 and 1 stated in the players preferences. To the extent these numbers are arbitrary, so is the value $1/3$.

Lemma 2 *If the play of the opponent is biased, then the player's payoff is greatest playing the opponent's bias, regardless of his signal.*

Let X be the signal for which $p_X < 1/3$. On signal X the player's best payoff is Y . If the signal is Y then necessarily $p_Y \geq 2/3$ and the the best payoff is Y .

- (d) $BSBS$, that is, each player plays is signal. In the unbiased or critical cases, since $p_X \geq 1/3$ no matter what X , a player can play his signal. So we have a N.E. In the biased case some player should be playing constant. So this is not a N.E.
- (e) $SBSB, SBBS, BSSB$. These are never N.E.'s. In the case of an unbiased coin we can find a player (i, t_i) not playing t_i and $p_{t_i} > 1/3$. In the case of a biased coin, one of the players must be playing a constant.

Note that N.E. given by *BSBS* has a positive probability of players going to different movies.

28.1 Formulate the exchange game as a Bayesian game and find all N.E.

Let S be a finite subset of $[0, 1]$. There are two players. The space Ω is $S \times S$ with each $s \in S$ drawn independently from a common distribution. The signal for player i is $\tau_i(s_1, s_2) = s_i$. The set of strategies is $A = A' \times A'$ where $A' = \{0, 1\}^{|S|}$ where $A'(s) = 1$ indicates that the player will exchange $s \in S$, otherwise he will not. Given an $(s_1, s_2) \in \Omega$, if both player are willing to exchange they do so. The payoff for i is s_i .

If both players are willing to exchange something, suppose v_i is the largest value player i is willing to exchange. If $v_1 \neq v_2$ this is not a N.E. W.l.o.g. suppose $v_1 > v_2$. The player $(1, v_1)$ would do better not to exchange. If $v_1 = v_2$ and at least one player has multiple values at which they would exchange than this is not a N.E. W.l.o.g. let player 2 have multiple values. Then player $(1, v_1)$ would be exchanging v_1 for a random value of expectation less than v_1 . If both players are willing to exchange $v_1 = v_2$ only, and these are not the smallest values in S , then the player $(1, s')$, where s' is the smallest value in S would do better to exchange.

Given that both players are willing to exchange something we have eliminated all but a mutual agreement to exchange minimum values from being N.E.'s. This is seen to be a N.E.

If only one player is willing to exchange something and this is not the minimum value than the other player on receiving signal the minimum value would be willing to exchange. Also neither player willing to exchange anything is a N.E.

Hence the four N.E. are: both, one or neither player willing to exchange the minimum value.

*28.2 Construct a two person Bayesian game in which increasing one player's information reduces his payoff (comparing the payoff in equilibrium for the respective games).

A two person game with a coin, $\Omega = \{H, T\}$. The coin is unbiased. Information functions: $\tau_1(\omega) = \omega$. For the limited information game, $\tau_2(\omega) = \perp$; for the full information game, $\tau_2(\omega) = \omega$. We construct a game where player 1 should play the coin, and so should player 2. Since player 2 does not know the coin, she can simply differ to player 1. This gets both players the best outcome. If player 2 knows the coin, however, she is tempted to play something bad for player 1 and that player must now take a defensive stance.

H	D	H	T	T	D	H	T
H	4, 4	0, 5	0, 0	H	0, 0	0, 0	0, 0
T	0, 0	0, 0	0, 0	T	4, 4	0, 5	0, 0
O	1, 1 3/4	2, 2	2, 1	O	1, 1 3/4	2, 1	2, 2

For the limited information game the strategy vector is indexed by $((1, H), (1, T), (2, \perp))$, and the unique N.E. is (H, T, D) , that is, player 1 plays the coin and player 2, uninformed of the coin, plays D. For the complete information game the vector is indexed by $((1, H), (1, T), (2, H), (2, T))$ and the unique N.E. is (O, O, H, T) . Player 2 overreaches, and player 1 reacts to limit the damage.

4 Mixed Strategy Nash Equilibrium

35.1 Formulate the game Guess the Average as a mixed strategy and find the Nash Equilibrium.

There are N players, and each has a distribution over the set $A = \{1, \dots, K\}$. The payoff is calculated by taking $2/3$ the average of the selected $a_i \in A$ for $i = 1, \dots, N$, and the k players with a_i closest to this number gain $1/k$.

The only N.E. is $a_i = 1$ for all i .

Consider v_i , the largest value that player i will play, that is, having non-zero probability in player i 's distribution. The average is not more than v_i . If $v_i > 1$ then the target value t is,

$$t = 2/3 \text{ avg } (v_j) \leq v_i - (1/3)v_i \leq v_i - 2/3.$$

Therefore decreasing the bid will either let i share the winnings, if he is not winning in this game, or be the sole winner, if he is sharing winnings with all players playing v_i .

