Adaptively Regularized Constrained Total Least-Squares Image Restoration

Wufan Chen, Ming Chen, and Jie Zhou

Abstract—In this paper, a novel algorithm for image restoration is proposed based on constrained total least-squares (CTLS) estimation, that is, adaptively regularized CTLS (ARCTLS). It is well known that in regularized CTLS (RCTLS) method, selecting a proper regularization parameter is very difficult. For solving this problem, we take the first-order partial derivative of the classic equation of RCTLS image restoration and do some simplification with it. Then, we deduce an approximate formula, which can be used to adaptively calculate the best regularization parameter along with the degraded image to be restored. We proved that the convergence and the stability of the solution could be well satisfied. The results of our experiments indicate that using this method can make an arbitrary initial parameter be an optimal one, which results in a good restored image of high quality.

Index Terms—Adaptively regularized, image restoration, regularized constrained total least-squares (CTLS).

I. INTRODUCTION

I mage restoration algorithms are an important field of image processing which has been widely studied and applied to many fields [1], [2]. A large number of science experiments have indicated that 85% of all information that humans can get is from vision. It is then essential to develop an effective algorithm of image restoration, by which the lost information from a degraded image can be recovered as efficiently as possible. In general, the quality of image restoration mainly depends on whether the noise resource causing an image to be degraded can be exactly known. Images can be blurred by atmospheric turbulence, relative motion between sensors and objects, longer exposures, and so on, but we may not know exactly the cause of blurring.

Fortunately, in most cases, a linear operator and an additive noise process can model the degradation process of an image as follows:

\[
g = Hf + n
\]  
(1)

where the imaging equation is written in matrix-vector form. \(g, f, n \in \mathbb{R}^N\) represent the observed degraded image, the original image, and the additive noise in the observed image, respectively. The matrix \(H \in \mathbb{R}^{N \times N}\) represents a linear distortion operator in the discrete form of point spread function (PSF), which is described in detail in Section II.

The ultimate aim of image restoration is to find the best approximation of an original image. Because of the presence of noise, resolving (1) is an ill-posed problem, which means that the solution may be nonexistent, not unique or unstable. Regularization is a general and very effective method for solving ill-posed problems [3]. Its basic idea is to use regularization parameter to trade off fidelity to the observed data and smoothness of the solution [3]–[6]. Regarding the solution of (1), there has been a lot of research done. In (1), when \(n = 0\) and \(H\) precisely models the blurring system, the total least-squares (TLS) technique is well used to solve this set of noise-contaminated equations [7], [8]. When the noise elements in both \(H\) and \(g\) are linearly related and have equal variances, the constrained TLS (CTLS) technique is most effective [9]. When both \(H\) and \(g\) are subject to the same errors, i.e., \(H = H + \Delta H\) and \(n = \Delta g\). Fan suggested that the regularized CTLS (RCTLS) scheme, which was more successful than the CTLS [10].

In recent years, Katsaggelos et al.’s research on image restoration are most comprehensive. In [5], Katsaggelos et al. proposed the idea of regularization function. One of the components of this function is the regularization parameter. It can adaptively be chosen so that the quality of image restoration is greatly improved; but because the solution of the simultaneous algebraic equations is obtained over the spatial domain, it requires a large number of flops. In [6], Katsaggelos et al., transformed the simultaneous algebraic equation from the spatial domain into the nonlinear function at each point in DFT domain, so that the implementation of RCTLS algorithm is very efficient, even for a very large image. However, the regularization parameter is selected by a trial-and-error based on visual inspection of each restored result, it is certainly not the best so that the high quality of image restoration cannot often be guaranteed.

In this paper, based on RCTLS theory, exploiting the concepts in [5] and [6], we propose a novel algorithm of image restoration. By using the circulant approximation for the PSF and the diagonalization properties of the discrete Fourier transform (DFT), the nonlinear function in the DFT domain in [6] is significantly modified. The novel algorithm of image restoration is characterized by the following.

1) Taking the first order derivative of the equation of image restoration and simplifying it, we obtain a regularization function. According to this function, the regularization parameter can be adaptively optimized step-by-step,
along with the degraded image to be restored. This is the idea of adaptively regularized CTLS (ARCTLS).

2) In the iterative process of calculating the minimum of the ARCTLS restoration function, the nonlinear function is locally linearized, i.e., the denominator in the function is replaced with the value obtained at previous iterative step and the regularization parameter is also calculated using the results at previous iterative step. This treatment changes the nonlinear function into a quadratic convex function, which reduces the calculation difficulty.

