Active Hypothesis Testing for Quickest Anomaly Detection

Kobi Cohen and Qing Zhao

Abstract—The problem of quickest detection of an anomalous process among \(M\) processes is considered. At each time, a subset of the processes can be observed, and the observations from each chosen process follow two different distributions, depending on whether the process is normal or abnormal. The objective is a sequential search strategy that minimizes the expected detection time subject to an error probability constraint. This problem can be considered as a special case of active hypothesis testing first considered by Chernoff in 1959 where a randomized strategy, referred to as the Chernoff test, was proposed and shown to be asymptotically optimal (as the error probability approaches zero) optimal. For the special case considered in this paper, we show that a simple deterministic test achieves asymptotic optimality and offers better performance in the finite regime. We further extend the problem to the case where multiple anomalous processes are present. In particular, we examine the case where only an upper bound on the number of anomalous processes is known.

Index Terms—Sequential detection, anomaly detection, dynamic search, active hypothesis testing, controlled sensing.

I. INTRODUCTION

We consider the problem of detecting a single anomalous process among \(M\) processes. Borrowing terminologies from target search, we refer to these processes as cells and the anomalous process as the target which can locate in any of the \(M\) cells. The decision maker is allowed to search for the target over \(K\) cells at a time (1 \(\leq K \leq M\)). The observations from searching a cell are i.i.d. realizations drawn from two different distributions \(f\) and \(g\), depending on whether the target is absent or present. The objective is a sequential search strategy that dynamically determines which cells to search at each time and when to terminate the search so that the expected detection time is minimized under a constraint on the probability of declaring a wrong cell as containing the target.

The problem under study applies to intrusion detection in cyber-systems when an intrusion to a subnet has been detected and the probability of each component being compromised is small (thus with high probability, there is only one abnormal component). It also finds applications in target search, fraud detection, and spectrum scanning in cognitive radio networks.

A. A Case of Active Hypothesis Testing

The above anomaly detection problem can be considered as a special case of the sequential design of experiments problem first studied by Chernoff in 1959 [1]. In this problem, a decision maker aims to infer the state of an underlying phenomenon by sequentially choosing the experiment (thus the observation model) to be conducted at each time among a set of available experiments. Chernoff focused on the case of binary hypotheses and showed that a randomized strategy, referred to as the Chernoff test, is asymptotically optimal as the maximum error probability diminishes. Specifically, the Chernoff test chooses the current experiment based on a distribution that depends on the past actions and observations. Variations and extensions of the problem and the Chernoff test were studied in [2]–[6], where the problem was referred to as controlled sensing for hypothesis testing in [3], [4] and active hypothesis testing in [5], [6] (see a more detailed discussion in Section I-C).

It is not difficult to see that the quickest anomaly detection problem considered in this paper is a special case of the active hypothesis testing problem considered in [1]–[6]. In particular, under each hypothesis that the target is located in the \(i^{th}\) \((i = 1, 2, \ldots, M)\) cell, the distribution (either \(f\) or \(g\)) of the next observation depends on the action of which cell to search (i.e., which experiment to carry out). The Chernoff test and its variations proposed in [2]–[6] thus directly apply to our problem. However, in contrast to the randomized nature of the Chernoff test and its variations, we show in this paper that a simple deterministic test achieves asymptotic optimality and offers better performance in the finite regime.

B. Main Results

Similar to [1]–[5], we focus on asymptotically optimal policies in terms of minimizing the detection time as the error probability approaches zero. The asymptotic optimality of the Chernoff test as shown in [1] requires that under any experiment, any pair of hypotheses are distinguishable (i.e., has positive Kullback-Liebler (KL) divergence). This assumption does not hold in the anomaly detection problem considered in this paper. For instance, under the experiment of searching the \(i^{th}\) cell, the hypotheses of the target being in the \(j^{th}\) \((j \neq i)\) and the \(k^{th}\) \((k \neq i)\) cells yield the same observation distribution \(f\). Nevertheless, we show in Theorem 1 that the Chernoff test preserves its asymptotic optimality for the problem at hand even without this positivity assumption on all KL divergences. As a result, it serves as a bench mark for comparison.

The Chernoff test, when applied directly to the anomaly detection problem, leads to a randomized cell selection rule: the cells to be search at the current time are drawn randomly...
C. Related Work

As discussed in Section I-A, the above anomaly detection problem can be considered as a special case of an active hypothesis testing first studied by Chernoff in [1], where the focus was on sequential binary composite hypothesis testing. The results in [1] were extended by Bessler to M-ary hypothesis in [2]. In [4], Nitinawarat et al. considered M-ary active hypothesis testing in both fixed sample size and sequential settings. Under the sequential setting, they extended the Chernoff test to relax the positivity assumption on all KL divergences as required in [1], [2]. Furthermore, they examined the asymptotic optimality of the Chernoff test under constraints on decision risks, a stronger condition than the error probability, and developed a modified Chernoff test to meet hard constraints on the decision risks. In [5], in addition to the asymptotic optimality adopted by Chernoff in [1], Nghshvar and Javidi examined active sequential hypothesis testing under the notion of non-zero information acquisition rate by letting the number of hypotheses approach infinity and under a stronger notion of asymptotic optimality. They further studied in [6] the roles of sequentiality and adaptivity in active hypothesis testing by characterizing the gain of sequential tests over fixed sample size tests and the gain of closed-loop policies over open-loop policies.

Target search or target whereabouts problems have been widely studied under various scenarios. Results under the sequential setting can be found in [7]–[10]. Specifically, optimal policies were derived for the problem of quickest search over Weiner processes under the model that a single process is observed at a time [7]–[9]. In [10], an optimal search strategy was established under a different search model, where the target must be contacted by one sensor and identified by another, while contact investigation must not be interrupted until the contact is identified. However, optimal policies under general distributions or with multi-process probing remain an open question. In this paper we address these questions under the asymptotic regime, as the error probability approaches zero. Target search under the setting of fixed sample size was considered in [11]–[14]. In [11]–[13], searching in a specific location provides a binary-valued measurement regarding the presence or absence of the target. Similar to this paper, Castanon considered in [14] continuous observations: the observations from a location without the target and with the target have distributions \( f \) and \( g \), respectively. Different from this paper where we consider sequential settings and obtain an asymptotically optimal policy that applies to general distributions, [14] focused on the fixed sample size setting and required a symmetry assumption on the distributions (specifically, \( f(x) = g(b - x) \) for some \( b \)) for the optimality of the proposed policy. The problem of universal outlier hypothesis testing was studied in [15]. Under this setting, a vector of observations containing coordinates with an outlier distribution is observed at each given time. The goal is to detect the coordinates with the outlier distribution based on a sequence of \( n \) i.i.d. vectors of observations.

Another related problem is sequential detection involving multiple independent processes [16]–[23]. An optimal threshold policy was established for the problem of quickly detecting an idle period over multiple independent ON/OFF processes in [16]. In [17], the problem of quickest detection of the emergence of primary users in multi-channel cognitive radio networks was considered. The problem of quickest search of idle channels when the idle/busy state of each channel is fixed over time was studied in [18]. In [19], [20], optimal index probing strategies were derived for the anomaly localization problem, where the objective is to minimize the expected cost incurred by the abnormal processes. In [21], the problem of identifying the first abnormal sequence among an infinite number of i.i.d. sequences was considered. An optimal cumulative sum (CUSUM) test was established under this setting. Further studies on this model can be found in [22], [23]. These studies all assume different models from that of this paper.

D. Organization

In Section II we describe the system model and problem formulation. A discussion on the randomized Chernoff test and its asymptotic optimality for the anomaly detection problem is provided in Section III. In Section IV we propose an asymptotically optimal deterministic policy to solve the problem. In Section V we extend the problem to the case where multiple anomalous processes are present. In Section VI we provide numerical examples to illustrate the performance of the proposed policy as compared with the Chernoff test. Section VII concludes the paper.
II. SYSTEM MODEL AND PROBLEM FORMULATION

A. System Model

Consider the following anomaly detection problem. A decision maker is required to detect the location of a single anomalous object (referred as a target) located in one of $M$ cells. If the target is in cell $m$, we say that hypothesis $H_m$ is true. The a priori probability that $H_m$ is true is denoted by $\pi_m$, where $\sum_{m=1}^{M} \pi_m = 1$. To avoid trivial solutions, it is assumed that $0 < \pi_m < 1$ for all $m$.

At each time, only $K$ ($1 \leq K \leq M$) cells can be observed. When cell $m$ is observed at time $n$, an observation $y_m(n)$ is independently drawn from a distribution $f(y)$; if hypothesis $m$ is true, $y_m(n)$ follows distribution $g(y)$.

Let $P_m$ be the probability measure under hypothesis $H_m$ and $E_m$ the operator of expectation with respect to the measure $P_m$.

We define the stopping rule $\tau$ as the time (i.e., detection delay) when the decision maker finalizes the search by declaring the location of the target. Let $\delta \in \{1,2,...,M\}$ be a decision rule, where $\delta = m$ if the decision maker declares that $H_m$ is true. Let $\phi(n) \in \{1,2,...,M\}^K$ be a selection rule indicating which $K$ cells are chosen to be observed at time $n$. The time series vector of selection rules is denoted by $\phi = (\phi(n), n = 1, 2, ...)$.

Let $y(n) = \{\phi(t), y_{\phi(t)}(t)\}_{t=1}^{n}$ be the set of all the available observations and the chosen cell indices up to time $n$.

A deterministic selection rule $\phi(n)$ at time $n$ is a mapping from $y(n-1)$ to $\{1, 2, ..., M\}^K$. A randomized selection rule $\phi(n)$ is a mapping from $y(n-1)$ to probability mass functions over $\{1, 2, ..., M\}^K$.

Definition 1: An admissible strategy $\Gamma$ for the sequential anomaly detection problem is given by the tuple $\Gamma = (\tau, \delta, \phi)$.

Let $P_{\tau}(\Gamma) = \sum_{m=1}^{M} \pi_m \alpha_m(\Gamma)$ be the probability of error under strategy $\Gamma$, where $\alpha_m(\Gamma) = P_m(\delta \neq m | \Gamma)$ is the probability of declaring $\delta \neq m$ when $H_m$ is true. Let $E(\tau | \Gamma) = \sum_{m=1}^{M} \pi_m E_m(\tau | \Gamma)$ be the average detection delay under $\Gamma$.

B. Objective

We adopt a Bayesian approach as in [1], [3] by assigning a cost of $c$ for each observation and a loss of 1 for a wrong declaration. The Bayes risk under strategy $\Gamma$ when hypothesis $H_m$ is true is given by:

$$R_m(\Gamma) \triangleq \alpha_m(\Gamma) + c E_m(\tau | \Gamma).$$

Note that $c$ represents the ratio of the sampling cost to the cost due to wrong decisions.

