

# Microalgal biomass surface productivity optimization based on a photobioreactor model

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**Abstract:** In this paper we predict and optimize the biomass surface productivity of microalgae in continuous culture under a constant light source. Surface biomass is identified as a key variable for assessing productivity: we provide both a mathematical and intuitive explanation. For reaching maximal productivities, biomass surface concentration must be such that growth at the culture bottom (assuming a planar geometry orthogonal to the light source) must be equal to respiration. Therefore the optimal biomass concentration depends both on the incident light and culture's depth. We then show how the chlorophyll/carbon ratio must also be carefully controlled to optimize light use in the photobioreactor. Finally, numerical results illustrate our theoretical approach.

*Keywords:* chemostat; microalgae; photobioreactor; model; light attenuation; productivity; biomass; optimization; continuous culture; bioenergy

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## 1. INTRODUCTION

Microalgae cultures have recently received high attention in the frameworks of food supplements, pharmaceutical production (Spolaore et al., 2006), as well as CO<sub>2</sub> fixation and biofuel production (Huntley and Redalje, 2007; Chisti, 2007). Their high actual photosynthetic yield compared to terrestrial plants (whose growth is limited by CO<sub>2</sub> availability) leads to large potential algal biomass productions of several tens of tons per hectare and per year.

The objective of this paper is to give new insights in the optimization of microalgal biomass production. The key aspect when dealing with photobioreactor is the optimization of light use. The pigment concentration (mainly chlorophyll) affects the light distribution and use within the reactor, so that for too high biomass, light in the photobioreactor is strongly attenuated and growth is low. Moreover, in case of nitrogen depreciation (necessary to produce biofuel (Chisti, 2007)) the pigments are affected and decrease, which modifies light distribution.

In order to find a solution to this non intuitive optimization problem, analysis based on a modelling approach is required. In this paper, we base our approach on a simplified macroscopic photobioreactor model, dealing not only with light diffusion, but also with nitrogen use and nitrogen/carbon ratio variation. In order to perform this mathematical analysis we must use models that are simple so that they are mathematically tractable; we use a simplified version of the model proposed in (Bernard et al., 2009, Sub.). Our results are obtained at steady state, and we prove that these equilibria are achievable. From this model, it is shown mathematically that biomass surface

concentration is the key parameter for surface productivity. Then global optimization is computed, and finally numerical results are presented.

The paper is structured as follows: first, we present the model dealing with both nitrogen limitation and light attenuation; then biomass concentration optimization is presented, followed by chlorophyll/carbon ratio optimization. A numerical result illustrates the approach. Finally, some ideas are given on how to choose the dilution rate in a continuous culture in order to maximize productivity.

## 2. A NEW DROOP PHOTOBIOREACTOR MODEL

### 2.1 Droop model presentation

We here focus on the growth of microalgae limited by inorganic nitrogen availability (provided by nitrate or ammonium), since such conditions are favourable for lipid production (Chisti, 2007) or other compounds such as astaxanthin (Aflalo et al., 2007). The Droop model (Droop, 1968, 1983) has been broadly used to take into account the ability of microalgae to uncouple substrate absorption and biomass ( $x$ ) growth (Droop, 1968, 1983). The growth rate is assumed to depend on the internal quota ( $q$ ) which, here, is the amount of intracellular nitrogen per biomass unit. This variable yield model writes thus:

$$\begin{cases} \dot{s} = Ds_{in} - \rho(s)x - Ds \\ \dot{q} = \rho(s) - \mu(q)q \\ \dot{x} = \mu(q)x - Dx \end{cases} \quad (1)$$

In this model the growth rate functions  $\mu$  and absorption functions  $\rho$  are generally taken as Michaelis-Menten and

Droop functions:

$$\begin{aligned}\rho(s) &= \rho_m \frac{s}{s + K_s} \\ \mu(q) &= \bar{\mu} \left(1 - \frac{Q_0}{q}\right)\end{aligned}\quad (2)$$

where  $\rho_m$  is the maximum uptake rate,  $K_s$  is the half saturation constant for substrate uptake,  $\bar{\mu}$  is the hypothetical maximum growth rate (at infinite internal nitrogen quota) and  $Q_0$  the minimal cell quota (below this level, no algal growth can take place).

It can be proved (Bernard and Gouzé, 1995) that the internal quota will stay between two bounds:

$$Q_0 \leq q \leq Q_m \quad (3)$$

Where  $Q_m = Q_0 + \frac{\rho_m}{\bar{\mu}}$  represents the maximum cell quota obtained in conditions of non limiting nutrients.