3) Because the minimization problem is changed into solving a quadratic convex function, letting the first order derivative of the convex function be zero, we can directly obtained its minimum value easier than using some classic optimization algorithms as in [6].

The rest of this paper is organized as follows. Section II describes the algorithm of ARCTLS in detail. How to solve the equation of image restoration in DFT domain is presented in Section III. To easily understand the generality of ARCTLS, the convergence and the stability of iterative solutions are analyzed in Section IV. In Section V, the results and analysis of our experiment are given. Finally, we present the conclusions and prospects of our research in Section VI.

II. WHAT ABOUT USING ADAPTIVE REGULARIZATION PARAMETER?

This section consists of two parts. The RCTLS method with constant regularization parameter is briefly introduced in Section II-A, and the new algorithm ARCTLS proposed in this paper is discussed in detail in Section II-B.

A. RCTLS Method with Constant Regularization Parameter

The \( N \times 1 \) PSF can be assumed via

\[
h = \tilde{h} + \Delta h
\]

(2)

where \( \tilde{h} \) and \( \Delta h \in \mathbb{R}^N \) are the known and the error-unknown components of the PSF. For convenience, \( \Delta h \) is referred to independent identically distributed (IID) noise, with zero mean and variance \( \sigma_h^2 \).

Provided the observed vector \( g \) is also subject to similar errors, i.e., \( g \) is blurred by IID zero-mean additive noise with variance \( \sigma_g^2 \) and furthermore, the noises in the observed data and the PSF are not correlated, (1) can be rewritten in matrix-vector form by

\[
g = Hf + \Delta g
\]

(3)

with

\[
H = \tilde{H} + \Delta H
\]

(4)

where \( \tilde{H} \) is considered as the known component of the \( N \times N \) PSF circulant matrix \( H \), while \( \Delta H \) is made up by \( \Delta h \) in (2) and is the error component of \( \tilde{H} \). Generally, the support size of the PSF is much smaller than that of the image, thereby using circulant convolution can only result in a very small increase in the size of the resulting matrix and vectors.

Specially speaking, when the blurring operator \( \tilde{H} \) is circulant, its error matrix \( \Delta H \) is also circulant, so that the elements in \( \Delta H \) are algebraically related. In [6], several instances have shown that in this case the TLS algorithm fails in solving (3) and (4). To overcome this difficulty, Katsaggelos et al. defined a new unknown “normalized noise vector” \( u \in \mathbb{R}^{2N} \) consisting of \( \Delta h \) and \( \Delta g \) defined as follows:

\[
u = \left[ \frac{\Delta h(0)}{\sigma_h}, \ldots, \frac{\Delta h(N-1)}{\sigma_h}, \frac{\Delta g(0)}{\sigma_g}, \ldots, \frac{\Delta g(N-1)}{\sigma_g} \right]^T.
\]

(5)

Here, the supporting region of \( \Delta h \) is assumed to be \( N \). But when the supporting region of \( \Delta h \) is \( M (M < N) \), the remaining should be replaced with zeros, which guarantees that the global analysis is still valid.

So (3) and (4) can be reformulated by

\[
\bar{H}f - g + \Delta Hf + \Delta g = 0
\]

\[
\frac{\bar{H}f - g + L u}{L} = 0
\]

(6)

where \( L \) is a \( N \times 2N \) matrix given by

\[
\left[ \begin{array}{cccccccc}
\sigma_h f(0) & \sigma_h f(N-1) & \ldots & \sigma_h f(1) & \sigma_g & \ldots & 0 \\
\sigma_h f(1) & \sigma_h f(0) & \ldots & \sigma_h f(2) & \sigma_g & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\sigma_h f(N-1) & \sigma_h f(N-2) & \ldots & \sigma_h f(0) & \sigma_g & \ldots & \sigma_g \\
\end{array} \right].
\]

(7)

Accordingly, the CTLS algorithm of image restoration can be rephrased as

\[
\min_{f,u} \left\{ \|u\|^2 \right\}
\]

subject to

\[
\bar{H}f - g + Lu = 0
\]

(9)

where \( \| \cdot \|^2 \) denotes the Euclidean norm, (9) is the constraint \( u \) and \( f \) satisfy.