35.2 Formulate the Investment Race as a strategic game and find its mixed strategy N.E.

There are two players who assign a probability measure α_i to $[0, 1]$. In a trial, the player with highest choice from $[0, 1]$ wins 1, or $1/2$ in a tie.

We argue that if α_i assigns zero weight above v_i , then the best response of the other player is any $a_j > v_i$. The payoff to j is the weight of α_i below a_j and $1/2$ the probability of $\alpha_i(a_j)$. This increases as a_j increases, and remains unchanged as $a_j > v_i$. If $v_i = 1$ then the best response is $a_i = 1$. The lemma states that in a N.E. any pure strategy in the support of the player's distribution is a best response. Therefore α_j has all weight at or about v_i . Likewise α_i has all weight about v_j . This implies they both assign probability 1 to playing value 1.

36.1 Formulate Guessing Right as a strategic game and find the mixed strategy N.E.

There are two players. The strategy set is $S \times S$ where $S = \{1, \dots, K\}$. Players choose distributions α_i over S . In a play, if $a_1 = a_2$ player 2 pays 1 to player 1, else nothing happens.

This enlightens us as to the nature of mixed strategies. Both players give a uniform distribution. This is a N.E. as each pure strategy is a best response against the other player's uniform distribution. Each pure strategy gives the same payoff, $1/K$. However a pure strategy cannot be a N.E. since the other player would react and modify his play to take advantage of this. Mixing dulls the response, it does not sharpen the gain.

Any other distribution cannot be a N.E. since given a non-uniform distribution, the other player will react, which will cause the original player to react again, and so forth. Let a and b be two plays of player 1 with unequal likelihoods, $\alpha_1(a) > \alpha_1(b)$. Then player 2's best response pure strategy would never play a , hence $\alpha_2(a) < \alpha_2(b)$ unless they are both zero. In any case the best response of player 1 must have $\alpha_1(a) < \alpha_1(b)$, or both zero. Which is a contradiction. So all choices of player 1 must be equally likely in any N.E.

Notes: At this point it might be good to set down the definitions, now that these appear clearer to me. Each player has a set A_i of *actions*. A *strategy* in a pure game is the choice for each player of an action, $(a_i) \in \prod_i A_i$. A strategy in a mixed game is a distribution α_i over A_i . Denoting the set of distributions on A_i by $\Delta(A_i)$, a strategy is also the choice by all players of a distribution, $(\alpha_i) \in \prod_i \Delta(A_i)$. A strategy α specifies $\alpha_i(a_j)$, for each player i the probability (or likelihood if we take a Bayesian viewpoint) that the player plays action a_j .

In each space $\prod_i A_i$ and $\prod_i \Delta(A_i)$ there is a possibly empty subset \mathcal{E} called the Nash Equilibrium. The definition of a N.E. is different for mixed than for pure strategies, but we have a helpful lemma: each pure strategy in the support of a player's mixed strategy, i.e., an action of non-zero likelihood, it must be a best-response in fact of the other player's strategies. All

pure strategies in the support of a player's mixed strategy have equal payoffs. The purpose of mixing is the force the opponents away from fixed responses, as the attempt to cover all contingencies.

The intuitive aspect of a N.E. is that it is stable. A player can hypothesize the opponents counter strategy to his strategy and modify his strategy, and so forth, in a "I know that you know that I know" infinite regress. In a N.E. this sort of thing does not get off the ground. However, I have had trouble writing proofs that reflect this aspect of the N.E., and will describe carefully in the next problem the technique I discovered to utilize this back-and-forth what-if game in the context of a proof.

36.3 Formulate Air Strike as a strategic game and find its mixed strategy N.E.

The common set of actions is $A = \{v_1, v_2, v_3\}$ where for player 1 action v_i is to strike target v_i , for player 2 action v_i is to defend v_i . The strategy is a distribution on A . Payoff is determined by 2 paying 1 v_i if 1 strikes v_i and 2 does not defend it; zero otherwise. The preference for destruction or defending the target is $v_1 > v_2 > v_3 > 0$. At times we will also consider the v_i to be real numbers. Which interpretation must be understood from context. For brevity we will denote 1's distribution by (p_i) , 2's distribution by (q_i) .