The average Bayes risk is given by:

$$R(\Gamma) = \sum_{m=1}^{M} \pi_m R_m(\Gamma) = P_{\tau}(\Gamma) + c E(\tau | \Gamma).$$

The objective is to find a strategy $\Gamma$ that minimizes the Bayes risk $R(\Gamma)$:

$$\inf_{\Gamma} R(\Gamma).$$

C. Notations

Let $I_m(n)$ be the indicator function, where $I_m(n) = 1$ if cell $m$ is observed at time $n$, and $I_m(n) = 0$ otherwise. Let

$$\ell_m(n) \triangleq \log \frac{g(y_m(n))}{f(y_m(n))},$$

and

$$S_m(n) \triangleq \sum_{t=1}^{n} \ell_m(t) I_m(t)$$

be the log-likelihood ratio (LLR) and the observed sum LLRs at time $n$ of cell $m$, respectively. We then define $m^{(i)}(n)$ as the index of the cell with the $i^{th}$ highest observed sum LLRs at time $n$.

Let

$$\Delta S(n) \triangleq S_{m^{(i)}(n)}(n) - S_{m^{(i)}(n)}(n)$$

denotes the difference between the highest and the second highest observed sum LLRs at time $n$. Finally, we define

$$I^*(M, K) \triangleq \max\begin{cases} D(g||f) + D(f||g), & \text{if } K = M, \\
\max \left\{ \frac{KD(f||g)}{M-1}, \frac{D(g||f) + (K-1)D(f||g)}{M-1} \right\}, & \text{if } K < M. \end{cases}$$

In subsequent sections we show that $I^*(M, K)$ plays the role of the rate function, which determines the asymptotically optimal performance of the test. Increasing $I^*(M, K)$ decreases the asymptotic lower bound on the Bayes risk. Note that $I^*(M, K)$ increases with $K$ (i.e., increasing the number of observed processes improves performance) and decreases with $M$ (i.e., increasing the number of hypotheses deteriorates performance).

III. THE CHERNOFF TEST

In this section, we present the Chernoff test proposed in [1] when it is applied to the anomaly detection problem. We then prove the asymptotic optimality of the Chernoff test for this problem when the positivity assumption on KL divergences as required in [1] no longer holds.

The Chernoff test has a randomized selection rule. Specifically, for a general $M$-ary active hypothesis testing problem, the action $u$ at time $n$ under the Chernoff test is drawn from a distribution $q(u)$ that depends on the past actions and observations:

$$q(u) = \arg \max_{\pi(u)} \min_{j \in M \setminus \{i(n)\}} \sum_{\nu} q(u) D(p^u_{\nu(j)} || p^u_{j}).$$

where $M$ is the set of the $M$ hypotheses, $\hat{i}(n)$ is the ML estimate of the true hypothesis at time $n$ based on past actions and observations, and $p^u_\nu$ is the observation distribution under hypothesis $j$ when action $u$ is taken.

It can be shown that the Chernoff test when applied to the anomaly detection problem works as follows. If $D(g||f) \geq$
Given in Appendix VIII-A.

Based on the argument of [4] and the proof of Theorem 2 among the set of cells \( M \setminus \{ m^{(1)}(n) \} \). If, however, \( D(g)[f] < D(f)[g]/(M - 1) \), all \( K \) cells are drawn randomly with equal probability among the set of cells \( M \setminus \{ m^{(1)}(n) \} \).

Even though the positivity assumption on KL divergences as required in the proof of the asymptotic optimality of the Chernoff test given in [1] no longer holds for the anomaly detection problem, we show in Theorem 1 below that the Chernoff test preserves its asymptotic optimality in this case. Note that in [4], a modified Chernoff test was developed in order to handle indistinguishable hypotheses under some (but not all) actions. The basic idea of this modified test is to replace the action distribution given in (8) with a uniform distribution over a subsequence of time instants that grows at a sublinear rate with time. This sequence of arbitrary actions that are independent of past observations will in general affect the performance in the finite regime. Theorem 1 below shows that this modification is unnecessary for the anomaly detection problem.

**Theorem 1:** Let \( R^* \) and \( R(\Gamma) \) be the Bayes risks under the Chernoff test and any other policy \( \Gamma \), respectively. Then,

\[
R^* \sim -\frac{c \log c}{I^*(M, K)} \sim \inf_{\Gamma} R(\Gamma) \quad \text{as} \quad c \to 0 .
\]

**Proof:** The proof is given in Appendix VIII-B and is based on the argument of [4] and the proof of Theorem 2 given in Appendix VIII-A.

IV. THE DETERMINISTIC DGF POLICY

In this section we propose a deterministic policy, referred to as the DGF policy, to solve (3). Theorem 2 shows that the DGF policy is asymptotically optimal in terms of minimizing the Bayes risk (2) as \( c \to 0 \).

A. The DGF policy:

At each time \( n \), the selection rule \( \phi(n) \) of the DGF policy chooses cells according to the order of their sum LLRs. Specifically, based on the relative order of \( D(g)[f] \) and \( D(f)[g]/(M - 1) \), either the \( K \) cells with the top \( K \) highest sum LLRs or those with the second to the \((K + 1)\)th highest sum LLRs are chosen, i.e.,

\[
\phi(n) = \begin{cases} 
  \{ m^{(1)}(n), m^{(2)}(n), \ldots, m^{(K)}(n) \} , & \text{if } D(g)[f] \geq \frac{D(f)[g]}{(M - 1)} \text{ or } K = M \\
  \{ m^{(2)}(n), m^{(3)}(n), \ldots, m^{(K+1)}(n) \} , & \text{if } D(g)[f] < \frac{D(f)[g]}{(M - 1)} \text{ and } K < M
\end{cases} \]

The stopping rule and decision rule under the DGF policy are given by:

\[
\tau = \inf \{ n : \Delta S(n) \geq -\log c \} ,
\]

and

\[
\delta = m^{(1)}(\tau) .
\]

The deterministic selection rule under the DGF policy is intuitively satisfying. Consider the case where \( K = 1 \). If cell \( m^{(1)}(n) \) is selected at each given time \( n \), the asymptotic detection time approaches \(-\log c/D(g)[f] \) since the true cell (say \( m \)) is observed at each given time with high probability (in the asymptotic regime) and the test is finalized once sufficient information is gathered from this cell (for a detailed asymptotic analysis see Appendix VIII-A). In this case, \( D(g)[f] \) determines the asymptotically optimal performance of the test, since \( E_{m}(\ell_{m}) = D(g)[f] \). On the other hand, if cell \( m^{(2)}(n) \) is selected at each given time \( n \), the asymptotic detection time approaches \((M - 1)\log c/D(f)[g] \) since one of the \( M - 1 \) false cells are observed at each given time with high probability and the test is finalized once sufficient information is gathered from all these cells. Since \( E_{m}(\ell_{j}) = -D(f)[g] \) for all \( j \neq m \), then \( D(f)[g]/(M - 1) \) determines the asymptotically optimal performance of the test. Therefore, the selection rule selects the strategy that minimizes the asymptotic detection time according to \( \max[D(g)[f], D(f)[g]/(M - 1)] \). Similar arguments apply to the case where \( K > 1 \).

B. Performance Analysis

The following main theorem shows that the DGF policy is asymptotically optimal in terms of minimizing the Bayes risk as \( c \to 0 \) zero:

**Theorem 2 (asymptotic optimality of the DGF policy):** Let \( R^* \) and \( R(\Gamma) \) be the Bayes risks under the DGF policy and any other policy \( \Gamma \), respectively. Then,

\[
R^* \sim -\frac{c \log c}{I^*(M, K)} \sim \inf_{\Gamma} R(\Gamma) \quad \text{as} \quad c \to 0 .
\]

**Proof:** For a detailed proof see Appendix VIII-A. We provide here a sketch of the proof. In App. VIII-A.1, we show that \(-c \log c/I^*(M, K)\) is an asymptotic lower bound on the Bayes risk that can be achieved by any policy \( \Gamma \). Then, we show in App. VIII-A.2 that the Bayes risk \( R^* \) under the DGF policy, approaches the asymptotic lower bound as \( c \to 0 \). Specifically, the asymptotic optimality property of DGF is based on Lemma 11, showing that the asymptotic expected search time approaches \( \frac{\log c}{I^*(M, K)} \), while the error probability is \( O(c) \) following Lemma 4.

Establishing the asymptotic expected sample size under DGF in Lemma 11 is mainly based on Lemmas 7, 8 and 10, which break the total expected search time into three major events \( \tau_1, \tau_2 = \tau_1 + \tau_2 \) and \( \tau = \tau_2 + n_{m} \). Roughly speaking, \( \tau_1 \) is the time when the sum LLRs of the true cell (say \( m \)) is the highest among all the cells for all \( n \geq \tau_1 \). \( \tau_2 \) is the time when sufficient information has been gathered to distinguish hypothesis \( m \) from at least one false hypothesis \( j \neq m \). \( \tau \) is the termination time, where sufficient information has been gathered to distinguish hypothesis \( m \) from all false hypotheses \( j \neq m \).

Lemma 8 shows that the second phase, which is the total amount of time between \( \tau_1 \) and \( \tau_2 \) determines the asymptotic detection time. Thus, \( E(\tau_2) \sim -\log c/I^*(M, K) \) as \( c \to 0 \). Lemma 7 shows that the first phase, which is the total amount of time from the beginning of the search until \( \tau_1 \).
occurs does not affect the asymptotic detection time. Thus, \( E(\tau_1)/E(\tau_2) \to 0 \). Note that differing from [4], where only polynomial decay of \( P_m(\tau_1 > n) \) was shown to handle indistinguishable hypotheses under some (but not all) actions when applying the extended randomized Chernoff test, Lemma 7 shows exponential decay of \( P_m(\tau_1 > n) \) under DGF. Lemma 10 shows that the third phase, which is the time between \( \tau_2 \) and \( \tau \) does not affect the asymptotic detection time. Thus, \( E(\tau_2)/E(\tau_3) \to 0 \). To prove Lemma 10, we define the dynamic range between the sum LLRs of false hypotheses at time \( t \) as \( DR(t) = \max_{f \neq m} S_f(t) - \min_{f \neq m} S_f(t) \). Lemma 9 shows that \( DR(\tau_2) \) is sufficiently small. Following Lemma 9, Lemma 10 shows that the time required to gather information between \( \tau_2 \) and \( \tau \) does not affect the asymptotic detection time.

The asymptotic performance (13) of the test is intuitively satisfying. For any fixed \( K \), the Bayes risk increases with \( M \), since increasing the number of cells increases the required number of observations before making a decision with sufficient reliability. On the other hand, for any fixed \( M \), the Bayes risk decreases with \( K \), since increasing the number of cells that can be observed at each given time reduces the total search time before making a decision with sufficient reliability.