The Droop model has been widely studied Bernard and Gouzé (1995, 2002) and validated (Sciandra and Ramani, 1994; Vatcheva et al., 2006). However, it cannot directly be used in the case of high density photobioreactors for two main reasons:

- In its rough form it does not include the effect of light intensity
- It does not account for the decrease of light due to the cell density

## 2.2 Adding light effect on growth

We first consider the case where light is homogeneous in the reactor, with an intensity  $I$ . In the next section we will consider the case with light attenuation in the reactor.

Light intensity has a direct effect on growth (photosynthesis), while nitrogen uptake can continue in the dark. Light can then be introduced into parameter  $\bar{\mu} = \bar{\mu}(I)$  (Bernard et al., 2009, Sub.):

$$\mu(q, I) = \bar{\mu}(I) \left(1 - \frac{Q_0}{q}\right) = \bar{\mu} \frac{I}{I + K_I} \left(1 - \frac{Q_0}{q}\right) \quad (4)$$

where  $K_I$  is the light half saturation coefficient.

In order to prevent unrealistic quota increase in the dark, we use the down regulation mechanism proposed by Lehman et al. (1975):

$$\rho(s, q) = \rho_m \frac{s}{s + K_s} \frac{Q_l - q}{Q_l - Q_0} \quad (5)$$

with  $Q_l > Q_0$ : the uptake rate stops for replete cells  $q = Q_l$ .

We also add a constant respiration term  $r$  that applies to both nitrogen and carbon, so that the nitrogen/carbon ratio  $q$  is not affected:

$$\dot{x} = (\mu(q, I) - r - D)x$$

One can verify that the  $q$  equation does not change by adding this respiration term.

Finally, the model including light effect reads:

$$\begin{cases} \dot{s} = Ds_{in} - \rho_m \frac{s}{s + K_s} \frac{Q_l - q}{Q_l - Q_0} x - Ds \\ \dot{q} = \rho_m \frac{s}{s + K_s} \frac{Q_l - q}{Q_l - Q_0} - \bar{\mu} \frac{I}{I + K_I} (q - Q_0) \\ \dot{x} = \bar{\mu} \frac{I}{I + K_I} \left(1 - \frac{Q_0}{q}\right) x - rx - Dx \end{cases} \quad (6)$$

In the next section, we will take into account the exponential light distribution within the reactor.

## 2.3 Light attenuation in the photobioreactor

Light is strongly attenuated by the biomass and its pigments in the photobioreactor. In this section we represent light distribution and its consequence on the growth rate. The biological and chemical concentrations are still assumed to be homogeneous within the photobioreactor.

We consider a planar geometry with perpendicular light source. For the sake of explanation simplicity we will consider a horizontal reactor and speak about "upper part" of the reactor on the side of incident light, and "reactor bottom" for the other side. However, our work is also valid for a vertical or diagonal geometry with perpendicular light source. The depth of the culture will be  $W$ .

At equilibrium, it has been shown (Bernard et al., 2009) that the chlorophyll concentration ( $Chl$ ) is proportional to the nitrogen content:

$$Chl = \gamma(I^*)qx$$

$\gamma(I^*)$  is a chlorophyll to nitrogen ratio, depending on the light  $I^*$  at which the microalgae have been photoadapted. The photoadaptation mechanism is presented in Bernard et al. (Sub.), but it induces a significant complexity and will be neglected in this study. We thus consider that  $\gamma$  is a constant value which does not depend on light.

To model light attenuation we use a Beer-Lambert law, where the attenuation depends on the chlorophyll content:

$$I(q, xz) = I_0 e^{-a_i \gamma q x z} \quad (7)$$

where  $I_0$  is the incident light,  $a_i$  is a light attenuation coefficient, and  $z$  is the depth, so that  $\gamma q x z$  represents the chlorophyll per surface unit present above depth  $z$ .

From (7) we can compute the average irradiance received by the cell culture:

$$\begin{aligned}\bar{I} &= \frac{I_0}{W} \int_0^W e^{-a_i \gamma q x z} dz \\ &= \frac{I_0}{a_i \gamma q x W} (1 - e^{-a_i \gamma q x W})\end{aligned}\quad (8)$$

which decreases with  $\gamma q x$ : this confirms the intuition that higher biomass or chlorophyll content leads to lower mean light in the reactor, due to stronger light attenuation.