As an exercise in the remark of the preceding note, we show that in any N.E. $p_1 \neq 0$. Let A' be the set of (mixed) strategies for which $p_1 = 0$. We show $E = A' \cap \mathcal{E} = \emptyset$. Given A' , each pure play by 2 should assign $q_1 = 0$, since a pure strategy of v_1 will strictly maximize his loss, and therefore should not be in the support of α_2 . Therefore $A' \supset A'' \supseteq E$ where A'' is the set of mixed strategies for which $p_1 = q_1 = 0$. In A'' , let v_j be a pure strategy of 1. By comparing its payoff to action v_1 , v_j is not a best response. Therefore nothing in A'' is a N.E. and $E = \emptyset$.

Looking at the pure strategies of 1, his profit is $v_i(1 - p_i)$ for action v_i . Setting them equal we have,

$$q_i = \frac{\delta - 2/v_i}{\delta} \quad \text{where} \quad \delta = 1/v_1 + 1/v_2 + 1/v_3$$

and these are in $[0, 1]$ if $1/v_1 + 1/v_2 \geq 1/v_3$.

Looking at the pure strategies of 2, his loss is $\sum_j v_j p_j - v_i p_i$ for action v_i . This reduces to $v_1 p_1 = v_2 p_2 = v_3 p_3$. Solving, we have $p_i = 1/(\delta v_i)$, with δ as above.

By the lemma for pure strategies, these are N.E.'s.

If $1/v_1 + 1/v_2 < 1/v_3$, the $p_3 = 0$, it is not worth protecting v_3 . Solving as above the N.E. is,

$$p = (v_2/(v_1 + v_2), v_1/(v_1 + v_2), 0), \quad q = (v_1/(v_1 + v_2), v_2/(v_1 + v_2), 0)$$

We have eliminated $p_1 = 0$ and $q_1 = 0$ as candidates for N.E. It remains to show that $p_2 = 0$ and $q_2 = 0$ are also unacceptable in a N.E. This is easy enough, as one player would shift attack or coverage from v_3 to v_2 .

36.3 Show that for all convex compact $X, Y \subset \mathbb{R}^k$ there exists $x^* \in X$ and $y^* \in Y$ such that $x^* \cdot y \leq x^* \cdot y^* \leq x \cdot y^*$ for all $x \in X$ and $y \in Y$.

Intuitively, propose a game with strategies in $X \times Y$, and payoff $x \cdot y$ for the y player and $-x \cdot y$ for the x player. A N.E. (x^*, y^*) would satisfy the inequality. The game satisfies the hypotheses of 20.3 so a N.E. does exist.

5 Extensive Games with Perfect Information

- 94.2 Suppose $G = \langle \{1, 2\}, (A_i), (\geq_i) \rangle$ is a strategic game and $A_i = \{a'_i, a''_i\}$. G is the strategic form of an extensive game with perfect information if and only if some row or column of the payoff table has both payoff's equivalent for both player 1 and 2.

The set of strategies for player i in an extensive game is defined as the map from non-terminal histories h to actions $A(h)$ defined on the subset of histories such that $P(h) = i$. The cardinality of the set of all strategies for player i is $\prod_{h'} |A(h')|$ with the product taken over $h' \in \{h \in H - Z \mid P(h') = i\}$. Under the supposition, this cardinality is 2, so each player has a single node at which to make a choice, and this choice is one of two actions. So for one of the player there is a choice that makes the game outcome indifferent to the other player's choice.

Conversely, we have the strategic game as given. Using the considerations of the preceding paragraph identify which player makes the first choice and which of his choices leads to the node where the second player makes his choice. Label edges and payoffs as required. (If the game begins with the player who does not make the first choice, this can be handled by letting that player have a node with only 1 action — essentially to pass to the second player without a choice.)

- 98.1 Give an example of a subgame perfect equilibrium of a Stackelberg game that does not correspond to a solution of the maximization problem.

Consider the game with terminal histories AB, AC, BD, BE and respective payoffs $(1, 1), (1, 2), (0, 1), (2, 1)$. The perfect eq. strategies are ACD and BCE . One of these has a payoff for player one of 2, the other 1. The maximization problem solves only to the payoff 2 solution.

- 99.1 Show that the one deviation property does not hold for games with infinite horizon.

The one person game with terminal histories $\{C^k S, k \geq 0\} \cup \{C^*\}$ with payoff $1/2^k$ for finite termination, 2 for infinite termination. The strategy $s(h) = S$ is subgame perfect for one deviation, but what is really optimal is $s(h) = C$.

- 100.1 Show that Kuhn's theorem does not hold for games that are not finite, even if they have finite horizon or finitely many possible actions at each play.

A one person game with histories $C^k S$ for all $k \geq 0$, and payoff k has no equilibrium.

- 100.2 A finite game satisfies the no indifference condition if for all terminal histories z and z' , if some player is indifferent between z and z' then all players are indifferent between them. Show that for such a game all players are indifferent between different subgame equilibria. Show also *interchangability*.