V. EXTENSION TO MULTIPLE ANOMALOUS PROCESSES

In this section we extend the results reported in previous sections to the case where multiple processes are abnormal. In Section V-A we consider the detection of \( L \) abnormal processes, where \( L \) is known. In Section V-B we consider the case where an unknown number \( \ell \geq 1 \) of abnormal processes are present and only an upper bound \( \ell \leq L \) is known.

Throughout this section, we define \( \mathcal{M}' \) as the set of all possible combinations of target locations, with cardinality \( \mathcal{M}' = |\mathcal{M}'| \) (i.e., a set of \( M' \) hypotheses, \( H_m' \), indicating that the locations of all targets are given by the \( (m')^{\text{th}} \) set in \( \mathcal{M}' \)) and \( \pi_m' \) as the \( a \) priori probability that \( H_m' \) is true. Here, the decision rule declares a set of target locations (i.e., hypothesis \( H_m' \)) and the error probability under policy \( \Gamma \) is defined as \( P_e(\Gamma) = \sum_{m'=1}^{M'} \pi_m' \alpha_m'(\Gamma) \), where \( \alpha_m'(\Gamma) = P_m'(\delta \neq H_m';\Gamma) \) is the probability of declaring \( \delta \neq H_m' \) when \( H_m' \) is true.

A. A Case of \( L \) Abnormal Processes

Consider the case where \( L \) abnormal processes are located among the \( M \) cells and \( L \) is known. In this case, the detection problem involves \( M' = \binom{M}{L} \) hypotheses. We show below that a variation of the DGF policy, dubbed the DGF(L) policy, is asymptotically optimal under this setting.

The stopping rule and decision rule under the DGF(L) policy are similar to that under the GDF policy:

\[
\tau = \inf \{ n : \Delta L S(n) \geq -\log_c \} ,
\]

where \( \Delta L S(n) \triangleq S_{m^{(1)}(n)}(n) - S_{m^{(L+1)}(n)}(n) \) and

\[
\delta = (m^{(1)}(\tau), m^{(2)}(\tau), ..., m^{(L)}(\tau)) .
\]

The selection rule under the DGF(L) policy is more involved and depends on the relative order of \( K \) and \( L \) (or \( M - L \)). Specifically,

\[
\phi(n) = \begin{cases} \phi_y(n) \quad \text{if} \; \frac{D(g||f)}{L} \geq \frac{D(f||g)}{M-L}, \\ \phi_f(n) \quad \text{if} \; \frac{D(g||f)}{L} < \frac{D(f||g)}{M-L}, \end{cases}
\]

where

\[
\phi_y(n) = \begin{cases} (m^{(1)}(n), m^{(2)}(n), ..., m^{(K)}(n)), \quad \text{if} \; K \geq L, \\ (m^{(L-K+1)}(n), m^{(L-K+2)}(n), ..., m^{(L)}(n)), \quad \text{if} \; K < L \end{cases}
\]

and

\[
\phi_f(n) = \begin{cases} (m^{(M-K+1)}(n), m^{(M-K+2)}(n), ..., m^{(M)}(n)), \quad \text{if} \; K > M - L, \\ (m^{(L+1)}(n), m^{(L+2)}(n), ..., m^{(L+K)}(n)), \quad \text{if} \; K \leq M - L \end{cases}
\]

It is not difficult to see that when \( L = 1 \), the DGF(L) policy degenerates to the DGF policy.

Next, we analyze the performance of the DGF(L) policy. Let

\[
I^*(M, K, L) \triangleq \begin{cases} I_{g}^*(M, K, L), \quad \text{if} \; \frac{D(g||f)}{L} \geq \frac{D(f||g)}{M-L}, \\ I_{f}^*(M, K, L), \quad \text{if} \; \frac{D(g||f)}{L} < \frac{D(f||g)}{M-L} \end{cases}
\]

where

\[
I_{g}^*(M, K, L) \triangleq \begin{cases} D(g||f) + \frac{(K-L)D(f||g)}{M-L}, \quad \text{if} \; K \geq L, \\ KD(f||g) \quad \text{if} \; K < L \end{cases}
\]

and

\[
I_{f}^*(M, K, L) \triangleq \begin{cases} D(f||g) + \frac{(K-M+L)D(g||f)}{L}, \quad \text{if} \; K > M - L, \\ KD(f||g) \quad \text{if} \; K \leq M - L \end{cases}
\]

The following theorem shows the asymptotically optimal performance of the DGF(L) policy:

**Theorem 3:** Let \( R^* \) be the Bayes risks under the DGF(L) policy and any other policy \( \Gamma \), respectively. Then,

\[
R^* \sim \frac{-c \log c}{I^*(M, K, L)} \sim \inf_{\Gamma} R(\Gamma) \quad \text{as} \quad c \to 0 .
\]
Proof: See Appendix VIII-C.

Note that in the DGF(L) policy, all \( L \) targets are declared simultaneously at the termination time of the detection. A modification to DGF(L) leads to a policy where abnormal processes are declared sequentially during the detection. Consider, for example, \( K = 1 \) and \( \frac{D(g||f)}{L} \geq \frac{D(f||g)}{M-L} \). It can be shown (with minor modifications to Theorem 3) that an asymptotically optimal policy is to test the cell with the largest sum LLRs and declare the first target once the largest sum LLRs exceeds the threshold \( -\log c \). The same procedure is then applied to the remaining \( M-1 \) cells. This repeats until \( L \) abnormal processes have been declared, at which point, the detection terminates. The asymptotic expected termination time is given by \( -L \log c / D(g||f) \) with \( P_e = O(c) \). Even though the total detection time remains the same order as under the DGF(L) policy, this modified version may be more appealing from a practical point of view. In particular, actions can be taken to fix each abnormal process the moment it is identified; the total impact to the system by these \( L \) abnormal processes can thus be reduced. If \( \frac{D(g||f)}{L} < \frac{D(f||g)}{M-L} \), it can be shown that an asymptotically optimal policy is to test the cell with the smallest sum LLRs and declare the first normal process once the smallest sum LLRs drops below \( \log c \). The same procedure is then applied to the remaining \( M-1 \) processes and is repeated until all \( M-1 \) objects are declared as normal (thus, the \( L \) remaining ones are declared as abnormal). The asymptotic expected termination time is given by \( -L \log c / D(f||g) \) with \( P_e = O(c) \). Even though in this case, the modified version also declares all \( L \) targets simultaneously at the termination time of the detection, the difference is that this modified version incurs much fewer switchings among processes than the DGF(L) policy. This may be more advantageous in some practical scenarios when switching among tested processes results in additional cost or delay. To see that the modified version incurs fewer switchings, we note that the modified version tests the process that the decision maker is most sure to be normal based on past observations while DGF(L) tests the process that the decision maker is least sure to be normal except the \( L \) processes currently considered as the targets (see the second line in (18) which shows that DGF(L) chooses the cell with the \((L+1)^{th}\) largest sum LLRs; the \( L \) processes with larger sum LLRs are the current maximum likelihood of the target locations).

### B. A Case of Unknown Number of Abnormal Processes

In this section, we consider the interesting case where the number \( \ell \) of abnormal processes (or targets) is unknown. It is only known that \( \ell \) is bounded by \( 1 \leq \ell \leq L \). We consider the case where \( K = 1 \). We also assume that the number of cells satisfies:

\[
M \geq \frac{L(D(g||f) + D(f||g))}{D(g||f)}. \tag{23}
\]

Note that (23) implies \( \frac{D(g||f)}{L} \geq \frac{D(f||g)}{M-L} \).

Throughout this section, we allow the decision maker to declare the target locations sequentially during the test (similar to the modified DGF(L) policy as discussed at the end of Section V-A). We refer to the detection time \( \tau_d \) as the time when the last target has been declared and to the termination time \( \tau \) as the time when the decision maker terminates the test. Note that \( \tau = \tau_d \) when the number \( \ell \) of targets is known (as discussed in previous sections). When \( \ell \) is unknown, however, \( \tau_d \leq \tau \) since the decision maker does not know whether it has already identified all targets at time \( \tau_d \). In general, the termination time \( \tau \) increases linearly with \( M \) under any policy with \( P_e = O(c) \) whenever \( \ell < L \). This is due to the fact that even if the \( \ell \) targets have been detected with sufficient reliability, the decision maker must verify whether there are other targets in the remaining \( M-\ell \) cells before terminating the test. On the other hand, following the modified DGF(L) policy, one would expect to achieve a detection time \( \tau_d \) less than \( -L \log c / D(g||f) \) for all \( \ell \leq L \), which is independent of the number \( M \) of total processes.

In scenarios with a large number of processes and \( L < < M \), a policy that focuses on minimizing the termination time \( \tau \), which grows linearly with \( M \), may not be practically appealing. It is desirable to have a policy that allows each abnormal process to be identified and fixed as quickly as possible during the test. In other words, it is desirable to have a policy that minimizes the detection time \( \tau_d \) rather than the termination time \( \tau \). In this case, even though the test continues after \( \tau_d \) to ensure there are no other targets, all abnormal processes have been fixed by the detection time \( \tau_d \) and cease to incur cost to the system. We thus modify the objective function to the following Bayes risk:

\[
R(\Gamma) \triangleq P_e(\Gamma) + cE(\tau_d|\Gamma),
\]

and we are interested in finding a strategy \( \Gamma \) that minimizes the Bayes risk (24).

Before presenting the desired solution for this case, we demonstrate with a specific example that even though the Chernoff test is asymptotically optimal in terms of minimizing the termination time \( \tau \), it is highly suboptimal in terms of minimizing the detection time \( \tau_d \). Assume that \( L = 2 \) is the upper bound on the number of targets, which can locate in any of \( M = 3 \) cells. As a result, the detection problem includes 6 hypotheses, \( H_1 = \{1\} \), \( H_2 = \{2\} \), \( H_3 = \{3\} \), \( H_4 = \{1,2\} \), \( H_5 = \{1,3\} \), \( H_6 = \{2,3\} \). The observation model under every hypothesis and cell selection is given in Table I. Assume that hypothesis \( H_1 \) is true and that \( \hat{H}(n) = H_1 \), where \( \hat{H}(n) \) is the ML estimate of the true hypothesis at time \( n \). Consider a deterministic policy that selects the cells according to the order of their sum LLRs and declares an object as target if \( S_m(n) > -\log c \) or normal if \( S_m(n) < \log c \).