In the reactor, growth rates vary with depth: in the upper part of the reactor, higher light causes higher growth than in the bottom part. The growth rate for a given depth  $z$  can be written:

$$\mu^z(q, I(q, xz)) = \bar{\mu} \frac{I(q, xz)}{I(q, xz) + K_I} \left(1 - \frac{Q_0}{q}\right) \quad (9)$$

Then, we compute the mean growth rate in the reactor:

$$\mu(q, I_0, x) = \frac{1}{W} \int_0^W \mu^z(q, I(q, xz)) dz$$

by the change of variable  $\chi = xz$  it can be rewritten

$$\mu(q, I_0, x) = \frac{1}{xW} \int_0^{xW} \mu^z(q, I(q, \chi)) d\chi \quad (10)$$

so that the mean growth rate depends on the surface biomass  $xW$ . It represents the total biomass present per

surface unit, and will be denoted  $X = xW$ . Finally, from (10), we compute the growth rate (now denoted  $\mu(q, I_0, X)$  after a slight abuse of notation):

$$\mu(q, I_0, X) = \frac{\bar{\mu}(1-Q_0/q)}{a_i \gamma q X} \ln \left( \frac{I_0 + K_I}{I_0 e^{-a_i \gamma q X} + K_I} \right) \quad (11)$$

which depends both on the surface biomass  $X$  and the nitrogen/carbon quota  $q$ , which is proportional to the chlorophyll/carbon quota at equilibrium. By construction  $\mu$  is decreasing with  $X$ : the more surface biomass, the more light attenuation, so that mean growth rate in the reactor is lower. It is interesting to note that  $\mu$  is increasing with  $q$  for low  $q$  values, and that it is decreasing with  $q$  for high  $q$  values: high chlorophyll concentrations lead to high light attenuation, so that mean growth in the reactor decreases.

It is also interesting to note that for a given  $q$  value, there is a maximal attainable biomass in the reactor, for which  $\mu(q, I_0, X) = r$ : if such a biomass is attained it cannot increase anymore due to respiration.

The simplified Droop Photobioreactor Model (DPM) is

$$\begin{cases} \dot{s} = Ds_{in} - \rho_m \frac{s}{s+K_s} \frac{Q_I - q}{Q_I - Q_0} x - Ds \\ \dot{q} = \rho_m \frac{s}{s+K_s} \frac{Q_I - q}{Q_I - Q_0} - \mu(q, I_0, X)q \\ \dot{x} = (\mu(q, I_0, X) - r - D)x \end{cases} \quad (12)$$

This model is a simplified version of the model presented by Bernard et al. (2009) when photoadaptation is neglected. It will be used in the rest of the paper.

### 3. OPTIMAL CONDITIONS FOR MAXIMIZING PRODUCTIVITY

#### 3.1 Choosing surface biomass

With model (12), our aim is to compute and optimize biomass surface productivity (units:  $mg[C]/m^2/day$ ):

$$P(q, I_0, X) = (\mu(q, I_0, X) - r)X \quad (13)$$

At equilibrium it is the product between dilution rate ( $D = \mu(I_0, q, X) - r$ ) and surface biomass.

*Important Remark:* productivity is a function of the nitrogen/carbon quota  $q$  and surface biomass  $X = xW$ : according to this model a thin culture ( $W$  small) with high biomass concentration  $x$  is equivalent to a deep culture ( $W$  high) with low biomass concentration  $x$ , if they have the same surface biomass.

The reactor can be seen as a solar panel with an energy yield, and losses. The panel's parameter is  $X$ :

- A low  $X$  (low biomass and thin culture) indicates that most light is not absorbed by the culture: the panel has a low energy yield.
- A high  $X$  (high biomass and deep culture) indicates that in the culture's bottom there is very little light: most light is absorbed, but in the reactor's bottom there are only respiration losses.

Thus, we must choose the best  $X$  value to maximize the panel's efficiency  $P$ .

*Theorem 1.* For given  $I_0$  and  $q$ , the optimal  $X$  surface biomass for maximizing productivity (13) is such that growth rate at depth  $W$  is equal to the respiration rate:

$$\mu^W(q, I(q, X_{opt})) = r \quad (14)$$

This optimal surface biomass concentration can thus be computed:

$$X_{opt}(q) = \frac{1}{a_i \gamma q} \ln \left( \frac{I_0}{K_I} \left( \frac{\tilde{\mu}(q)}{r} - 1 \right) \right) \quad (15)$$

*Proof:* For a given biomass surface concentration  $X$ , productivity can be written from (13) and (10)

$$\begin{aligned} P(q, I_0, X) &= \int_0^X (\mu^z(q, I(q, \chi)) - r) d\chi \\ &= \int_0^{X_{opt}} (\mu^z - r) d\chi + \int_{X_{opt}}^X (\mu^z - r) d\chi \\ &= P(q, I_0, X_{opt}) + \int_{X_{opt}}^X (\mu^z - r) d\chi \end{aligned}$$

where the first term is the productivity  $P(q, I_0, X_{opt})$  with  $X_{opt}$  chosen according to (14), and the second term is always negative because  $\mu^z$  decreases with  $X$ :

- If  $X$  is lower than  $X_{opt}$ , then this term would "remove" microalgae that grow more than they respire:  $\mu^z(q, I(q, \chi)) > r, \forall \chi < X_{opt}$ .
- If  $X$  is higher than  $X_{opt}$ , then this term would "add" microalgae that respired more than they grow:  $\mu^z(q, I(q, \chi)) < r, \forall \chi > X_{opt}$ .

so that  $X_{opt}$  maximizes surface productivity.