According to mathematical theory, given the observed data \( g \) and the knowledge of the PSF in (2), the determination of the original image is an inverse problem. If the inverse transform does not exist, the problem is then referred to be singular. On the other hand, the inverse transformation of the inverse problem is existence but not unique, which implies that there may exist a set of solutions. However, this may not be acceptable for an actual physical problem. Even if the inverse transformation is existence and unique, it will be ill conditioned. This means that any small disturbance in observed data \( g \) will produce a very large perturbation in the restored image \( f \). In fact, so long as the physical problem is existence, in other words, the signals in an image are not dominated by noise, we can always manage to get a proper solution, selecting \( f \) in (3) through an efficient method.
TABLE I
RESULTS FOR SET 1 OF EXPERIMENTS

<table>
<thead>
<tr>
<th></th>
<th>RCTLS</th>
<th>LL-D.F.P</th>
<th>ARCTLS</th>
</tr>
</thead>
<tbody>
<tr>
<td>ISNR</td>
<td>1.17</td>
<td>2.50</td>
<td>4.012</td>
</tr>
</tbody>
</table>

One of the most powerful schemes is the regularization technique, i.e., the RCTLS algorithm [1], [4]. According to the regularization approach in [6], the minimization in (8) is replaced with the following minimization of

$$\min_{\hat{f},u} \left\{ |u|^2 + \lambda |Q\hat{f}|^2 \right\}$$

subject to

$$\overline{H} \hat{f} - g + Lu = 0$$

where $Q$ is the regularization operator, and $\lambda$ is the regularization parameter. The regularization operator $Q$ is capable of incorporating prior knowledge about $f$ into the restoration process [4], [11], and the selection of $Q$ is usually based on the smoothness of $f$. In addition, the regularization parameter can be selected to trade off fidelity to the observed data and smoothness of the solution.

B. Adaptively Regularized CTLS Method

The RCTLS technique is powerful, but if the selected regularization parameter $\lambda$ is unsuitable, then the quality of image restoration could not be very high. For an unsuitable $\lambda$ in the RCTLS method, most of optimization algorithms, like Davidon–Fletcher–Powell method, may only yield a local minimum but not a global one.

In this paper, the function of ARCTLS used in the iterative process is introduced. This approach enables the above difficulties in the execution of RCTLS scheme to be effectively removed.

First, image restoration can be simply reformularized in the space domain as the following equation:

$$\min_{\hat{f}} L(\hat{f})$$

where

$$L(\hat{f}) = |H\hat{f} - g|^2 + \lambda(\hat{f})|Q\hat{f}|^2.$$  \hspace{1cm} (13)

We can get

$$\frac{dL(\hat{f})}{d\hat{f}} = 2H^T(H\hat{f} - g) + \lambda(\hat{f})|Q\hat{f}|^2 = 0$$

and there exists a constraint condition

$$\lambda(\hat{f})|_{H\hat{f} = g} = 0$$

so the solution of the equation is

$$\lambda(\hat{f}) = \frac{2f^T H^T (H\hat{f} - g) + C}{|Q\hat{f}|^2}$$

because the first term of the right side can satisfy the condition, then $C = 0$. In order to enable $\lambda(\hat{f})$ to be rigid positive and assure $\hat{f}$ to be numerically stable in the iterative process, it is assumed that

$$\lambda(\hat{f}) = \alpha \frac{(H\hat{f} - g)^T (H\hat{f} - g)}{g^T g}$$

where $\alpha (\geq 1)$ is an adjustable factor of $\lambda(\hat{f})$'s step length. The function $\lambda(\hat{f})$ in (17) describes the proportional relation of the error energy to the observed vector’s energy, and $\lambda(\hat{f})$ is obviously a quadratic function of $\hat{f}$. With the progress of the iterative process, the error energy will become smaller and the regularization parameter $\lambda(\hat{f})$ will more and more approach the best. Of course, the correction factor $\alpha$ is also very important. The smaller $\alpha$ is, the slower the change of $\lambda(\hat{f})$ is, i.e., $\lambda(\hat{f})$ can be adjusted to or very closely to optimal, in this case the restored image is of high quality; otherwise, the larger $\alpha$ is, the faster the change of $\lambda(\hat{f})$ is. In this case, $\lambda(\hat{f})$ can only be adjusted to a suboptimal value and the quality of image restoration is rather too low.

We said that the function $\lambda(\hat{f})$ of ARCTLS strategy is established based on the principle of minimizing the error energy. For any image to be restored, an optimal regularization parameter $\lambda$
not only exists but also accelerates the convergence of the iterative process.