Let s and s' be two subgame perfect equilibria. We must show that for every subgame h every player is indifferent between the outcomes at h following strategies $s|h$ and $s'|h$. We do this by induction.

For games of maximum depth zero there is only one outcome and the result holds trivially. Assume the theorem for games of depth less than l . A game of depth l has subgames following from its initial move of depth less than l . The strategies s and s' induce subgame perfect equilibria on each of the subgames, and by induction all players are indifferent between the outcomes of s and s' on any strict subgame of the game.

Since s and s' both induce subgame perfect equilibria on the subgame entered by s on the first move, the outcome of s is equal to the outcome of s' on that particular subgame. Since s' maximizes the outcome for the initial player over all the outcomes in the immediate subgames, s cannot be more preferable than s' , to the initial player. Reversing the roles of s and s' we conclude that the outcomes of playing each are equal for the first player. By the no indifference condition, they are equal for all players. This completes the induction.

Proof of the interchangability property. We reduce to the case where s and s' differ only in the strategy of player i , that is, $s_{-i} = s'_{-i}$ but $s_i \neq s'_i$. Consider any history h where $P(h) = i$. The next action in s is $s(h)$ and the next action in s' is $s'(h)$. The subgame at h has perfect equilibria induced by s and s' , hence all players are indifferent between outcomes given by $s|(h \cdot s(h))$ and $s'|(h \cdot s(h))$ as well as between $s|(h \cdot s'(h))$ and $s'|(h \cdot s'(h))$. But more is true, since s and s' differ only in player i 's choices, the game at h is maximized for player i against the same adversarial choices, so player i is indifferent between $s|(h \cdot s(h))$ and $s|(h \cdot s'(h))$. By the no indifference property, all players are indifferent between these two subgames. So we can replace $s(h)$ by $s'(h)$ in strategy s . Doing this for all histories at which i plays, we succeed in interchanging the play of i in strategies s and s' .

- 101.1 Show that a SPE of a game Γ is also the SPE of Γ modified by making h terminal and assigning outcome the SPE outcome in Γ of playing forward from h , for h not a played history in the SPE.

This follows from the fact that no outcomes are changed by the modification. The construction of a SPE depends only on the outcome by following a strategy after a history h , not on any details of how the outcome is reached.

- 101.2 Let s be a strategy for an extensive game Γ . Create a new game Γ' by deleting all histories $h \cdot a' \cdot h'$ where $a' \neq s(h)$ but $a' \in A(h)$. Show that if s is a SPE in Γ , it remains so in Γ' , suitably restricted.

At h , $s(h)$ was no less preferable than a' . Removing a' does not change this. At all histories prefixes of h , they need only consider the outcomes by following $s(h)$. Thus s would remain an SPE in the modified game.

- 101.3 Armies 1 and 2 attempt to occupy an island. Army 1 has K battalions, army 2 has L battalions. The army which does not possess the island can attack, after which both armies lose 1 battalion, and if the attacker has a battalion left, he now occupies the island; or the army can concede. Analyze this situation as an extensive game using the notation of perfect subgame equilibrium.

The histories are $H = \{A, C\}^+$ where A is Attack and C is Concede; the player function P maps histories of even length to 1, of odd length to 2. The outcome is defined on $(k.l.m)$, where $0 \leq k \leq K$, the number of battalions remaining to army 1, $0 \leq l \leq L$, the number of battalions remaining to army 2, and $m \in \{0, 1, 2\}$ the label of the army occupying the island, where 0 signifies neither player occupies. The outcome is calculated by attaching $(K, L, 2)$ to the initial history and having the update $(a_1, a_2, j) \rightarrow (a_1 - 1, a_2 - 1, i)$ when i attacks and $a_i > 1$ and $a_j > 0$. If $a_i = 1$ and $a_j > 0$ then the outcome is $(a_1 - 1, a_2 - 1, 0)$. The preference is,

$$(k + 2.l, 2) >_1 (k, l, 1) >_1 (k + 1, l, 2)$$

for player 1, and a likewise for player 2.

Suppose $K > L$. If $L = 1$ and it is 1's play, it attacks and 2 cannot counter attack, which is a better outcome than 1 immediately conceding, so 1 attacks. If it is 2's play it will be wiped out if it attacks, so it concedes immediately. For $L > 1$, by induction in each sub-game 1 attacks and 2 concedes (or is wiped out). If it is 1's play, and it attacks 2 concedes, by induction, which is a better outcome for 1, so it attacks. If it is 2's play and he does not concede 1 will attack, by induction, and 2 is worse off, so he concedes. This completes the induction.