It is then possible to compute this optimal surface biomass from (14), (9) and (7)

$$\tilde{\mu}(q) \frac{I_0 e^{-a_i \gamma q X_{opt}}}{I_0 e^{-a_i \gamma q X_{opt}} + K_I} = r$$

(with notation  $\tilde{\mu}(q) = \bar{\mu}(1 - Q_0/q)$ ) we obtain

$$I_0 e^{-a_i \gamma q X_{opt}} = K_I \frac{r}{\tilde{\mu}(q) - r} \quad (16)$$

which gives the light intensity at depth  $W$ , and then leads to (15).

Since  $X_{opt}(q)$  needs to be positive, it only exists for  $q > \frac{I_0 \bar{\mu}}{I_0 \bar{\mu} - r(I_0 + K_I)} Q_0$ , that is the minimal quota that ensures that growth can compensate respiration at the reactor's surface.  $\square$

#### 3.2 Optimization with both surface biomass and nitrogen quota

We are then left with the choice of an optimal nitrogen/carbon value  $q$ , which can be controlled by adjusting  $D$  and  $s_{in}$  (see section 4).

- A low quota leads to low potential growth rate  $\tilde{\mu}(q)$
- A high quota leads to high  $\tilde{\mu}(q)$  potential growth rate, but also to higher light attenuation, so that the mean growth rate  $\mu(I_0, q, X)$  in the reactor can be lower.

Thus, an optimal intermediate value must be found.

*Theorem 2.* There exists only one value  $q_{opt}$  maximizing  $P$ , provided that the following condition is satisfied:

$$I_0 > K_I \left[ \frac{\bar{\mu}}{\bar{\mu} - 2r} e^{\frac{r^2}{\bar{\mu} - 2r}} - 1 \right] \quad (17)$$

*Important Remark:* (17) is, in practice, always true since  $I_0 \gg K_I$  and  $\bar{\mu} \gg r$ .

*Proof:* First, we compute the growth rate at optimal surface biomass concentration (15), from (11) and (16)

$$\mu(q, I_0, X_{opt}) = \tilde{\mu}(q) \frac{\ln \left( \left(1 + \frac{I_0}{K_I}\right) \left(1 - \frac{r}{\tilde{\mu}(q)}\right) \right)}{a_i \gamma q X_{opt}(q)} \quad (18)$$

Then we compute the corresponding productivity

$$\begin{aligned} P(q, I_0, X_{opt}) &= (\mu(q, I_0, X_{opt}(q)) - r) X_{opt}(q) \\ &= \frac{\tilde{\mu}(q)}{a_i \gamma q} \ln \left( \left(1 + \frac{I_0}{K_I}\right) \left(1 - \frac{r}{\tilde{\mu}(q)}\right) \right) \\ &\quad - \frac{r}{a_i \gamma q} \ln \left( \frac{I_0}{K_I} \left( \frac{\tilde{\mu}(q)}{r} - 1 \right) \right) \end{aligned}$$

Its derivative with respect to  $q$  is then computed, to find the optimal  $q$  value:

$$\frac{\partial P(q, I_0, X_{opt})}{\partial q} = \frac{1}{a_i \gamma q^2} f(q)$$

with

$$\begin{aligned} f(q) &= \tilde{\mu} \left( \frac{2Q_0}{q} - 1 \right) \ln \left( \left(1 + \frac{I_0}{K_I}\right) \left(1 - \frac{r}{\tilde{\mu}(q)}\right) \right) \\ &\quad + r \ln \left( \frac{I_0}{K_I} \left( \frac{\tilde{\mu}(q)}{r} - 1 \right) \right) \end{aligned}$$

which can only be equal to zero for values of  $q$  higher than  $2Q_0$ , because both logarithms are positive by construction: the first one comes from the growth rate, and the second one from the optimal biomass surface concentration.