According to the regularization scheme in this paper, the high quality of image restoration is performed by the minimization of

\[
\min_{f, u} \left\{ \|u\|_2^2 + \lambda(f)\|Qf\|_2^2 \right\}
\]

subject to

\[
\bar{H}f - g + Lu = 0.
\]

Similarly, the RCTLS scheme with variable regularization parameter in (18) and (19) is simplified by transforming it into an unconstrained optimization scheme. First, (19) can be rewritten by

\[
u = -L^+(Hf - g)
\]

where \(L^+\) is the generalized inverse of \(L\) and can be denoted as

\[
L^+ = L^T (L L^T)^{-1}.
\]

Substituting (20) and (21) into (18) and noticing \(\|A\|^2 = A^T A\), it is easy to see that the minimization of (18) and (19) is
Fig. 3. Set 2 of experiments: (a) degraded image, (b) RCTLS scheme, (c) LL-DFP scheme, and (d) ARCTLS scheme.

equivalent to the minimization of a nonlinear function defined by

\[ P(f) = (\mathbf{H}f - g)^T (L^+)^T (L^+) (\mathbf{H}f - g) \]
\[ + \lambda(f) \left( f^T Q^T Q f \right). \]  

then (22) becomes

\[ P(f) = (\mathbf{H}f - g)^T (LL^T)^{-1} (\mathbf{H}f - g) \]
\[ + \lambda(f) \left( f^T Q^T Q f \right). \]  

Noting that

\[ (L^+)^T (L^+) = (LL^+)^{-1} \]  

How can we find a value of \( f \) to minimize \( P(f) \) so as to optimize the quality of restored image? The next section will discuss this in detail.
III. How to Solve the Equation of Image Restoration in DFT Domain

It is hardly possible to minimize the unconstrained equation (24) in normal ways because there are two key difficulties.

1) The nonlinear elements in $(LL^T)^{-1}$ and the quadratic function $\lambda(f)$ have destined $P(f)$ to be high order nonlinear function. The first derivative of this function is still high order nonlinear function. Normal optimization algorithm cannot assure that the numerical solution of (24) is a global optimal function.

2) The system of the simultaneous equations structured by $P(f)$ is too large to execute it on a computer. For example, if a $N \times N$ image is to be restored, the matrix in (24) would be of the size $N^2 \times N^2$.

$P(f)$ with constant regularization parameter in [6] was simplified by using the diagonalization properties of the DFT for circulant matrices. This simplification is very beneficial to directly obtain the solution at each point in DFT domain using some normal optimization algorithms, just like Davidon–Fletcher–Powell approach; thereby, the second one of above two difficulties is solved, but the first one is not. In this paper, similarly, the minimization of $P(f)$ in spatial domain is transformed into one in DFT domain. We can get the equation in DFT domain

$$\min \{ P(F_i) \}, \quad \text{for } i = 0, 1, \cdots, N - 1$$

where the $P(F_i)$ are locally linearized as

$$P(F_i^{k+1}) = \frac{|H_i F_i^{k+1} - G_i|^2}{\sigma^2_i |f_i|^2 + \sigma^2_o} + \lambda(F_k)|Q_i|^2 |F_i^{k+1}|^2$$

and

$$\lambda(F_k) = \lambda(F_0, F_1, \cdots, F_k, F_{N-1}).$$

In (26) and (27), $k$ is the $k$th step iteration, $| \cdot |$ denotes the modulus of a complex quantity, the unknown $H_i$ and the known $G_i$ are the DFT coefficients of $f$ and $g$ at point $i$, respectively. $H_i$ and $Q_i$ are the eigenvalues of the circulant matrices $H$ and $Q$, which can be easily obtained using the DFT. But (27) shows that the regularization parameter $\lambda(F)$ in the DFT domain is the function of all $F_i$, $i = 0, 1, \cdots, N - 1$, i.e., $\lambda$ is unique throughout the DFT domain in each iteration. Specially, via locally linearized treatment, (26) becomes a quadratic convex function in each iteration. Letting the first order derivative of this quadratic convex function be zero, we can directly obtain global optimal solution under the condition of local linearization.

In the execution of ARCTLS strategy, the main steps are given as follows.

Step 1) When $k = 0$, let $F_0 = G_i$, for $i = 0, 1, \cdots, N - 1$, as the initial iteration values, and $\alpha$ is given according to person’s willing.

Step 2) $\lambda(F_k)$ can be calculated in (31), or is arbitrarily given.

Step 3) From (29), to find $F_i^{k+1}, i = 0, 1, \cdots, N - 1$.

Step 4) From (31), to find $\lambda(F_k^{k+1})$ and to calculate the convergence criterion by

$$\frac{1}{N-1} \sum_{i=0}^{N-1} \left( F_i^{k+1} - F_i^k \right) \leq \varepsilon.$$ 

The very small positive numbers $\varepsilon$ is given according to person’s willing; if (32) is not satisfied, then return to Step 3; otherwise, end the iterative process.