Suppose $K < L$. If $K = 1$ and it is 1's play, army 1 has a worse outcome to attack, so it concedes. If it is 2's play, it can attack and 1 cannot counter-attack, so it has a better outcome, so it attacks. For $K > 1$, by induction, in each sub-game 1 concedes and 2 attacks. If it is 1's play, and it attacks, 2 will counter attack, by induction, and 1's outcome must be worse, so 1 concedes. If it is 2's play, and it attacks, 1 will concede, by induction, and it's outcome is better, so 2 attacks. This

For $K = L$ consider $K = L = 1$, the player to play must concede. If $K = L = 2$ then the player to play will attack, because for $K = L = 1$ the opponent must concede. For $K = L > 2$ assume that in each sub-game if the players battalions are odd he concedes, if even he attacks. If $K = L$ is even and it is player 1's turn, if he attacks then by induction 2 concedes, for a better outcome, so he attacks. Same for player 2. If $K = L$ is odd and it is player 1's turn, if he attacks then player 2 will counter attack and the result is worse for 1, so he concedes. Same for player 2. This completes the induction.

6 Extensions of Extensive Games with Perfect Information

*102.1 A problem on extensive games with perfect information and chance moves.

103.1 A Pie Sharing game. Player one shares a pie among the three players. Players 2 and 3 simultaneous either accept or reject the sharing. If they both accept, all players get the share dictated by player 1's choice. Else all get zero. Formulate and find sub-game perfect equilibria.

The set of all terminal histories is $\{(p_1, p_2, p_3), (X_2, X_3)\}$ where $X_j \in \{Y, N\}$ and $p_i \in [0, 1]$ subject to $p_1 + p_2 + p_3 = 1$. The player function is 1 on the initial history, else $\{2, 3\}$. The payoff is (p_1, p_2, p_3) if $X_1 = X_2 = Y$, else $(0, 0, 0)$. Player i prefers the outcome of greatest numerical value in position i . For terminal history h this will be denoted $p_i(h)$.

The sub-game perfect equilibria for players 2 and 3 are $\{N, N\}$, $\{Y, Y\}$ and $\{N, Y\}$ if $p_2 = 0$ and $\{Y, N\}$ if $p_3 = 0$. The sub-game perfect equilibria for player 1 depends on the set $E = \{h \mid h = (p, (Y, Y))\}$ where p is any action. If the set is empty, then any action is a sub-game perfect equilibria. Else, any h such that $p_1(h) = \sup_{h' \in E} p_1(h')$ is a sub-game perfect equilibria.

Note well the strategy in which player 2 and 3 agree to accept when $p_2 = p_3$ and $p_1 < 1$ does not give any equilibria, since the set E does not contain a history which attains the supremum.

103.2 Stop and Continue. Formulate and find all sub-game perfect equilibrium.

The set of terminal histories is $H = \{(X, (m, n))\}$ where $X \in \{S, C\}$ and $m, n \in \mathbb{Z}^+$. The player function is $P(\emptyset) = 1$, else $\{1, 2\}$. The payoff is $(1, 1)$ for all histories of the form $(S, (m, n))$, and (mn, mn) else.

There is no sub-game equilibrium for $\Gamma(C)$ unless $m = n = 0$. Therefore the sub-game perfect equilibrium is $(S, (0, 0))$.

*103.3 Show that the one deviation property holds for extensive games with simultaneous moves, but the Kuhn's theorem does not.

As for Kuhn's theorem, take any two person strategic game with two players and have it be an extensive game where the players state their action simultaneously. Since there are such strategic games (matching pennies) without equilibria, the extensive game which simulates it will have no equilibria.

To do: one deviation proof.

7 Coalitional Games: The Core

259.3 *A production economy.* Let W be a set of workers, and c a capitalist. The production function $f : \mathbb{N} \rightarrow \mathbb{R}$ is concave and $f(0) = 0$. A coalitional game has player $N = W \cup \{c\}$ and a value function on the subsets of N ,

$$v(S) = \begin{cases} 0 & \text{if } c \notin S \\ f(|S \cap W|) & \text{if } c \in S \end{cases}$$

Show that the core is,

$$\{x \in \mathbb{R}^N \mid 0 \leq x_i \leq f(w) - f(w-1), \sum x_i = f(w), i \in W\}$$

where $w = |W|$ and give an interpretation.

It is necessary for x in the core that $x_i \leq f(w) - f(w-1)$. If not, consider $S = W \cup c - w_i$ for a worker w_i with $w_i > f(w) - f(w-1)$. Then $x(S) = f(w) - x_i < f(w) - f(w-1) + f(w-1) = v(S)$ and the set S should "defect" — their worth is greater than the sum of their pay.