<table>
<thead>
<tr>
<th>cell 1</th>
<th>cell 2</th>
<th>cell 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H_1 ) = {1}</td>
<td>g</td>
<td>f</td>
</tr>
<tr>
<td>( H_2 ) = {2}</td>
<td>f</td>
<td>g</td>
</tr>
<tr>
<td>( H_3 ) = {3}</td>
<td>f</td>
<td>g</td>
</tr>
<tr>
<td>( H_4 ) = {1,2}</td>
<td>g</td>
<td>f</td>
</tr>
<tr>
<td>( H_5 ) = {1,3}</td>
<td>g</td>
<td>f</td>
</tr>
<tr>
<td>( H_6 ) = {2,3}</td>
<td>g</td>
<td>g</td>
</tr>
</tbody>
</table>
This policy achieves $\tau_d \sim -\log c/D(g||f)$ (since cell 1 is first identified as a target with high probability) and $\tau \sim -\log c/D(g||f) - 2\log c/D(f||g)$ (since the number of targets is unknown and $L = 2$, thus the decision maker must continue testing the normal processes before terminating the test). On the other hand, the Chernoff test (which aims to minimize the termination time) will not select cell 1 at time $n$, since $H_4$ or $H_5$ minimizes (8), i.e., $D(p_{12}||p_{12})_R = D(p_{12}||p_{12})_R = D(g||g) = 0$. It can be verified that selecting randomly cells 2 or 3 with equal probability $1/(M-1) = 1/2$ maximizes (8) and achieves a rate function $D(f||g)/(M-1) = D(f||g)/2$, which results in $\tau = \tau_d \sim 2\log c/D(f||g)$, which is greater than the detection time under the above deterministic policy. Intuitively speaking, once cells 2, 3 are identified as normal, cell 1 is identified as abnormal (because at least 1 target is present). Therefore, the Chernoff test observes cells 2, 3 to minimize the detection time $\tau$ (by not testing cell 1), while increasing the detection time $\tau_d$.

Next, we present the DGF($\ell, L$) policy to minimize the Bayes risk (24). Let $T(n)$ be the set of cells satisfying $S_m(n) \geq -\log c$ at time $n$. Define

$$\hat{n}^{(1)}(n) = \arg \max_{m \in T(n)} S_m(n).$$

The selection rule under the DGF($\ell, L$) policy is given by:

$$\phi(n) = \hat{n}^{(1)}(n).$$

The stopping rule and decision rule under the DGF($\ell, L$) policy are given by:

$$\tau = \inf \{ n : |S_m(n)| \geq -\log c \ \forall m \},$$

and

$$\delta = T(\tau_d).$$

Note that $\delta$ denotes the target locations and the complete set is declared at time $\tau_d$. Since the number of targets is unknown, the decision maker continues taking observations to verify that there is no other target. The test is terminated at time $\tau$.

The following theorem shows the asymptotically optimal performance of the DGF($\ell, L$) policy:

**Theorem 4:** Let $\ell \leq L$ be the number of targets, $K = 1$ and assume that (23) holds. Let $R^*$ and $R(\Gamma)$ be the Bayes risks (24) under the DGF($\ell, L$) policy and any other policy $\Gamma$, respectively. Then,

$$R^* \sim -\ell c \log c / D(g||f) \sim \inf_{\Gamma} R(\Gamma) \quad \text{as} \quad c \to 0.$$  \hspace{1cm} (29)

**Proof:** See Appendix VIII-D.

### VI. Numerical Examples

In this section we present numerical examples to illustrate the performance of the proposed deterministic policy as compared to the Chernoff test. We simulated a single anomalous object (i.e., target) located in one of $M$ cells with the following parameters: The *a priori* probability that the target is present in cell $m$ was set to $\pi_m = 1/M$ for all $1 \leq m \leq M$. When cell $m$ is observed at time $n$, an observation $y_{mn}(n)$ is independently drawn from a distribution $f \sim \exp(\lambda_f)$ or $g \sim \exp(\lambda_g)$, depending on whether the target is absent or present, respectively. It can be verified that:

$$D(g||f) = \log(\lambda_g) - \log(\lambda_f) + \frac{\lambda_M}{\lambda_g} - 1,$$

$$D(f||g) = \log(\lambda_f) - \log(\lambda_g) + \frac{\lambda_M}{\lambda_f} - 1.$$

Let $R_{DGF}, R_{CH}$ be the Bayes risks under the DGF policy and the Chernoff test, respectively. Let $R_{LB} = -\ell c \log c / (\ell M)$ be the asymptotic lower bound on the Bayes risk as $c \to 0$. We define:

$$L_{DGF} \triangleq \frac{R_{DGF} - R_{LB}}{R_{LB}},$$

$$L_{CH} \triangleq \frac{R_{CH} - R_{LB}}{R_{LB}},$$

as the relative loss in terms of Bayes risk under the DGF policy and the Chernoff test, respectively, as compared to the asymptotic lower bound. Following Theorems 1, 2, we expect both $L_{DGF}$ and $L_{CH}$ to approach 0 as $c \to 0$. $L_{DGF}$ and $L_{CH}$ serve as performance measures of the tests in the finite regime.

First, we consider the case where $M = 5$ and $K = 1$. Note that when $K = 1$ and $D(g||f) \geq D(f||g)/(M - 1)$, the Chernoff test coincides with the DGF policy: they both select cell $m^{(1)}(n)$. When $D(g||f) < D(f||g)/(M - 1)$, however, the proposed policy selects cell $m^{(2)}(n)$, while the Chernoff test selects cell $j \neq m^{(1)}(n)$ randomly at each given time $n$. We set $\lambda_f = 0.5, \lambda_g = 10$ and obtain $D(g||f) \approx 2.05, D(f||g)/(M - 1) \approx 4$. As a result, the Chernoff test and the DGF policy have different cell selection rules. The performance of the Algorithms are presented in Fig. 1(a), 1(b). In Fig. 1(a), the asymptotic lower bound on the expected sample size and the average sample sizes achieved by the algorithms are presented as a function of $c$ (log-scale). In Fig. 1(b), $L_{DGF}$ and $L_{CH}$ are presented as a function of $c$. Although both schemes approach the asymptotic lower bound as $c \to 0$, it can be seen that the DGF policy significantly outperforms the Chernoff test in the finite regime for all values of $c$.

Next, we consider the case where $M = 5$ and $K = 2$ (i.e., two cells are observed at a time). In this case, the DGF policy selects cells $m^{(1)}(n)$ and $m^{(2)}(n)$ at each given time $n$ only if $D(g||f) \geq D(f||g)/(M - 1).$ Otherwise, it selects cells $m^{(2)}(n)$ and $m^{(3)}(n)$. The Chernoff test selects cells $m^{(1)}(n) \neq m^{(1)}(n)$ randomly at each given time $n$ only if $D(g||f) \geq D(f||g)/(M - 1)$. Otherwise, it selects cells $i,j \neq m^{(1)}(n)$ randomly. First, we set $\lambda_f = 2, \lambda_g = 10$ and obtain $D(g||f) \approx 0.8, D(f||g)/(M - 1) \approx 0.6$. The performance of the algorithms in this case are presented in Fig. 2(a), 2(b). Next, we set $\lambda_f = 0.5, \lambda_g = 10$ and obtain $D(g||f) \approx 2.05, D(f||g)/(M - 1) \approx 4$. The performance of the Algorithms in this case are presented in Fig. 3(a), 3(b). In Fig. 2(a), 3(a), the asymptotic lower bound on the expected sample size and the average sample sizes achieved by the algorithms are presented as a function of the cost per observation $c$. In Fig. 2(b), 3(b), $L_{DGF}$ and $L_{CH}$ are...
presented as a function of $c$. It can be seen that the DGF policy significantly outscores the Chernoff test in the finite regime for all values of $c$ under all cases. These results demonstrate the advantage of using the deterministic selection rule applied by the DGF policy instead of the randomized Chernoff test for the anomaly detection problem.

**VII. CONCLUSION**

The problem of quickest detection of an anomalous process (i.e., target) among $M$ processes (i.e., cells) was investigated. Due to resource constraints, only a subset of the cells can be observed at a time. The objective is a search strategy that minimizes the expected search time subject to an error probability constraint. The observations from searching a cell are realizations drawn from two different distributions $f$ or $g$, depending on whether the target is absent or present, respectively. A simple deterministic policy was established to solve the Bayesian formulation of the search problem, where a cost of $c$ per observation and a loss of 1 for wrong decisions are assigned. It is shown that the proposed index policy is asymptotically optimal in terms of minimizing the Bayes risk as $c$ approaches zero.

The problem was further extended to handle the case where multiple anomalous processes are present. In particular, the interesting case where only an upper bound on the number of anomalous processes is known was considered. We showed that existing methods may not be practically appealing under the latter setting. Hence, we proposed a modified optimization problem for this case. Asymptotically optimal deterministic policies were developed for these cases as well.

(a) Average sample sizes achieved by the algorithms and the asymptotic lower bound as a function of the cost per observation.

(b) The loss in terms of Bayes risk under the DGF policy and the Chernoff test as compared to the asymptotic lower bound. $L_{DGF}, L_{Ch}$ approach 0 as $c \to 0$

Fig. 1. Algorithms’ performance for $M = 5, K = 1, \lambda_f = 0.5, \lambda_g = 10$

(a) Average sample sizes achieved by the algorithms and the asymptotic lower bound as a function of the cost per observation.

(b) The loss in terms of Bayes risk under the DGF policy and the Chernoff test as compared to the asymptotic lower bound. $L_{DGF}, L_{Ch}$ approach 0 as $c \to 0$

Fig. 2. Algorithms’ performance for $M = 5, K = 2, \lambda_f = 2, \lambda_g = 10$
Fig. 3. Algorithms’ performance for j

Let DGF policy as

− that the asymptotic expected search time approaches

optimality property of DGF is based on Lemma 11, showing

that the Bayes risk

achieved by any policy

while the error probability is

O

(a) Average sample sizes achieved by the algorithms and the asymptotic lower bound as a function of the cost per observation.

(b) The loss in terms of Bayes risk under the DGF policy and the Chernoff test as compared to the asymptotic lower bound. \(L_{DGF}, L_{Ch}\) approach 0 as \(c \to 0\)

VIII. APPENDIX

A. Proof of Theorem 2

In this appendix we prove the asymptotic optimality of the DGF policy as \(c \to 0\). In App. VIII-A.1, we show that \(\frac{-c \log c}{\Gamma(M,K)}\) is an asymptotic lower bound on the Bayes risk that can be achieved by any policy \(\Gamma\). Then, we show in App. VIII-A.2 that the Bayes risk \(R^*\) under the DGF policy, approaches the asymptotic lower bound as \(c \to 0\). Specifically, the asymptotic optimality property of DGF is based on Lemma 11, showing that the asymptotic expected search time approaches \(-\log c\), while the error probability is \(O(c)\) following Lemma 4.