IV. Error Estimation Based on ARCTLS Strategy

In this section, the error estimation includes the analysis of the existence and stability of approximate solution. Because of the strong nonlinear character of the term $(LL^T)^{-1}$ in (24), it is impossible to discuss straightforwardly the formula (24) in the spatial domain. Utilizing the diagonalization properties of circulant matrices in DFT domain and based on ARCTLS strategy, we can treat the intractable problems in a special way.

A. Existence of Approximate Solution

From the discussion in Sections II and III, It is clear that the minimization of (18) and (19) is equivalent to the one of (24) in the spatial domain, furthermore, (24) is also equivalent to (26) or (29) in the DFT domain.

Taking the modular operation to (29), we have

$$A_i |F_i^{k+1}|^4 + B_i |F_i^k|^2 + C_i \cdot |F_i^{k+1}|^2 = D_i$$

where $H_i^*$ is the conjugate of $H_i$. 

$$\Omega (F_k) = \frac{\sigma^2_i |f_i|^2}{\sigma^2_i |f_i|^2 + \sigma^2_o} \quad \text{in (30)}$$

In (29), the numerators are constant over $k$ because both the circulant matrix of the PSF and the observed image are known; but the denominators are variable; in (31) $E$ is the total energy of the observed image and both $E$ and $\alpha$ are constant in the iterative process. As above, the quality of image restoration actually depends on two variables: $\Omega$ in (30) and $\lambda$ in (31).

In the execution of ARCTLS strategy, the main steps are given as follows.

Step 1) When $k = 0$, let $F_0^k = G_i$, for $i = 0, 1, \cdots, N - 1$, as the initial iteration values, and $\alpha$ is given according to person’s willing.

Step 2) $\lambda(F_k)$ can be calculated in (31), or is arbitrarily given ($0 < \lambda < 1$).

Step 3) From (29), to find $F_i^{k+1}, i = 0, 1, \cdots, N - 1$.

Step 4) From (31), to find $\lambda(F_k^{k+1})$ and to calculate the convergence criterion by

$$\frac{1}{N-1} \sum_{i=0}^{N-1} \left( F_i^{k+1} - F_i^k \right) \leq \varepsilon.$$ 

The very small positive numbers $\varepsilon$ is given according to person’s willing; if (32) is not satisfied, then return to Step 3; otherwise, end the iterative process.
TABLE II
RESULTS FOR SET 2 OF EXPERIMENTS

<table>
<thead>
<tr>
<th></th>
<th>$\sigma_n^2 = 2.3 \times 10^{-6}$</th>
<th>$\sigma_n^2 = 0.1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>ISNR</td>
<td>RCTLS</td>
<td>LL-D.F.P</td>
</tr>
<tr>
<td>2.25</td>
<td>2.44</td>
<td>3.33</td>
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TABLE III
COMPUTATION TIME FOR ALL THREE SCHEMES

<table>
<thead>
<tr>
<th></th>
<th>RCTLS</th>
<th>LL-D.F.P</th>
<th>ARCTLS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Set 1</td>
<td>112.6</td>
<td>54.3</td>
<td>35.5</td>
</tr>
<tr>
<td>Set 2</td>
<td>115.3</td>
<td>56.1</td>
<td>34.3</td>
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</table>

TABLE IV
ADAPTIVE CORRECTION FOR ANY INITIAL VALUE $\lambda (F^{0})$

<table>
<thead>
<tr>
<th>Initial value $\lambda$</th>
<th>0.5</th>
<th>0.06</th>
<th>0.0001</th>
</tr>
</thead>
<tbody>
<tr>
<td>Iteration step</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>4.574x10^{-2}</td>
<td>3.03x10^{-2}</td>
<td>8.565x10^{-2}</td>
</tr>
<tr>
<td>2</td>
<td>2.903x10^{-2}</td>
<td>2.728x10^{-2}</td>
<td>2.280x10^{-2}</td>
</tr>
<tr>
<td>3</td>
<td>2.711x10^{-2}</td>
<td>2.687x10^{-2}</td>
<td>2.617x10^{-2}</td>
</tr>
<tr>
<td>4</td>
<td>2.684x10^{-2}</td>
<td>2.681x10^{-2}</td>
<td>2.678x10^{-2}</td>
</tr>
<tr>
<td>5</td>
<td>2.680x10^{-2}</td>
<td>2.680x10^{-2}</td>
<td>2.679x10^{-2}</td>
</tr>
<tr>
<td>ISNR</td>
<td>4.012</td>
<td>4.012</td>
<td>4.012</td>
</tr>
</tbody>
</table>