We can show that an extremum of  $P$  corresponds to a maximal productivity, and that it is unique, by demonstrating that when  $\frac{\partial P(q, I_0, X_{opt})}{\partial q} = 0$  is achieved (equivalent to  $f(q) = 0$ ), the second derivative  $\frac{\partial^2 P(q, I_0, X_{opt})}{\partial q^2}$  is negative.

$$\frac{\partial^2 P(q, I_0, X_{opt})}{\partial q^2} = -\frac{r}{a_i \gamma q^3} f(q) + \frac{r}{a_i \gamma q^2} \frac{\partial f}{\partial q}(q)$$

When  $f(q) = 0$ , we know that  $q > 2Q_0$ , and that the sign of this second derivative is the sign of  $\frac{\partial f}{\partial q}(q)$

$$\begin{aligned} \frac{\partial f}{\partial q}(q) &= \frac{\tilde{\mu}}{r} \frac{Q_0}{q^2} \left[ \frac{r^2}{\tilde{\mu}(q) - r} - 2 \ln \left( \left(1 + \frac{I_0}{K_I}\right) \left(1 - \frac{r}{\tilde{\mu}(q)}\right) \right) \right. \\ &\quad \left. - \left(1 - \frac{2Q_0}{q}\right) \frac{r}{\tilde{\mu}(q) - r} \frac{\tilde{\mu}}{\tilde{\mu}(q)} \right] \end{aligned}$$

which should be negative when  $f(q) = 0$ , because the third term will be negative ( $q > 2Q_0$  and  $\tilde{\mu}(q) > r$ ), and the first term is small compared to the second one. Let us clarify precise conditions for  $\frac{\partial f}{\partial q}(q) < 0$  to hold:

$$\frac{r^2}{\tilde{\mu}(q) - r} < 2 \ln \left( \left(1 + \frac{I_0}{K_I}\right) \left(1 - \frac{r}{\tilde{\mu}(q)}\right) \right) \quad (19)$$

Starting from (19) and with

$$\mu(q) > \mu(2Q_0) = \frac{\tilde{\mu}}{2}$$

we have that

$$\frac{r^2}{\tilde{\mu}(q) - r} < \frac{r^2}{\tilde{\mu}/2 - r}$$

and also

$$1 - \frac{r}{\tilde{\mu}(q)} > 1 - \frac{2r}{\tilde{\mu}}$$

so that (20) implies (19)

$$\frac{r^2}{\tilde{\mu}/2 - r} < 2 \ln \left( \left(1 + \frac{I_0}{K_I}\right) \left(1 - \frac{2r}{\tilde{\mu}}\right) \right) \quad (20)$$

From this inequality, we can compute the condition (17) which shows that a "high enough"  $I_0$  incident light ensures this unicity property.  $\square$

## 4. OPTIMAL CONTROL FOR A PHOTOBIOREACTOR

### 4.1 Choosing $D$ and $s_{in}$ to maximize productivity

Having identified optimal  $X_{opt}$  and  $q_{opt}$ , we now have to verify that an equilibrium with  $X = X_{opt}$  and  $q = q_{opt}$  can be achieved through an appropriated choice of  $D^*$  and  $s_{in}^*$ . The  $\dot{x} = 0$  equation imposes the choice of the dilution rate:

$$D^* = \mu_{opt} - r$$

with

$$\mu_{opt} = \mu(q_{opt}, I_0, X_{opt}(q_{opt}))$$

$D^*$  is positive since  $\mu > \mu^z(q_{opt}, I(q_{opt}, X_{opt})) = r$  (see Theorem 1).

The equilibrium substrate concentration can then be computed from  $\dot{q} = 0$ :

$$\rho_m \frac{s}{s + K_s} \frac{Q_l - q_{opt}}{Q_l - Q_0} = \mu_{opt} q_{opt}$$

so that, at equilibrium:

$$s_{opt} = K_s \frac{\mu_{opt} (1 - Q_0/Q_l) q_{opt}}{\rho_m - (\rho_m/Q_l + \mu_{opt} (1 - Q_0/Q_l)) q_{opt}}$$

which must be positive: this point will be developed later.

Finally we obtain  $s_{in}$  from the Droop nitrogen mass balance equality (Bastin and Dochain, 1990)

$$s_{in} = s_{opt} + q_{opt} X_{opt} / W$$

which is the optimal input substrate value to maximize surface productivity.