Throughout the appendix we use the following notations:

Let

\[ N_j(n) = \sum_{t=1}^{n} 1_j(t) \]  

be the number of times that cell \(j\) has been observed up to time \(n\). We define

\[ \Delta S_{m,j}(n) = S_m(n) - S_j(n), \]  

as the difference between the observed sum of LLRs of cells \(m\) and \(j\). Let

\[ \Delta S_m(n) = \min_{j \neq m} S_{m,j}. \]  

Thus,

\[ \Delta S(n) = S_m(\tau(n)) - S_{m}(\tau(n)) = \max_m \Delta S_m(n). \]  

Without loss of generality we prove the theorem when hypothesis \(m\) is true. For convenience, we define

\[ \tilde{\ell}_k(i) = \begin{cases} \ell_k(i) - D(g||f), & \text{if } k = m, \\ \ell_k(i) + D(f||g), & \text{if } k \neq m. \end{cases} \]  

Note that \(\tilde{\ell}_k(i)\) is a zero-mean r.v under hypothesis \(H_m\).

1) The Asymptotic Lower bound on the Bayes risk:

The asymptotic lower bound on the Bayes risk is shown in

Theorem 5 and is mainly based on Lemmas 1, 3, provided

below. Lemma 1 shows that under \(P_m\), \(\Delta S_m(\tau)\), defined in

(32), must be large enough to obtain a sufficiently small error \(\alpha_m\). Lemma 3 implies that \(\tau\) must be large enough to obtain

a sufficiently large \(\Delta S_m(\tau)\).

Lemma 1: Assume that \(\alpha_m(\Gamma) = O(-c \log c)\) for all \(m\). Let \(0 < \epsilon < 1\). Then:

\[ P_m(\Delta S_m(\tau) < -(1-\epsilon) \log c | \Gamma) = O(-c^\epsilon \log c), \]  

for all \(m\) and for any policy \(\Gamma\).

Proof: Note that:

\[ P_m(\Delta S_m(\tau) < -(1-\epsilon) \log c | \Gamma) \]

\[ = P_m(\Delta S_m(\tau) < -(1-\epsilon) \log c, \delta = m | \Gamma) \]

\[ + P_m(\Delta S_m(\tau) < -(1-\epsilon) \log c, \delta \neq m | \Gamma) \]

\[ \leq P_m(\Delta S_m(\tau) < -(1-\epsilon) \log c, \delta = m | \Gamma) + \alpha_m(\Gamma) \]  

Note that \(\alpha_m(\Gamma) = O(-c \log c)\) as conditioned by the Lemma. Next, we upper bound the term \(P_m(\Delta S_m(\tau) < -(1-\epsilon) \log c, \delta = m | \Gamma)\). Let \(R_m\) be the decision region where \(H_m\) is accepted at time \(n\) and \(\Delta S_{m,j}(n) < -(1-\epsilon) \log c\) for some \(j \neq m\). Let \(N_m(y), N_j(y), N_{\neq m,j}(y)\) be the sets of time indices for an observation vector \(y\) (where \(y = y(n)\) is a time series of observations up to time \(n\), collected from \(K\) cells at each time), containing the time indices when cells \(m, j\) and all \(i \neq m, j\) are observed, respectively. Thus, for all \(j \neq m\) there
exists $G > 0$ such that:

\[ -Ge^\log c \geq P_j(\delta = m|\Gamma) \]

\[ \geq P_j(\Delta S_{m,j}(\tau) \leq -(1 - \epsilon) \log c , \delta = m|\Gamma) \]

\[ = \sum_{n=1}^{\infty} \int_{R_n} \left[ \prod_{i \in N_m(g)} f(y_m(i)) \prod_{i \in N_j(g)} g(y_j(i)) \prod_{i \in N_{m,j}(g)} f(y_{m,j}(i)) \right] dy \]

\[ = \sum_{n=1}^{\infty} \int_{R_n} \left[ \prod_{i \in N_m(g)} g(y_m(i)) \prod_{i \in N_j(g)} f(y_j(i)) \prod_{i \in N_{m,j}(g)} f(y_{m,j}(i)) \right] dy \times \]

\[ \sum_{n=1}^{\infty} \int_{R_n} \exp \{-\Delta S_{m,j}(\tau)\} \times \]

\[ \prod_{i \in N_m(g)} g(y_m(i)) \prod_{i \in N_j(g)} f(y_j(i)) \prod_{i \in N_{m,j}(g)} f(y_{m,j}(i)) \] \[ \geq e^{1-\epsilon} P_m(\Delta S_{m,j}(\tau) < -(1 - \epsilon) \log c , \delta = m|\Gamma) . \]

Thus,

\[ P_m(\Delta S_{m,j}(\tau) < -(1 - \epsilon) \log c , \delta = m|\Gamma) = O(-e^\epsilon \log c ) \forall j \neq m . \]

As a result,

\[ P_m(\Delta S_m(\tau) < -(1 - \epsilon) \log c , \delta = m|\Gamma) \]

\[ \leq \sum_{j \neq m} P_m(\Delta S_{m,j}(\tau) < -(1 - \epsilon) \log c , \delta = m|\Gamma) \]

\[ = O(-e^\epsilon \log c ) . \]

Finally,

\[ P_m(\Delta S_m(\tau) < -(1 - \epsilon) \log c|\Gamma) = O(-e^\epsilon \log c ) . \]

Lemma 2: Assume that

\[ I^*(M, K) = D(g||f) + \frac{(K-1)D(f||g)}{M-1} . \]

Define the following function:

\[ d(t) \triangleq t \left[ D(g||f) + \frac{K^2}{M-1}D(f||g) \right] . \]

Then, $d(t)$ is monotonically increasing with $t$.

Proof: Note that $I^*(M, K) = D(g||f) + \frac{(K-1)D(f||g)}{M-1}$ implies:

\[ D(g||f) + \frac{(K-1)D(f||g)}{M-1} \geq \frac{KD(f||g)}{M-1} \]

\[ \iff D(g||f) \geq \frac{D(f||g)}{M-1} . \]

Differentiation $d(t)$ with respect to $t$ yields:

\[ \frac{\partial d(t)}{\partial t} = D(g||f) - \frac{D(f||g)}{M-1} \geq 0 , \]

which completes the proof.

Lemma 3: For any fixed $\epsilon > 0$,

\[ P_m \left( \max_{1 \leq i \leq n} \Delta S_m(t) \geq n \left( I^*(M, K) + \epsilon \right) | \Gamma \right) \to 0 \]

as $n \to \infty , (42)$

for all $m$ and for any policy $\Gamma$.

Proof: It should be noted that polynomial decay of a similar condition under a binary composite hypothesis testing was shown in [1, Lemma 5] using a variation of the Kolmogorov’s inequality. Here, we use a different approach to show exponential decay of (42). Let $j^*(t) = \arg \min_{j \neq m} N_j(t')$ denotes the cell (except cell $m$) which has been observed the lowest number of times up to time $t$. Let

\[ \Delta S^*_m(t) \triangleq S_m(t) - S_j^*(t) . \]

Since $\Delta S_m(t) \leq \Delta S^*_m(t)$ for all $m$ and $t$, we have:

\[ P_m \left( \max_{1 \leq i \leq n} \Delta S_m(t) \geq n \left( I^*(M, K) + \epsilon \right) | \Gamma \right) \]

\[ \leq P_m \left( \max_{1 \leq i \leq n} \Delta S^*_m(t) \geq n \left( I^*(M, K) + \epsilon \right) | \Gamma \right) \]

Define

\[ W_m(t) \triangleq \sum_{i=1}^{t} \tilde{e}_m(i) - \sum_{i=1}^{t} \tilde{e}_{j^*(i)}(i) . \]

Next, we consider three cases:

Case 1 : $K = M$:

In this case $I^*(M, K) = D(g||f) + D(f||g)$. Furthermore, note that $N_j(t) = t$, for all $j$ and $t$. Thus,

\[ \Delta S^*_m(t) = W^*_m(t) + \sum_{i=1}^{t} D(g||f) + D(f||g) \]

\[ \leq W^*_m(t) + nI^*(M, K) . \]

Therefore,

\[ \Delta S^*_m(t) \geq n \left( I^*(M, K) + \epsilon \right) \]

implies

\[ W^*_m(t) \geq n \epsilon . \]
Using the i.i.d. property of $\ell_k(t)$ across the time series and applying the Chernoff bound yield\footnote{Note that when $K = M$, then $N_m(t) = N_j(t) = t$ for all $t$. However, we use the loosened bound in (46), since it applies to cases 2, 3 below.}:

$$P_m \left( \max_{1 \leq t \leq n} \Delta S_m(t) \geq n (I^*(M, K) + \epsilon) | \Gamma \right)$$
$$\leq P_m \left( \max_{1 \leq t \leq n} W^*_m(t) \geq n \epsilon | \Gamma \right)$$
$$\leq \sum_{t=1}^{n} P_m \left( \sum_{r=1}^{N_{m}(t)} \sum_{r=1}^{N_{j^*(t)}(t)} \ell_m(r) + \ell_{j^*(t)}(r) \geq n \epsilon | \Gamma \right)$$
$$\leq \sum_{t=1}^{n} \sum_{N_m(t)=0}^{t} \sum_{N_{j^*(t)}(t)=0}^{t} \left[ E_m \left( e^{t (\ell_m(1) - \epsilon/2)} \right) N_m(t) \right] \times$$
$$\left[ E_m \left( e^{(-\ell_{j^*(t)}(1) - \epsilon/2)} \right) N_{j^*(t)}(t) \right] \times$$
$$\exp \left\{ -\frac{\epsilon}{2} \left( 2n - N_m(t) - N_{j^*(t)}(t) \right) \right\},$$

(46)

for all $s > 0$.

Clearly, a moment generating function (MGF) is equal to one at $s = 0$. Furthermore, since $E_m(\ell_m(1) - \epsilon/2) = -\epsilon/2 < 0$ and $E_m(-\ell_{j^*(t)}(1) - \epsilon/2) = -\epsilon/2 < 0$ are strictly negative, differentiating the MGFs of $\ell_m(1) - \epsilon/2$ and $-\ell_{j^*(t)}(1) - \epsilon/2$ with respect to $s$ yields strictly negative derivatives at $s = 0$. Hence, there exist $s > 0$ and $\gamma' > 0$ such that $E_m\left(e^{s(\ell_m(1) - \epsilon/2)}\right), E_m\left(e^{s(-\ell_{j^*(t)}(1) - \epsilon/2)}\right)$ and $e^{-se/2}$ are strictly less than $e^{-\gamma'} < 1$. Since $2n - N_m(t) - N_{j^*(t)}(t) \geq 0$, there exist $C > 0$ and $\gamma > 0$, such that summing over $t, N_m(t), N_{j^*(t)}(t)$ yields:

$$P_m \left( \max_{1 \leq t \leq n} \Delta S_m(t) \geq n (I^*(M, K) + \epsilon) | \Gamma \right)$$
$$\leq C e^{-\gamma n} \to 0 \text{ as } n \to \infty,$$

(47)

**Case 2 :** $K < M$ and $I^*(M, K) = \frac{K D(f) | g}{M-1}$.

Note that:

$$\Delta S^*_m(t) = W^*_m(t) + N_m(t)D(g|f) + N_{j^*(t)}(t)D(f|g)$$
$$\leq W^*_m(t) + D(f|g) \left[ \frac{K_n - N_m(t)}{M-1} + N_{j^*(t)}(t) \right]$$

(48)

The last inequality holds since $\frac{K D(f) | g}{M-1} \geq D(g|f) + \frac{(K-1) D(f) | g}{M-1}$ implies $D(g|f) \leq \frac{D(f|g)}{M-1}$.