where

$$A_i = \lambda^2 \sigma_\theta^2 |Q_i|^4$$
$$B_i = 2\lambda \sigma_\theta^2 |H_i|^2 |Q_i|^2 + 2\lambda^2 \sigma_\theta^2 \sigma_\phi^2 |Q_i|^2$$
$$C_i = 2\lambda \sigma_\theta^2 |H_i|^2 |Q_i|^2 + \lambda^2 \sigma_\theta^2 |Q_i|^4 + |H_i|^4$$
$$D_i = |H_i^T G_i|^2$$

when $|F_i^{n+1}| \approx |F_i^{n}|$, and let $|F_i^{n}| = x_i$, then (33) can be rewritten as

$$A_i x_i^3 + B_i x_i^2 + C_i x_i + D_i = 0 \quad \text{for } i = 0, 1, \cdots, N-1.$$  \hspace{1cm} (34)

where $A_i$, $B_i$, $C_i$, and $D_i$ are positive constants in each step iteration.

According to Descartes’ rule of signs, regarding any equation with real coefficients, there is no positive solution if the signs of all the coefficients are the same; otherwise, there is only one positive one if the signs alter only once. In (34), the parameters $A_i$, $B_i$, $C_i$, and $D_i$ are all positive. Changing the position of $D_i$ from the right to the left, we know the signs of the coefficients in (34) change only once, so there exists only one positive solution. But because the coefficients of (34) are unstable, we cannot determine whether the solution is a real one according to Sturm’s criterion. Fortunately, under the physical condition, $|F_i| \geq 0$, $i = 0, 1, \cdots, N-1$, so the positive solution must be real.

For each in $N$ equations, the analysis are given below.

1. On the principle of mathematics, (34) has at least six approximation solutions, but only one is positive.
2. Only when $D_i = 0$, $x_i = 0$. Because the alteration of the signs among $A_i$, $B_i$, $C_i$ would not happen.
3. If the magnitudes of $A_i$, $B_i$, $C_i$ can be suitably matched, the best approximate solution is certainly found.

It was said previously that the approximate solution or exact solution actually exists for any actual problem of physics. To a great extent, the approximate technique for ill-posed problems usually can decide whether some approximate solution is good or not. It is the ARCTLS method that makes the variable $\lambda$ to be adjusted to realize optimal match among $A_i$, $B_i$, $C_i$ in (33), which assure the high quality of image restoration.

B. Stability of Approximate Solution

So-called “stability of approximate solution” indicates that any small disturbance of the known operator and vector will not result in the large changes of the solution, which often appears in numerical iterative process. According to the equivalent relations above mentioned, setting the derivative of $P(f)$ in (24) with respect to $f$ equal to zero, we have

$$\frac{\partial P(f)}{\partial f} = 2 \left[ \frac{\partial}{\partial f} (H - (H^T f - g) (LL^T)^{-1} \cdot \frac{1}{2} \left( \frac{\partial L}{\partial f} L^T + L \frac{\partial L^T}{\partial f} \right) (LL^T)^{-1} \cdot (H^T f - g) \right] + 2\beta (H^T f - g)^T (H^T f - g) Q f = 0 \quad (35)$$

where $\beta = \alpha/E$. Now, suppose that the degraded operator $H$ and the observed vector $g$ are both contaminated by small noise vectors $\Delta H$ and $\Delta g$, that is

$$H = H + \Delta H \quad (36)$$
$$g = g + \Delta g \quad (37)$$

then $f_0$, the undisturbed (noise-free) system of linear equations $H f = g$, has a consistent solution. In this case, it will create an error vector, that is

$$f = f_0 + \Delta f \quad (38)$$

and

$$H f_0 = g. \quad (39)$$

Substituting (36)–(39) into (35) and neglecting the higher order terms in $\Delta f$, $\Delta H$, and $\Delta g$ following [9], we obtain from (35)

$$[H^T (L_0 L_0^T)^{-1} + \beta \left( f_0^T Q g f_0 \right) H^T ] H \Delta f = \left[ H^T (L_0 L_0^T)^{-1} + \beta \left( f_0^T Q g f_0 \right) H^T \right] \cdot (H f_0 + g) \quad (40)$$
where $L_0$ is the same as one in (7). Observing carefully (40), we discover from (6)

$$
\mathbf{H} \Delta \mathbf{f} = \mathbf{H} \mathbf{f}_0 + \Delta \mathbf{g} = L_0 \mathbf{u}.
$$