### 4.2 Optimal equilibrium attainability

In some cases (depending on incident light and microalgal parameters) the computed  $s_{opt}$  is negative. This is caused by

$$\mu_{opt} q_{opt} > \rho_m \frac{Q_l - q_{opt}}{Q_l - Q_0} \quad (21)$$

so that the optimal  $q_{opt}$  quota cannot be attained under biomass surface concentration  $X_{opt}$  (see the  $\dot{q}$  dynamics). In such cases, we must reformulate our optimization problem:

$$\begin{aligned} \max_{(q, X)} \quad & (\mu(q, I_0, X) - r) X \\ \text{such that} \quad & \mu(q, I_0, X) q \leq \rho_m \frac{Q_l - q}{Q_l - Q_0} \end{aligned} \quad (22)$$

*Theorem 3.* The solution of the optimization problem (22) is the one provided by Theorems 1 and 2 ( $q_{opt}, X_{opt}$ ) or lies on the following constraint:

$$\mu(q, I_0, X) q = \rho_m \frac{Q_l - q}{Q_l - Q_0} \quad (23)$$

*Proof:* • If  $(q_{opt}, X_{opt})$  is such that the constraint is valid, we have already demonstrated that this couple is optimal (see Figure 2 for a numerical example).

• If  $(q_{opt}, X_{opt})$  does not verify the constraint (example on Figure 3), the closer from  $X_{opt}(q)$  is  $X$ , the higher is  $P$  (see Theorem 1's demonstration). The optimal solution lies thus as close as possible to  $X = X_{opt}(q)$ , *i.e.* either on  $X = X_{opt}(q)$  or on the constraint (23).

Let us denote  $C$ -curve the part of the  $X = X_{opt}(q)$  curve that verifies the constraint.

Because  $\mu(q, I_0, X)q$  is increasing with  $q$  and  $\rho_m \frac{Q_l - q}{Q_l - Q_0}$  is decreasing with  $q$ , we know that any  $q$  value on the  $C$ -curve verifying the constraint is lower than  $q_{opt}$  (which is too high to verify the constraint). We also know from Theorem 2 that on the  $C$ -curve, the derivative of  $P$  with respect to  $q$  is positive for any  $q < q_{opt}$ . Because of that, on the  $C$ -curve, the maximal  $P$  is attained for the highest possible  $q$ : it is the  $q$  value lying both on the  $C$ -curve and constraint (23). This demonstrates that the optimal productivity will lie on the constraint.

In this last case  $s_{in}$  should be chosen infinite to have  $s$  infinite and  $\rho(s) = \rho_m$ , so that we can be on the constraint. The identification of the optimal  $(X, q)$  couple is then not straightforward, and this case will be the topic of further analysis.  $\square$

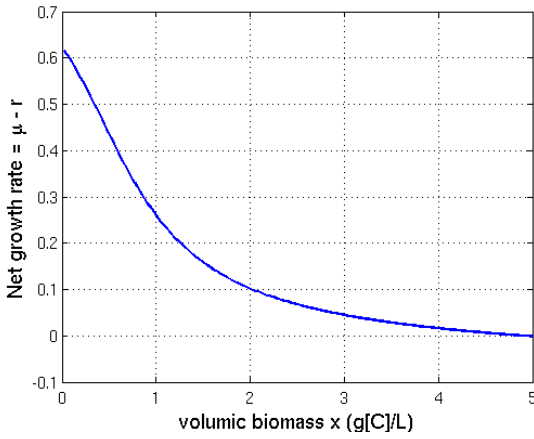


Fig. 1. Mean net growth rate ( $\mu - r$ ) in the reactor computed from (11), with  $q = 0,1g[N]/g[C]$  and  $I_0 = 100\mu$  mol quanta.  $m^{-2}s^{-1}$ . It is affected by light attenuation, caused by high biomass concentration.

## 5. NUMERICAL RESULTS

In the previous sections we have determined that there is a unique optimal  $(X_{opt}(q_{opt}), q_{opt})$  couple that maximizes the surface productivity  $P$ . In this section we illustrate this result with parameters for the microalgae *IsochrYSIS galbana*.

### 5.1 Microalgae parameters

The parameter values are taken from Bernard et al. (Sub.) to show productivity results predicted by the model for *I. galbana*. However, parameters  $K_I$  and  $\gamma$  (which may vary through photoacclimation) are computed from average values of the photoadaptation model in Bernard et al. (Sub.).

For all the simulations we use culture's depth  $W = 0,1m$ .

The growth rate predicted for such a strain is computed and plotted in Figure 1, for  $q = 0,1g[N]/g[C]$  and  $I_0 = 100\mu$  mol quanta.  $m^{-2}s^{-1}$ . We see on this figure that because of respiration, there is a maximal biomass  $x = 4.93g[C]/L$  for which respiration is equal to growth ( $\mu(X) - r = 0$ ).

Table 1. Parameter values of the lipid model for *I.galbana* culture.