The above proof shows $v(S) \leq x(S)$ for S containing the capitalist c and $|S \cap W| = w - 1$. We proceed by induction to show $v(S) \leq x(S)$ for all $S \ni c$. (Sets S which do not contain c satisfy the core requirement, trivially.) Let S have k workers and the theorem is true for all sets with k or larger workers,

$$\begin{aligned} x(S - w_i) &= x(S) - x_i \geq x(S) - f(w) + f(w-1) \\ &\geq x(S) - f(k) + f(k-1) \quad (\text{because } f \text{ concave}) \\ &\geq f(k) - f(k) + f(k-1) \quad (\text{induction hypoth.}) \\ &= f(k-1) = v(S) \end{aligned}$$

where we have used the concavity of f and the induction hypothesis.

A worker's pay is upper bound by the marginal increase in productivity of the last worker employed. All excess due to economies of scale accrues to the capital holder.

260.2 *A market for an indivisible good.* In a market for an indivisible good a set of buyers is B and sellers is L . Each seller holds one unit of the good and has reservation price of 0; each buyers wishes to purchase one unit of the good and has a reservation price of 1. In the model $N = B \cup L$ and $v(S) = \min\{|S \cap B|, |S \cap L|\}$.

Calculate and interpret the core when $|B| = |L|$.

Considering pairs of one seller and one buyer, we have $x_b + x_l \geq 1$ for any buyer b and seller l . Summing over all pairs thus created, it cannot ever be that the inequality is strict, else the sum would be larger than $|B|$. It also cannot be that $x_b \neq x_{b'}$ for two buyers, since then pairing the lesser valued buyer with the lesser valued seller the sum would be less than one.

We verify that assigning x_b to every buyer and x_l to every seller, and $x_b + x_l = 1$ is in the core.

The interpretation is that there is an single market clearing price, and when buyers and sellers are in balance, that is, supply equals demand, the price is anywhere between the value at which the sellers are prepared to sell and the value at which the buyers are prepared to buy.

260.4 *Convex games.* For a convex payoff $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$. Assign values $x_i = v(S_i \cup i) - v(S_i)$ where $S_i = \{1, \dots, i-1\}$. Show x is in the core.

We need to define $v(\emptyset) = 0$ (the given definition of v is over non-empty subsets). The theorem is trivial for a set system with just a single element 1: $x_1 = v(1)$ and that's the end of it.

Continue by induction. Suppose the theorem true for sets of $n-1$ members. We use the notation $[i]$ for the set $\{1, \dots, i\}$. Let $|N| = n$. Considering the construction, the assignment of x_i and v for the restricted system $[n-1]$ is unchanged, and applying the induction hypothesis, $v(R) \leq x(R)$ for all $R \subseteq [n-1]$. We consider $R' \ni n$ and show for these sets $v(R') \leq x(R')$ as well.

For $R' \ni n$ write $R' = R \cup \{n\}$ where $n \notin R$. Using convexity,

$$v(R \cup n) + v([n-1]) \leq v([n]) + v(R)$$

Rearranging,

$$v(R \cup n) - v(R) \leq v([n]) - v([n-1]) = x_n$$

So the increment to v upon adding element n is less than x_n . Rearranging and using the induction hypothesis,

$$v(R') = v(R \cup n) \leq x_n + v(R) \leq x_n + x(R) = x(R').$$

Q.E.D.

261.1 *Simple games.* A coalitional game with trans. payoff is simple if $v(S) \in \{0, 1\}$ for all S , and $v(N) = 1$. If $v(S) = 1$ then S is said to be winning, and a veto player is a player in all winning coalitions. Show that if there is no veto player then the core is empty; show also that if the set of veto players is non-empty then the core is the set of all feasible payoff profiles that give zero to all other participants.

Note that since $v(S) \geq 0$ for all S then $x_i \geq 0$ for all i . If there is no veto player there must be disjoint S and T which are both winning. Then $x(S), x(T) \geq 1$, so $x(N) \geq x(S \cup T) \geq 2$, which is a contradiction.

Let x be in the a feasible payoff assigning zero to any non-veto player. Let V be the set of veto players. Since $v(N) = x(N) = 1$ and $x(N) = x(V) + x(N \setminus V) = x(V) + 0$, we have $x(V) = 1$. If $v(S) = 1$ then $V \subseteq S$ by definition of V , then $x(S) = x(V) = 1$. So $x(S) \geq v(S)$ for all S .

*261.2 *Zerosum games.* A coalitional game with transferable payoff is zerosum if $v(S) + v(N \setminus S) = v(N)$ for every coalition S ; it is additive if $v(S) + v(T) = v(S \cup T)$ for all disjoint S and T . Show that a zerosum game that is not additive has an empty core.