Due to the structure of $L_0 \mathbf{u}$, expanding it as follows:

$$
L_0 \mathbf{u} = \begin{pmatrix}
    f(0) & f(N-1) & \cdots & f(1) \\
    f(1) & f(0) & \cdots & f(2) \\
    \vdots & \vdots & \ddots & \vdots \\
    f(N-1) & f(N-2) & \cdots & f(0)
\end{pmatrix}
\begin{pmatrix}
    \Delta h(0) \\
    \Delta h(1) \\
    \vdots \\
    \Delta h(N-1)
\end{pmatrix}
+ \begin{pmatrix}
    \Delta g(0) \\
    \Delta g(1) \\
    \vdots \\
    \Delta g(N-1)
\end{pmatrix} = F_0 \Delta \mathbf{h} + \Delta \mathbf{g}.
$$

Equation (42) can be denoted as

$$
\mathbf{H} \Delta \mathbf{f} = F_0 \Delta \mathbf{h} + \Delta \mathbf{g}
$$

where $\Delta \mathbf{h}$ is the same as one in (2), $F_0$ in (42) is the circulant matrix built by the elements of the consistent solution $f_0$. Taking Fourier transformation to (43), we obtain

$$
\mathbf{W} \mathbf{H} \mathbf{W}^H (\Delta \mathbf{f}) = \mathbf{W} F_0 \mathbf{W} \mathbf{W}^H (\Delta \mathbf{h}) + \mathbf{W} (\Delta \mathbf{g})
$$

furthermore

$$
\mathbf{H}_i \Delta \mathbf{f}_i = F_0^c \Delta \mathbf{h}_i + \Delta \mathbf{g}_i \quad i = 0, 1, \cdots, N-1
$$

where $\mathbf{W}$ is the $N \times N$ DFT matrix, $\mathbf{W}^H$ is the Hermitian of $\mathbf{W}$; $\mathbf{H}_i$ and $F_0^c$ are the eigenvalues of matrices $\mathbf{H}$ and $F_0^c$; $\Delta \mathbf{h}_i$ and $\Delta \mathbf{g}_i$ are the DFT coefficients corresponding to $\Delta \mathbf{h}_i$, $\Delta \mathbf{h}_i$, and $\Delta \mathbf{g}_i$, respectively. Taking the modular operation to (44), we can get

$$
|\mathbf{H}_i| |\Delta \mathbf{f}_i| \leq |F_0^c| |\Delta \mathbf{h}_i| + |\Delta \mathbf{g}_i| $$

or

$$
|\Delta \mathbf{f}_i| \leq \frac{1}{|\mathbf{H}_i|} \left[ |F_0^c| |\Delta \mathbf{h}_i| + |\Delta \mathbf{g}_i| \right] \quad i = 0, 1, \cdots, N-1.
$$

Obviously, (45) clearly shows that the effect of a small noise vector $\Delta \mathbf{h}_i$ and $\Delta \mathbf{g}_i$ on the error vector $\Delta \mathbf{f}_i$ is bounded. Iterative process at the consistent solution is achieved. In the other words, ARCTLS strategy proposed in this paper are stable and convergence in the iterative process. Specially, it is very significant that the stability is independent on the magnitude of the adjustable factor $\alpha$. Of course, a suitable $\alpha$ is desired to obtain the “best” approximate solution for the high quality of image restoration.

V. EXPERIMENT RESULTS

To evidence the advantage of the proposed algorithm based on the ARCTLS theory, this section provided two numerical experiment results, which were compared with the ones given by use of the RCTLS algorithm and the locally linearized Davidon–Fletcher–Powell (DFP) optimization technique, i.e., RCTLS and LL-DFP. For convenience, we took $256 \times 256$ “Lena” original image as example in Fig. 1, and made some engagements below.

The PSF, used to blur $256 \times 256$ “Lena” standard image, is still the Gaussian-shaped and expressed as

$$
h(i, j) = \exp \left\{ -\frac{i^2 + j^2}{2\sigma^2} \right\} \quad i, j = 0, 1, \cdots, N-1
$$

where $\sigma$ is constant, which assures the response system be lossless. Due to the high-pass character of Laplace operator filter, the Laplace operator is selected as the regularization operator.

No matter which restoration scheme is used, the observed data are utilized as the initial values in the iterative process. As objective measurement of the performance for different scheme, the improvement in signal-to-noise-ratio (ISNR) is employed, and defined by

$$
\text{ISNR} = 20 \log \frac{|\mathbf{f} - \mathbf{g}|^2}{||\mathbf{f} - \mathbf{\bar{f}}||^2}
$$

where $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{\bar{f}}$ are the original, degraded and restored images, respectively.