Note that

$$N_{j^*(t)}(t) \leq \frac{Kt - N_m(t)}{M-1} \leq \frac{K n - N_m(t)}{M-1}. $$

Hence,

$$\Delta S^*_m(t) \leq W^*_m(t) + D(f|g) \frac{K n}{M-1} = W^*_m(t) + n I^*(M, K).$$

(49)

Therefore,

$$\Delta S^*_m(t) \geq n (I^*(M, K) + \epsilon)$$

implies

$$W^*_m(t) \geq n \epsilon.$$

The rest of the proof is similar to Case 1.

**Case 3 :** $K < M$ and $I^*(M, K) = D(g|f) + \frac{(K-1) D(f) | g}{M-1}$.

Note that:

$$\Delta S^*_m(t) = W^*_m(t) + N_m(t)D(g|f) + N_{j^*(t)}(t)D(f|g)$$
$$\leq W^*_m(t) + N_m(t)D(g|f) + \frac{K n - N_m(t)}{M-1}D(f|g)$$
$$= W^*_m(t) + N_m(t) \left[ D(g|f) + \frac{K n}{M-1} - D(f|g) \right]$$

(50)

Since $0 \leq N_m(t) \leq n$, by Lemma 2 we have:

$$\Delta S^*_m(t) \leq W^*_m(t) + n \left[ D(g|f) + \frac{K n}{M-1} - D(f|g) \right]$$

(51)

$$= W^*_m(t) + n I^*(M, K).$$

The rest of the proof is similar to Case 1. Hence, (42) follows.

The following theorem provides the lower bound on the Bayes risk (2):

**Theorem 5:** Let $R(\Gamma) = O(-c \log c)$ be the Bayes risk under policy $\Gamma$. Then,

$$R(\Gamma) \geq -(1 + o(1)) \frac{c \log(c)}{I^*(M, K)}.$$  

(52)

for any policy $\Gamma$.

**Proof:** To show the lower bound on the Bayes risk, we use a similar argument as in [1]. For any $\epsilon > 0$ let $n_c = \frac{\log c}{I^*(M, K) + \epsilon}$. Note that

$$P_m \left( \tau \leq n_c | \Gamma \right)$$
$$= P_m \left( \tau \leq n_c, \Delta S_m(\tau) \geq (1 - \epsilon) \log c | \Gamma \right)$$
$$+ P_m \left( \tau \leq n_c, \Delta S_m(\tau) < (1 - \epsilon) \log c | \Gamma \right)$$

(53)

The first term in the last inequality approaches zero as $c \to 0$ by Lemma 3. The second term in the last inequality approaches zero as $c \to 0$ by Lemma 1 for any policy $\Gamma$ that achieves $R(\Gamma) = O(-c \log c)$. As a result, the expected sample size under policy $\Gamma$ satisfies $E_m(\tau | \Gamma) \geq -(1 + o(1)) \log(c) / I^*(M, K)$. Hence, $R(\Gamma) \geq -(1 + o(1)) \frac{c \log(c)}{I^*(M, K)}$.
2) Asymptotic Optimality of the DGF policy:

In this section we show that the DGF policy achieves the lower bound on the Bayes risk (52) as $c \to 0$. We mainly focus on the more interesting case where $K \geq 2$ cells are observed at a time. The case where $K = 1$ is simpler and follows with minor modifications. Below, the proof follows the structure discussed in Section IV-B.

**Lemma 4:** Assume that the DGF policy is implemented. Then, the error probability is upper bounded by:

$$P_e \leq (M - 1)c . \quad (54)$$

**Proof:** Let $\alpha_{m,j} = P_m(\delta = j)$ for all $j \neq m$. Thus, $\alpha_m = \sum_{j \neq m} \alpha_{m,j}$. Note that accepting $H_j$ (i.e., $\Delta S_{j,m}(n) \geq -\log c$) implies $\Delta S_{j,m}(n) \geq -\log c$. Let $R_j(n)$ be the subset of the sample space where $\Delta S_{j,m}(n) \geq -\log c$ occurs at time $n$. Let $N_m(y)$, $N_j(y)$, $N_{\neq m,j}(y)$ be the sets of time indices for an observation vector $y$ (where $y = y(n)$ is a time series of observations up to time $n$, collected from $K$ cells at each time), containing the time indices when cells $m, j$ and all $i \neq m, j$ are observed, respectively. Thus, $\alpha_{m,j} = P_m(\delta = j) = P_m(\Delta S_{j}(\tau) \geq -\log c) \leq P_m(\Delta S_{j,m}(\tau) \geq -\log c)$

$$= \sum_{n=1}^{\infty} \int_{R_j(n)} \left[ \prod_{i \in N_m(y)} g(y_m(i)) \prod_{i \in N_j(y)} f(y_j(i)) \prod_{i \in N_{\neq m,j}(y)} f(y_{\neq m,j}(i)) \right] dy$$

$$= \sum_{n=1}^{\infty} \int_{R_j(n)} \left[ \prod_{i \in N_m(y)} g(y_m(i)) \prod_{i \in N_j(y)} f(y_j(i)) \prod_{i \in N_{\neq m,j}(y)} f(y_{\neq m,j}(i)) \right] \times$$

$$\times \left[ \prod_{i \in N_m(y)} f(y_m(i)) \prod_{i \in N_j(y)} g(y_j(i)) \prod_{i \in N_{\neq m_j}(y)} f(y_{\neq m,j}(i)) \right] dy$$

$$= \sum_{n=1}^{\infty} \int_{R_j(n)} \exp \{-\Delta S_{j,m}(\tau)\} \times$$

$$\left[ \prod_{i \in N_m(y)} f(y_m(i)) \prod_{i \in N_j(y)} g(y_j(i)) \prod_{i \in N_{\neq m,j}(y)} f(y_{\neq m,j}(i)) \right] dy$$

$$\leq c P_j(\Delta S_{j,m}(\tau) \geq -\log c) \leq c . \quad (55)$$

Finally, $\alpha_m = \sum_{j \neq m} \alpha_{m,j} \leq (M - 1)c . \quad (56)$

**Lemma 5:** Assume that the DGF policy is implemented. Fix $0 < q < 1$. Then, there exist $C > 0$ and $\gamma > 0$ such that

$$P_m(S_j(n) \geq S_m(n), N_j(n) \geq qn) \leq Ce^{-\gamma n} , \quad (56)$$

for $m = 1, 2, \ldots, M$ and $j \neq m$.

**Proof:** Applying the Chernoff bound and using the i.i.d. property of $\ell_j(t)$, $\ell_m(t)$ across time yield:

$$P_m(S_j(n) \geq S_m(n), N_j(n) \geq qn) \leq \sum_{n=1}^{\infty} \sum_{n=1}^{N_j(\tau)} \sum_{n=1}^{N_m(\tau)} \left[ E_m \left( e^{s\ell_j(1)} \right)^{N_j} \right]^{N_m}$$

$$\leq \sum_{n=1}^{\infty} \sum_{n=1}^{N_j(\tau)} \sum_{n=1}^{N_m(\tau)} \left[ E_m \left( e^{s\ell_j(1)} \right)^{N_j} \right]^{N_m}$$

for all $s > 0$.

Clearly, a moment generating function (MGF) is equal to one at $s = 0$. Furthermore, since $E_m(\ell_j(1)) = -D(f||g) < 0$ and $E_m(\ell_m(1)) = -D(g||f) < 0$ are strictly negative, differentiating the MGFs of $\ell_j(1), \ell_m(1)$ with respect to $s$ yields strictly negative derivatives at $s = 0$. As a result, there exist $s > 0$ and $\gamma > 0$ such that

$$P_m(S_j(n) \geq S_m(n), N_j(n) \geq qn) \leq \sum_{n=1}^{\infty} e^{-\gamma N_j} \sum_{n=1}^{\infty} e^{-\gamma N_m} \leq Ce^{-\gamma m} . \quad (58)$$

**Lemma 6:** Assume that the DGF policy is implemented. Fix $0 < q < 1$. Then, there exists $\gamma > 0$ such that

$$P_m(S_j(n) \geq S_m(n), N_j(n) \geq qn) \leq e^{-\gamma n} , \quad (59)$$

for $m = 1, 2, \ldots, M$ and $j \neq m$.

**Proof:** The proof is similar to the proof of Lemma 5.

**Lemma 7:** Assume that the DGF policy is implemented. Then, there exist $C > 0$ and $\gamma > 0$ such that

$$P_m(\tau_1 > n) \leq Ce^{-\gamma n} , \quad (60)$$

for $m = 1, 2, \ldots, M$.

**Proof:** Note that differing from [4], where only polynomial decay of $P_m(\tau_1 > n)$ was shown to handle indistinguishable hypotheses under some (but not all) actions when applying the extended randomized Chernoff test, Lemma 7 shows exponential decay of $P_m(\tau_1 > n)$ under DGF. We focus on the case where $M > 2$. The case of $M = 2$ is simpler and follows with minor modifications. Note that:

$$P_m(\tau_1 > n) \leq P_m \left( \max_{j \neq m} \sup_{t \geq n} (S_j(t) - S_m(t)) \geq 0 \right)$$

$$\leq \sum_{j \neq m} \sum_{t \geq n} P_m(S_j(t) \geq S_m(t)) . \quad (61)$$
Therefore, it suffices to show that there exist $C > 0$ and $\gamma > 0$ such that $P_m (S_j(n) \geq S_m(n)) \leq Ce^{-\gamma n}$.

**Step 1:** Bounding each term in the summation on the RHS of (61):

Let:

$$\rho = \frac{1}{4M-6}.$$  

Note that $0 < \rho \leq 1/6$.