261.3 *Pollute the lake.* A set of N factories are to decide between treating waste before release into their common water source (a lake) or on intake. Treatment before release cost b , and treatment before use costs kc where k is the number of factories that release their waste without treatment. Analyze this as a coalition game with transferable payoff. Assume $c \leq b \leq nc$.

We have to define a payoff function v . The overall cost of pollution grows quadratically in the number of polluters, since each factory must clear each other factory's waste. However if $c < b$ a single factory might find it marginally less expensive to pollute. It seems that the tension in this game is to redistribute the value of all N factories agreeing to treat waste before release so that small coalitions do not find it advantageous to break from the entirety and pollute, as well as medium coalitions don't break with the entirety since they are not receiving the full return of benefit by not polluting.

These considerations suggest,

$$v(S) = s |nc - ((n - s)c + b)| = s |sc - b|$$

where nc is the cost to a single factory in S if all factories pollute and $(n - s)c + b$ is the cost to a single factory in S if all s factories in S agree to treat but all factories in $N \setminus S$ pollute.

For the maximum coalition N , if $b < nc$ then there is value in agreeing not to pollute. If $b = nc$ the coalition is indifferent. In the ignored case, $b > nc$, it is never worth anyone's money to treat. In the other ignored case $b < c$, it is never worth anyone's money not to treat, even if other factories do not treat.

The payoff $v(N)$ is distributed to each factory so that if $sc < b$ for some coalition, that is, they would do better locally to pollute, their payoff transferred from the value of the entirety prevents them from doing so. They are bribed into staying in the entirety decision not to pollute. If $sc > b$ for some coalition, their payoff transferred from the entirety keeps them from breaking with the entirety to reap the benefits amongst themselves of their treating waste. They are fairly compensated for the value of their decision to treat. These two sides of a decision to break from the entirety motivate the absolute value in the value function.

For any coalition S it is the sign of $sc - b$ which determines the common behavior of all participants in S . We either have a coalition breaking from the entirety and deciding to pollute, all of them, or a coalition breaking from the entirety, continuing an agreement to treat, but redistributing amongst themselves the value of this agreement as it accrues to the members of this coalition only.

$ N = 3$	$ S = 3$	$ S = 2$	$ S = 1$
$b = c$	$6c$	$2c$	0
$b = 2c$	$3c$	0	(c)
$b = 3c$	0	$(2c)$	$(2c)$

As the above table shows, for $N = 3$, there is no core if $b = 3c$, since there is no payoff to distribute and a required bribe to coalitions of size 2 and 1 (numbers in parenthesis were negative before the absolute value), a unique core of (c, c, c) in the case $b = 2c$, where the

total value $3c$ just compensates each factory for the cost of not polluting as a coalition of 1, and a wide core for $b = c$, where even a coalition of 1 is indifferent between treating and polluting. For instance, $(4c, 2c, 0)$, where one factory receives nothing, is in the core.

A necessary condition for a non-empty core is $nv(\{1\}) \leq v(N)$, since we have to have enough value to keep coalitions of one from defecting, and the core is unique with equality. This solves as $b \leq c(n + 1)/2$. One can also work out the necessity of $v(S) \leq |S|v(N)/|N|$, a “critical” core where all factories receive equal payout. It is the same bound. So there is a unique core when $b = c(n + 1)/2$. For $b > c(n + 1)/2$ the core is empty; else the core has positive volume.

263.2 Let $N = \{1, 2, 3, 4\}$. Show the game N, v in which,

$$v(S) = \begin{cases} 1 & \text{if } S = N \\ 3/4 & \text{if } S \text{ is } \{1, 2\}, \{1, 3\}, \{1, 4\} \text{ or } \{2, 3, 4\} \\ 0 & \text{otherwise} \end{cases}$$

has an empty core, by using that fact that there exists a balanced collection of weights which is 0 for all coalitions other than $S = \{1, 2\}, \{1, 3\}, \{1, 4\}$ or $\{2, 3, 4\}$.

For a game to have nonempty core, all balanced collections must make the game balanced. Letting $\lambda_S = 1/3$ for $S = \{1, 2\}, \{1, 3\}, \{1, 4\}$ and $\lambda_S = 2/3$ for $S = \{2, 3, 4\}$, we have a balanced collection, but $\sum \lambda_S v(S) = 3 \cdot 1/3 \cdot 3/4 + 2/3 \cdot 3/4 = 5/4 > v(N)$.

265.2 Consider the market with transferable payoff like that of the previous example in which there are five agents $\omega_1 = \omega_2 = (2, 0)$, and $\omega_3 = \omega_4 = \omega_5 = (0, 1)$. Find the coalition form of this market and calculate the core. Suppose agents 3, 4 and 5 form a syndicate. Does the core predict that the formation of the syndicate benefits its members? Interpret the answer.