Set 1 of experiments: in this set of experiments, the Gaussian-shaped PSF used to blur the original image had variance $\sigma^2 = 6.25$; the PSF used for restoration by all three algorithms was the previous one contaminated by additive white Gaussian noise of variance $\sigma^2 = 8 \times 10^{-7}$; for both PSF’s, their region of supporting was designed as $29 \times 29$ pixels. The Gaussian noise $\Delta \mathbf{g}_i$ enforced to the observed vector, had variance $\sigma^2 = 1.0$. We utilized three different schemes, RCTLS, LL-DFP, and ARCTLS, to restore this degraded image. The corresponding ISNR values are listed in Table I. The degraded and restored images are shown in Fig. 2(a)–(d), respectively.

Set 2 of experiments: in this set of experiments, we tested our approach in the other way. In most cases, the true PSF is not exactly known indeed. Of course, it is basically efficient to assume that the PSF is the Gaussian-shaped one. The PSF used to blur the original image had $\sigma^2 = 9$ and was corrupted by additive white noise, and the PSF used by all three algorithms had $\sigma^2 = 16$ and was assumed to be perturbed by Gaussian noise of variance $\sigma^2 = 2.3 \times 10^{-6}$. For both PSF’s, their region of supporting was $31 \times 31$ pixel. The observed data were blurred with Gaussian noise of variance $\sigma^2 = 0.1$. Similarly, we adopted three different schemes, RCTLS, LL-DFP, and ARCTLS, to restore the degraded image. The corresponding ISNR values are listed in Table II, and the degraded and restored images are shown in Fig. 3(a)–(d), respectively.

In two sets of experiments, it is specially emphasized that LL-DFP scheme is meant to minimize the formula (26) and to estimate it with the Davidon–Fletcher–Powell iterative optimization approach, but the parameter $\lambda$ is constant and selected by a trial and error method. On the other hand, the computation time on an IBM PC 350-P90 for all three schemes is listed in Table III, which is benefit to appraising reasonably those approaches. According to the quality of image restoration and computation cost, by comparison, it is very clearly seen that the LL-DFP scheme is better than RCTLS scheme, but the ARCTLS scheme is the best. There are many reasons to support this conclusion.

1) Although the DFP approach is optimal iterative one, the minimization function in the DFT domain in [6] is non-
convex, which can not guarantee that the iterative solution is global optimal. In addition, the regularization parameter $\lambda$ is constant and chosen by a trial and error method based only on visual inspection of the results, so that the computation cost of the RCTLS is considerably large, and the quality of image restoration cannot obviously be improved.

2) The locally linearized formula (26) in this paper is a convex function in each step iteration; when $\lambda$ is given, the iteration solution determined by the DFP approach is global optimal and the iteration convergence is accelerated. Therefore, the better quality of image restoration and the less computation cost can be obtained. It is certain that if a suitable $\lambda$ can be given from the visual inspection, the quality will be further promoted.

3) For the proposed ARCTLS scheme, the locally linearized formula (26) has the property of convex function, so (29) enables us to directly get global optimal solution. The ARCTLS formula (30) created from the principle of energy approach can encourage the rate of convergence and adjust the reasonable match among $A_2, B_2, C_2$ in (34) so as to find the “best” approximate solution. By use of the ARCTLS scheme, the high quality of image restoration can be realized under very low computation cost.

In Section III, we said that the initial value $\lambda(F^0)$ can be calculated in (31) or is arbitrarily given. We did a set of experiments, for any given $\lambda(F^0)$’s, an optimal $\lambda(F)$ is rapidly obtained in the ARCTLS strategy shown in Table IV. This set of experiment proves again that the ARCTLS strategy is a wonderful invention indeed. From our experience, the adjustable factor of the step length $\alpha$ is selected in the interval $(10–20)$.

VI. CONCLUSIONS

Based on the RCTLS theory, we have proposed a novel effective technique of image restoration, i.e., ARCTLS scheme in this paper. Using the locally linearized method, we can directly get the derivative of RCTLS restoration function in DFT domain, which greatly decrease the complex and the computation time of RCTLS algorithm and explicitly increase the quality of restored image. Through the ARCTLS scheme, we can conveniently derive a suitable regularization parameter, which reduces the subjectivity in the process of image restoration and assures that we can always get the best quality of restored image for any initial value of regularization parameter in an interval. The theory of the existence and stability for approximate solution has been successfully proved in the special way in this paper.

In the future work, we will study how to utilize the prior information of the degraded image to determine the adjustable factor $\alpha$ well. By solving this problem, this method can be more efficient and powerful in image restoration or other field of image processing.

REFERENCES


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