Thus,

$$P_m (S_j(n) \geq S_m(n)) \leq P_m (S_j(n) \geq S_m(n), N_j(n) < \rho n, N_m(n) < \rho n)$$

$$+ P_m (S_j(n) \geq S_m(n), N_j(n) \geq \rho n)$$

$$+ P_m (S_j(n) \geq S_m(n), N_m(n) \geq \rho n)$$  

(62)

By Lemmas 5, 6, there exist $\gamma_1 > 0$ and $C_1 > 0$ such that the second and the third terms on the RHS are upper bounded by $C_1 e^{-\gamma_1 n}$. In the case of $K = M$ the first term on the RHS equals zero (since $N_j(n) = N_m(n) = n$ surely). Hence, it remains to show that the first term on the RHS decreases exponentially with $n$ for $K < M$. Note that the event $(N_j(n) < \rho n, N_m(n) < \rho n)$ implies that at least $\tilde{n} = n - N_j(n) - N_m(n) = n(1 - 2\rho)$ times cells $j, m$ are not observed. Let $N_r(n)$ be the number of times when cell $r \neq j, m$ has been observed and cells $j, m$ have not been observed up to time $n$. We refer to each such time as $r \neq j, m$-probing time. There exists a cell $r \neq j, m$ such that $\tilde{N}_r(n) \geq \frac{\tilde{n}}{M-2} = \frac{n(1-2\rho)}{M-2}$. Hence, (62) can be upper bounded by:

$$P_m (S_j(n) \geq S_m(n)) \leq \sum_{r \neq j, m} P_m (\tilde{N}_r(n) > \frac{n(1-2\rho)}{M-2}, N_j(n) < \rho n, N_m(n) < \rho n)$$

$$+ 2C_1 e^{-\gamma_1 n}$$  

(63)

It remains to show that each term in the summation on the RHS of (63) decreases exponentially with $n$.

**Step 2:** Bounding each term in the summation on the RHS of (63):

Let $\tilde{t}_1^r, \tilde{t}_2^r, ..., \tilde{t}_{\tilde{N}_r(n)}^r$ be the $r \neq j, m$-probing time indices and let

$$\zeta \triangleq \frac{1 - 2\rho}{2(M-2)}.$$  

Note that the event $\tilde{N}_r(n) > \frac{n(1-2\rho)}{M-2}$ implies the following:

- At every $r \neq j, m$-probing time, $S_j(n) \leq S_r(n)$ or $S_m(n) \leq S_r(n)$ must occur (otherwise, if $S_j(n) > S_r(n)$ and $S_m(n) > S_r(n)$ then $j$ or $m$ are observed).

- After $\zeta n$ $r \neq j, m$-probing times have passed, then another $\zeta n$ $r \neq j, m$-probing times must occur.

- In particular, at time $\tilde{t}_\zeta^r$ the following holds: $S_j(\tilde{t}_\zeta^r) \leq S_r(\tilde{t}_\zeta^r)$ or $S_m(\tilde{t}_\zeta^r) \leq S_r(\tilde{t}_\zeta^r)$ must occur.

- $N_r(\tilde{t}_\zeta^r) \geq \zeta n$.

Therefore, using the i.i.d. property of $\ell_r(t), \ell_j(t)$ across time we have:

$$P_m (\tilde{N}_r(n) > \frac{n(1-2\rho)}{M-2}, N_j(n) < \rho n, N_m(n) < \rho n)$$

$$\leq \sum_{\tilde{t} = 0}^{\tilde{n}} P_m \left( \inf_{n' \leq \rho n} \sum_{i=1}^{\tilde{t}} \ell_j(i) \leq \sum_{i=1}^{\tilde{t}} \ell_r(i) \right)$$

$$+ \sum_{\tilde{t} = 0}^{\tilde{n}} P_m \left( \inf_{n' \leq \rho n} \sum_{i=1}^{\tilde{t}} \ell_m(i) \leq \sum_{i=1}^{\tilde{t}} \ell_r(i) \right)$$

$$\leq \sum_{\tilde{t} = 0}^{\tilde{n}} \sum_{n' = 0}^{\rho n} \sum_{i=1}^{\tilde{t}} \ell_j(i) \leq \sum_{i=1}^{\tilde{t}} \ell_r(i)$$

(64)

$$+ \sum_{\tilde{t} = 0}^{\tilde{n}} \sum_{n' = 0}^{\rho n} \sum_{i=1}^{\tilde{t}} \ell_m(i) \leq \sum_{i=1}^{\tilde{t}} \ell_r(i)$$

$$= \sum_{q=0}^{\rho n} \sum_{n' = 0}^{\rho n} \sum_{i=1}^{\tilde{t}} \ell_j(i) \leq \sum_{i=1}^{\tilde{t}} \ell_r(i)$$

$$+ \sum_{q=0}^{\rho n} \sum_{n' = 0}^{\rho n} \sum_{i=1}^{\tilde{t}} \ell_m(i) \leq \sum_{i=1}^{\tilde{t}} \ell_r(i)$$

**Step 3:** Bounding the first term on the RHS of (64):

Note that

$$\sum_{i=1}^{\rho n} \ell_r(i) + \sum_{i=1}^{\rho n} -\ell_j(i)$$

$$= \sum_{i=1}^{\rho n} \ell_r(i) + \sum_{i=1}^{\rho n} -\ell_j(i) - D(f||g) (\zeta n + n' - 2 (\rho n - n'))$$

(65)

and

$$\zeta n + n' \geq \zeta n + q - n' - 2 (\rho n - n')$$

$$= n (\zeta - 2\rho) + q + n' = \frac{1}{4(M-2)} n + q + n'$$

for all $n' \leq \rho n$.

Therefore,

$$\sum_{i=1}^{\rho n} \ell_r(i) + \sum_{i=1}^{\rho n} -\ell_j(i) \geq 0$$

(66)

implies

$$\sum_{i=1}^{\rho n} \ell_r(i) + \sum_{i=1}^{\rho n} -\ell_j(i) \geq C_1 (n + q + n')$$

(67)

where $C_1 = \frac{D(f||g)}{4(M-2)} > 0$.

Applying the Chernoff bound and using the i.i.d. property of
\( \ell_r(t), \ell_j(t) \) across the time series yield:

\[
P_m \left( \sum_{i=1}^{n'} \ell_j(i) \leq \sum_{i=1}^{n} \ell_r(i) \right)
\leq P_m \left( \sum_{i=1}^{n'} \tilde{\ell}_j(i) + \sum_{i=1}^{n} -\tilde{\ell}_r(i) \geq C(n + q + n') \right)
\leq \left[ E_m \left( e^{s(\tilde{\ell}_r(1) - C)} \right) \right]^{\zeta n + q} \left[ E_m \left( e^{s(\tilde{\ell}_j(1))} \right) \right]^{n'} \times e^{-sC(n + q + n')}
= \left[ E_m \left( e^{s(\tilde{\ell}_r(1) - C)} \right) \right]^{\zeta n + q} \times \left[ E_m \left( e^{s(\tilde{\ell}_j(1))} \right) \right]^{n'} \times e^{-sC(n - \zeta n)}.
\]

(68)

for all \( s > 0 \).

Clearly, a moment generating function (MGF) is equal to one when \( s = 0 \). Furthermore, since \( E_m(\tilde{\ell}_r(1) - C) = -C < 0 \) and \( E_m(-\tilde{\ell}_j(1) - C) = -C < 0 \) are strictly negative, differentiating the MGF of \( \tilde{\ell}_r(1) - C, \tilde{\ell}_j(1) - C \) with respect to \( s \) yields strictly negative derivatives at \( s = 0 \). Hence, there exist \( s > 0 \) and \( \gamma_2 > 0 \) such that \( E_m(e^{s(\tilde{\ell}_r(1) - C)}) \) and \( e^{-sC} \) are strictly less than \( e^{-\gamma_2} < 1 \). Hence,

\[
P_m \left( \sum_{i=1}^{n'} \ell_j(i) \leq \sum_{i=1}^{n} \ell_r(i) \right) \leq e^{-\gamma_2(n + q + n')}.
\]

(69)

Hence,

\[
\sum_{q=0}^{n-\zeta n} \sum_{n'=0}^{\rho n} P_m \left( \sum_{i=1}^{n'} \ell_j(i) \leq \sum_{i=1}^{n} \ell_r(i) \right) \leq e^{-\gamma_2n} \sum_{q=0}^{n-\zeta n} e^{-\gamma_2q} \sum_{n'=0}^{\rho n} e^{-\gamma_2n'} \leq C_2 e^{-\gamma_2n},
\]

where \( C_2 = (1 - e^{-\gamma_2})^{-2} \).

**Step 4: Bounding the second term on the RHS of (64):**

Applying the Chernoff bound and using the i.i.d. property of \( \ell_r(t), \ell_m(t) \) across the time series yield:

\[
P_m \left( \sum_{i=1}^{n'} \ell_r(i) + \sum_{i=1}^{n} -\ell_m(i) \geq 0 \right)
\leq \left[ E_m \left( e^{s\ell_r(1)} \right) \right]^{\zeta n + q} \left[ E_m \left( e^{s(-\ell_m(1))} \right) \right]^{n'}.
\]

(71)

for all \( s > 0 \).

Since \( E_m(\ell_r(1)) = -D(\|f\| < 0 \) and \( E_m(-\ell_m(1)) = -D(\|g\|) < 0 \) are strictly negative, there exist \( s > 0 \) and \( \gamma_3 > 0 \) such that \( E_m(e^{s\ell_r(1)}) \) and \( E_m(e^{s(-\ell_m(1))}) \) are strictly less than \( e^{-\gamma_3} < 1 \). Hence,

\[
P_m \left( \sum_{i=1}^{n'} \ell_r(i) + \sum_{i=1}^{n} -\ell_m(i) \geq 0 \right) \leq e^{-\gamma_3(n + q + n')}.
\]

(72)

Finally, there exists \( \gamma_3 = \zeta \gamma_3 > 0 \) such that

\[
\sum_{q=0}^{n-\zeta n} \sum_{n'=0}^{\rho n} P_m \left( \sum_{i=1}^{n'} \ell_m(i) \leq \sum_{i=1}^{n} \ell_r(i) \right) \leq e^{-\gamma_3n} \sum_{q=0}^{n-\zeta n} e^{-\gamma_2q/\zeta} \sum_{n'=0}^{\rho n} e^{-\gamma_2n'/\zeta} \leq \frac{e^{-\gamma_3n}}{(1 - e^{-\gamma_2/\zeta})^2},
\]

which completes the proof.