Parameter	Value	Unit
$Q_0$	0.05	$g[N].g[C]^{-1}$
$Q_l$	0.25	$g[N].g[C]^{-1}$
$\bar{\mu}$	1.7	$d^{-1}$
$\rho_m$	0.073	$g[N].g[C]^{-1}.d^{-1}$
$K_s$	0.0012	$g[N]/m^3$
$r$	0.07	$day^{-1}$
$K_I$	20	$\mu$ mol quanta. $m^{-2}s^{-1}$
$a_i$	16.2	$m^2/g[Chl]$
$\gamma$	0.25	$g[Chl]/g[N]$

Condition (17) for the unicity of a maximum productivity is verified for  $I_0 > 3,5\mu$  mol quanta.  $m^{-2}s^{-1}$  which is small compared to incident light intensities ranging usually between  $100\mu$  mol quanta.  $m^{-2}s^{-1}$  to  $3000\mu$  mol quanta.  $m^{-2}s^{-1}$ .

### 5.2 Productivity prediction

Productivity was computed with these parameters and with  $I_0 = 100\mu$  mol quanta.  $m^{-2}s^{-1}$  (Figure 2) and  $2000\mu$  mol quanta.  $m^{-2}s^{-1}$  (Figure 3).

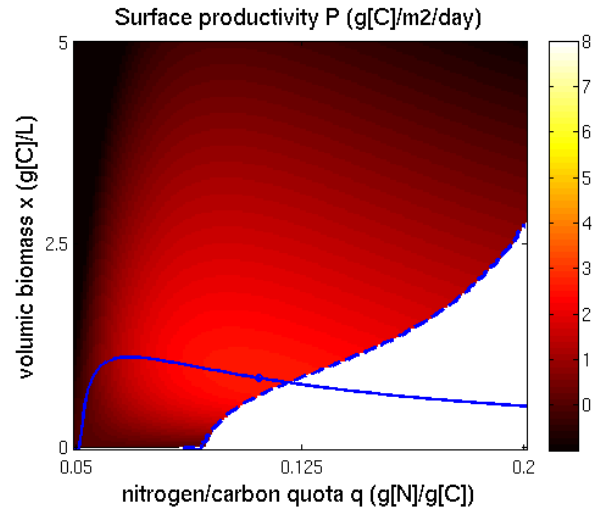


Fig. 2. Surface biomass productivity for  $I_0 = 100\mu$  mol quanta.  $m^{-2}s^{-1}$ . The  $X = X_{opt}(q)$  curve is represented by a solid line, and the optimal  $(q_{opt}, X_{opt})$  couple is represented by a circle. Dotted line indicates constraint (23).

On Figure 2 we see that there is an optimal  $(q_{opt}, X_{opt})$  couple which maximizes productivity. Note that, contrarily to the usually assumed hypothesis that higher nitrogen/carbon ratio leads to higher productivities, we see here that the optimal conditions do not correspond to nitrogen replete microalgae: for  $X = X_{opt}$ ,  $q > q_{opt}$  lead to suboptimal productivities. Thus, depending on the species and culture conditions, this result suggests that it could be advantageous to have slightly nitrogen limited microalgae, so that light attenuation by chlorophyll is weaker and light is used more efficiently in the photobioreactor.

In Figure 3 we see that the optimal  $(q_{opt}, X_{opt})$  couple (circle) depends on the incident irradiance (it is not the same as in the previous figure), and that higher light intensities lead to higher productivities.

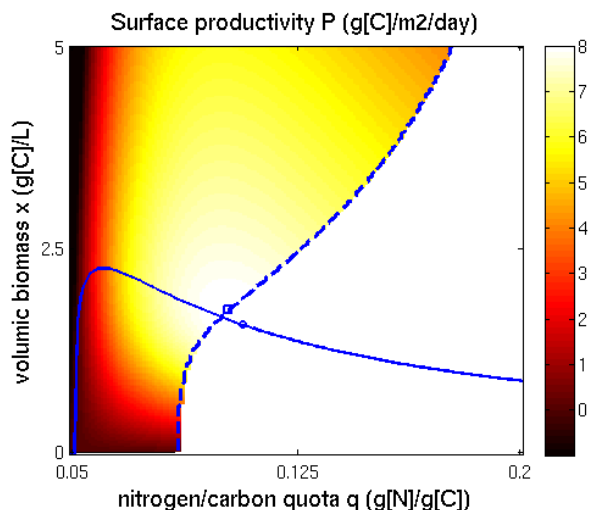


Fig. 3. Surface biomass productivity for  $I_0 = 2000\mu \text{ mol quanta. } m^{-2}s^{-1}$ . The  $X = X_{opt}(q)$  curve is represented by a solid line, and the optimal  $(q_{opt}, X_{opt})$  couple is represented by a circle. This optimal couple does not respect the constraint (dotted line) of the optimization problem (22). The real optimum (square) thus lies on the constraint.