We are to define the value function v as the allocation (z_i) consistent with endowments ω_i maximizing total production $\sum f_i(z_i)$. As in the example to which this problem refers, each unit of ketchup and mayonnaise makes one unit of russian dressing, which is really valuable. Hence, $v(S) = \min(2|S \cap K|, |S \cap M|)$. Then $v(N) = 3$ and the allocation $(0, 0, 1, 1, 1)$ is in the core (by checking).

We know that $v(\{1, 3, 4\}) = 2$ and $v(\{2, 5\}) = 1$. So that means that $z_2 + z_5 \geq 1$ and $z_1 + z_3 + z_4 \geq 2$. Since the sum is 3, both inequalities are equalities. Swapping ketchup or mayonnaise players, given that we have equalities, then $z_1 = z_2$ and $z_3 = z_4 = z_5$. Solving $z_1 + 2z_3 = 2$, $z_1 + z_3 = 1$ we have $z_1 = z_2 = 0$ and $z_3 = z_4 = z_5 = 1$. So the core is unique.

By forming a syndicate, the three person game will get 3 units for the syndicate, and 0 each for the ketchup players. We no longer have to check how each mayonnaise player in the mayonnaise syndicate is doing. The syndicate can allocate the 3 units as it pleases.

I find the interpretation part difficult. Returning to the non-syndicate case, what does the payoff vector $(0, 0, 1, 1, 1)$ mean? By combining our worthless ketchup and mayonnaise to make really valuable russian dressing, we can earn 3 units (and be stuck with 1 unit of worthless ketchup). Too bad for the ketchup wealthy, but the core enforces that he cannot receive value beyond what can be matched with mayonnaise.

The game doesn't seem to describe how this match-up proceeds. Perhaps the ketchup players transfer 3 units of ketchup to the mayonnaise players and so the mayonnaise people end up with 3 units of really valuable russian dressing. Perhaps 3 dollars are injected into the system and the mayonnaise people buy the ketchup and end up with 3 units of russian dressing,

which they surrender in return for the 3 dollars injected. Perhaps the mayonnaise players sell the russian dressing, keep half a dollar and give half a dollar to their ketchup supplier.

As long as the mayonnaise players remain in control of their one unit of mayonnaise, I don't see how that makes any difference. If they surrender control of their mayonnaise to the syndicate, then the payoff to individuals is a matter of policy within the syndicate. Individual mayonnaise players may do better or worse.

267.2 Let $N, l, (\omega_i), (f_i)$ be a market with transferable payoff for which $\sum \omega_i \leq 0$. Let $X_i = \{(z_i, y_i), y_i \leq f_i(z_i)\}$ and $\{z_i^*\}$ be a solution to,

$$\max_{\{z_i\}_{i \in N}} \left\{ \sum_{i \in N} f_i(z_i) : \sum_{i \in Z} z_i \leq \sum_{i \in N} \omega_i \right\}$$

Show that the hyperplane that separates $\sum_{i \in N} X_i$ from $\{(z, y), z \leq \sum z_i^*, y \geq \sum f_i(z_i^*)\}$ defines competitive prices.

Consider the case where the f_i are strictly increasing. (Else, since they are convex they are constant after some point.) Then the two surfaces intersect in the point (z^*, y^*) where $z^* = \sum z_i^*$ and $y^* = \sum f_i(z_i^*)$. The separating hyperplane puts $\sum X_i$ strictly below the hyperplane passing through (z^*, y^*) except for this one point. The hyperplane is,

$$y = p^* \cdot z + (y^* - p^* \cdot z^*)$$

and every point $(z, y) \in \sum X_i$ has $y \leq p^* \cdot z + \alpha$, where $\alpha = y^* - p^* \cdot z^*$.

Suppose we have not maximized all $f_i(z_i) - p^*(z_i - \omega_i)$. Fix one offending player j for which $z'_j > z_j^*$ and,

$$f_j(z'_j) - p^*(z'_j - \omega_j) > f_j(z_j^*) - p^*(z_j^* - \omega_j)$$

therefore,

$$f_j(z'_j) - f_j(z_j^*) > p^*(z_j^* - z'_j)$$

and $(z^* - z_j^* + z'_j, y^* - f_j(z_j^*) + f_j(z'_j))$ is contained in $\sum X_i$. The point on the hyperplane at this z' is,

$$p^* \cdot (z^* - z_j^* + z'_j) + y^* - p^* \cdot z^* = y^* + p^*(z'_j - z_j^*) < y^* + f_j(z'_j) - f_j(z_j^*) = y'$$

contradicting that $\sum X_i$ is never above the hyperplane.