Contrarily to the previous example, here the constraint of optimization problem (22) is not verified, so that the theoretical optimum cannot be attained: the real optimum (determined numerically) lies on the constraint, as predicted by Theorem 3.

## 6. CONCLUSION

In this paper we determined the optimal conditions for maximizing the biomass surface productivity, and redefined the optimization problem for the case where the optimal equilibrium is not attainable. The identified optimal conditions correspond to both optimal biomass surface concentration  $X_{opt}$  so that light is used optimally (section 3.1), and optimal nitrogen/carbon ratio. This work clearly follows the work of Cornet and Dussap (2009); Takache et al. (2009) who computed optimal biomass concentration from a light diffusion model (neglecting also photoacclimation) and experimentally validated the obtained productivities.

In order to get analytical results we have kept a trade-off between model simplicity to handle mathematical analysis, and model complexity to capture the main phenomena driving a photobioreactor's productivity: nitrogen absorption, growth depending both on the nitrogen/carbon quota and on the light attenuation induced by chlorophyll.

A key result shown by this approach is that slight nitrogen depreciation can enhance photobioreactor productivity. This is of particular importance since nitrogen limitation is known to stimulate lipids production. The model analysis also predicts that surface biomass  $X = xW$  drives productivity, so that a thin reactor (low  $W$ ) with dense biomass  $x$  or a deep reactor (high  $W$ ) with low biomass concentration  $x$  should have the same productivity, if their surface biomass is the same. These results must now be verified with dedicated experiments.

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## REFERENCES

- Aflalo, C., Meshulam, Y., Zarka, A., and Boussiba, S. (2007). On the relative efficiency of two- vs. one-stage production of astaxanthin by the green alga *haematococcus pluvialis*. *Biotech. and Bioeng.*, 98(300-305).
- Bastin, G. and Dochain, D. (1990). *On-line estimation and adaptive control of bioreactors*. Elsevier, Amsterdam.
- Bernard, O. and Gouzé, J.L. (1995). Transient Behavior of Biological Loop Models, with Application to the Droop Model. *Mathematical Biosciences*, 127(1), 19–43.
- Bernard, O. and Gouzé, J.L. (2002). Global qualitative behavior of a class of nonlinear biological systems: application to the qualitative validation of phytoplankton growth models. *Artif. Intel.*, 136, 29–59.
- Bernard, O., Masci, P., Mairet, F., and Sciandra, A. (Sub.). A photobioreactor model in nitrogen limited conditions. *submitted*.
- Bernard, O., Masci, P., and Sciandra, A. (2009). A photobioreactor model in nitrogen limited conditions. In *Proceedings of the Mathmod 09 conference*. Vienna, Austria.
- Chisti, Y. (2007). Biodiesel from microalgae. *Biotechnology Advances*, 25, 294–306.
- Cornet, J.F. and Dussap, C.G. (2009). A simple and reliable formula for assessment of maximum volumetric productivities in photobioreactors. *Biotechnol Prog*, 25(2), 424–435.
- Droop, M. (1968). Vitamin B12 and marine ecology. IV. the kinetics of uptake growth and inhibition in *Monochrysis lutheri*. *J. Mar. Biol. Assoc.*, 48(3), 689–733.
- Droop, M. (1983). 25 years of algal growth kinetics, a personal view. *Botanica marina*, 16, 99–112.
- Huntley, M. and Redalje, D. (2007). Co2 mitigation and renewable oil from photosynthetic microbes: A new appraisal. *Mitigation et Adaptation Strategies for Global Change*, 12, 573 – 608.
- Lehman, J.T., Botkin, D.B., and Likens, G. (1975). The assumptions and rationales of a computer model of phytoplankton population dynamics. *Limn. & Oceanogr.*, 20, 343–364.
- Sciandra, A. and Ramani, P. (1994). The limitations of continuous cultures with low rates of medium renewal per cell. *J. Exp. Mar. Biol. Ecol.*, 178, 1–15.
- Spolaore, P., Joannis-Cassan, C., Duran, E., and Isambert, A. (2006). Commercial applications of microalgae. *J. of Biosci. and Bioeng.*, 101(2), 87 – 96.
- Takache, H., Christophe, G., Cornet, J.F., and Pruvost, J. (2009). Experimental and theoretical assessment of maximum productivities for the microalgae *chlamydomonas reinhardtii* in two different geometries of photobioreactors. *Biotechnol Prog*, 26(2), 431–440.
- Vatcheva, I., deJong, H., Bernard, O., and Mars, N. (2006). Experiment selection for the discrimination of semi-quantitative models of dynamical systems. *Artif. Intel.*, 170, 472–506.