For the next lemmas we define

\[
D'(f\|g) \triangleq (K - 1)D(f\|g)/(M - 1).
\]

(74)

**Definition 3:** \( \tau_2 \) is defined as follows:

1. If \( K = M, \tau_2 \) denotes the smallest integer such that \( \sum_{i=t_1+1}^{n} \ell_m(i) \leq M(\|f\|) \leq \frac{D(f\|g)}{M(M, K)} \log c \) and \( \sum_{i=t_1+1}^{n} \ell_r(i) \leq M(\|f\|) \leq \frac{D(f\|g)}{M(M, K)} \log c \) for some \( j \neq m \) for all \( n \geq \tau_2 \).
2. If \( K < M \) and \( I^*(M, K) = KD(f\|g)/(M - 1), \tau_2 \) denotes the smallest integer such that \( \sum_{i=t_1+1}^{n} \ell_m(i) \leq M(\|f\|) \leq \frac{D(f\|g)}{M(M, K)} \log c \) for some \( j \neq m \) for all \( n \geq \tau_2 \).
3. If \( K < M \) and \( I^*(M, K) = D(f\|g) + (K - 1)D(f\|g)/(M - 1), \tau_2 \) denotes the smallest integer such that \( \sum_{i=t_1+1}^{n} \ell_m(i) \leq M(\|f\|) \leq \frac{D(f\|g)}{M(M, K)} \log c \) and \( \sum_{i=t_1+1}^{n} \ell_r(i) \leq M(\|f\|) \leq \frac{D(f\|g)}{M(M, K)} \log c \) for some \( j \neq m \) for all \( n \geq \tau_2 \).

**Definition 4:** \( n_2 \triangleq \tau_2 - \tau_1 \) denotes the total amount of time between \( \tau_1 \) and \( \tau_2 \).

**Lemma 8:** Assume that the DGF policy is implemented. Then, for every fixed \( \epsilon > 0 \) there exist \( C > 0 \) and \( \gamma > 0 \) such that

\[
P_m (n_2 > n) \leq Ce^{-\gamma n} \forall n > -(1 + \epsilon) \log c/\gamma^*(M, K),
\]

(75)

for all \( m = 1, 2, ..., M \).

**Proof:** We prove the theorem for three cases:

**Case 1:** \( K = M \):
In this case \( I^*(M, K) = D(g\|f) + D(f\|g) \) and \( 1_k(t) = 1 \) for all \( k, t \). Let \( \tau_2^n \) and \( \tau_2^2 \) for \( j \neq m \) be the smallest integers such that \( \sum_{i=t_1+1}^{n} \ell_m(i) \geq -\frac{D(f\|g)}{M(M, K)} \log c \) for all \( n > \tau_2^n \) and \( \sum_{i=t_1+1}^{n} \ell_r(i) \leq -\frac{D(f\|g)}{M(M, K)} \log c \) for all \( n > \tau_2^2 \), respectively. Similarly, \( n_2^2 \) denotes the total amount of time between \( \tau_1 \) and \( \tau_2^2 \). Clearly, \( n_2 \leq \max(k(n_2^2)) \). As a result,

\[
P_m (n_2 > n) \leq \sum_{k=1}^{M} P_m (n_2^k > n).
\]

(76)
Thus it remains to show that $P_m(n^2 > n)$ decreases exponentially with $n$. Next, we prove the lemma for cell $m$. The proof for cell $j \neq m$ follows with minor modifications. Let $\epsilon_1 = D(g||f)/c(1 + \epsilon) > 0$. Thus,

$$\sum_{i=\tau_{1+1}}^{t+\tau_1} \tilde{\ell}_m(i) + \frac{D(g||f)}{I^*(M,K)} \log c$$

$$= \sum_{i=\tau_{1+1}}^{t+\tau_1} \tilde{\ell}_m(i) + tD(g||f) + \frac{D(g||f)}{I^*(M,K)} \log c$$

$$\geq \sum_{i=\tau_{1+1}}^{t+\tau_1} \tilde{\ell}_m(i) + t\epsilon_1,$$

for all $t \geq n > -(1 + \epsilon) \log c/I^*(M,K)$.

As a result,

$$\sum_{i=\tau_{1+1}}^{t+\tau_1} \ell_m(i) \leq -\frac{D(g||f)}{I^*(M,K)} \log c.$$

(78)

implies

$$\sum_{i=\tau_{1+1}}^{t+\tau_1} \tilde{\ell}_m(i) \leq -t\epsilon_1.$$

(79)

Hence, for any $\epsilon > 0$ there exists $\epsilon_1 > 0$ such that

$$P_m(n^2 > n) \leq P_m \left( \inf \sum_{i=\tau_{1+1}}^{t+\tau_1} \ell_m(i) \leq -\frac{D(g||f)}{I^*(M,K)} \log c \right)$$

$$\leq \sum_{t=n}^{\infty} P_m \left( \sum_{i=\tau_{1+1}}^{t+\tau_1} \ell_m(i) \leq -\frac{D(g||f)}{I^*(M,K)} \log c \right)$$

$$\leq \sum_{t=n}^{\infty} P_m \left( \sum_{i=\tau_{1+1}}^{t+\tau_1} -\tilde{\ell}_m(i) \geq t\epsilon_1 \right).$$

(80)

for all $t \geq n > -(1 + \epsilon) \log c/I^*(M,K)$.

By applying the Chernoff bound, it can be shown that there exists $\gamma_1 > 0$ such that $P_m \left( \sum_{i=\tau_{1+1}}^{t+\tau_1} -\tilde{\ell}_m(i) \geq t\epsilon_1 \right) \leq e^{-\gamma_1 t}$ for all $t \geq n > -(1 + \epsilon) \log c/I^*(M,K)$. Hence, there exist $C_1 > 0$ and $\gamma_1 > 0$ such that $P_m(n^2 > n) \leq C_1 e^{-\gamma_1 n}$ for all $n > -(1 + \epsilon) \log c/I^*(M,K)$.

**Case 2** : $K < M$ and $I^*(M,K) = \frac{K \cdot D(f||g)}{M-1}$.

In this case, the cell with the highest index is not observed for all $n$. As a result, cell $m$ is not observed for all $n \geq \tau_1$ since $S_m(n) > S_j(n)$ for all $j \neq m$ for all $n \geq \tau_1$. Therefore, a similar argument as in Case 1 applies to cell $m$. Next, we focus on cell $j^* (\tau_1 + t) \neq m$ as in Case 2. Note that in this case $N_{j^* (\tau_1 + t)} (\tau_1 + t) \geq \frac{(K-1)D(f||g)}{M-1}$ (since $t$ observations are taken from cell $m$ and $(K-1)t$ observations are taken from $M-1$ cells for $j \neq m$). Thus, (82) can be developed for this case as well with minor modifications. Specifically, for any $\epsilon > 0$ there exists $\gamma_1 > 0$ such that

$$\sum_{i=\tau_{1+1}}^{\tau_1+t} \ell_{j^* (\tau_1 + t)} (i) 1_{j^* (\tau_1 + t)} (i) - \frac{D(f||g)}{I^*(M,K)} \log c$$

$$= \sum_{i=\tau_{1+1}}^{\tau_1+t} \tilde{\ell}_{j^* (\tau_1 + t)} (i) 1_{j^* (\tau_1 + t)} (i) - N_{j^* (\tau_1 + t)} (\tau_1 + t) D(f||g) \log c$$

$$\leq \sum_{i=\tau_{1+1}}^{\tau_1+t} -D(f||g) \left( 1 - \frac{(M-1)D(f||g)}{M-1} \right) \left( 1 - \frac{I^*(M,K)}{t} \right) \log c$$

(82)

$$\leq \sum_{i=\tau_{1+1}}^{\tau_1+t} \tilde{\ell}_{j^* (\tau_1 + t)} (i) 1_{j^* (\tau_1 + t)} (i) - t\epsilon_1.$$

(83)

Note that $N_{j^* (\tau_1 + t)} (\tau_1 + t) \geq \frac{Kt}{M-1}$ (since $Kt$ observations are taken from $M-1$ cells). Thus, for any $\epsilon > 0$ there exists $\epsilon_1 > 0$ such that:

$$\sum_{i=\tau_{1+1}}^{\tau_1+t} \ell_{j^* (\tau_1 + t)} (i) 1_{j^* (\tau_1 + t)} (i) - \frac{D(f||g)}{I^*(M,K)} \log c$$

$$= \sum_{i=\tau_{1+1}}^{\tau_1+t} \tilde{\ell}_{j^* (\tau_1 + t)} (i) 1_{j^* (\tau_1 + t)} (i) - N_{j^* (\tau_1 + t)} (\tau_1 + t) D(f||g) \log c$$

$$\leq \sum_{i=\tau_{1+1}}^{\tau_1+t} \tilde{\ell}_{j^* (\tau_1 + t)} (i) 1_{j^* (\tau_1 + t)} (i) - \frac{D(f||g)}{I^*(M,K)} \left( 1 - \frac{(M-1)D(f||g)}{M-1} \right) \log c$$

(82)

$$\leq \sum_{i=\tau_{1+1}}^{\tau_1+t} \tilde{\ell}_{j^* (\tau_1 + t)} (i) 1_{j^* (\tau_1 + t)} (i) - t\epsilon_1.$$

(83)
for all $t \geq n > -(1 + \epsilon) \log c/\ell^\ast(M, K)$.

The rest of the proof follows by applying the Chernoﬀ bound.

Definition 5: The dynamic range of $S_j(n)$ at time $t$ is defined as follows:

\begin{equation}
\text{DR}(t) = \max_{j \neq m} S_j(t) - \min_{j \neq m} S_j(t) .
\end{equation}

Lemma 9: Assume that the DGF policy is implemented. Then, for every fixed $\epsilon_1 > 0, \epsilon_2 > 0$ there exist $C > 0$ and $\gamma > 0$ such that

\begin{equation}
P_m(\text{DR}(t_2) > \epsilon_1 n) \leq C e^{-\gamma n}
\end{equation}

for all $m = 1, 2, \ldots, M$.

Proof: Note that

\begin{equation}
P_m(\text{DR}(t_2) > \epsilon_1 n) \leq P_m(t_2 > n) + P_m(\text{DR}(t_2) > \epsilon_1 n, t_2 \leq n)
\end{equation}

Since $t_2 = \tau_1 + n_2$, applying Lemmas 7, 8 implies that the first term on the RHS of (86) decreases exponentially with $n$ for all $n > -(1 + \epsilon_2) \log c/\ell^\ast(M, K)$ for every fixed $\epsilon_2 > 0$. It remains to show that the second term on the RHS of (86) decreases exponentially with $n$. Let $j = \arg \max_{j \neq m} S_j(t_2), \tilde{j} = \arg \min_{j \neq m} S_j(t_2)$. Let $t_0$ be the smallest integer such that $S_j(t) \leq S_j(t)$ for all $t_0 < t \leq t_2$. As a result, $\text{DR}(t_2) > \epsilon_1 n$ implies

\begin{equation}
\sum_{t=t_0}^{t_2} \tilde{\ell}_j 1_j(t) - \tilde{\ell}_2 1_2(t) > \epsilon_1 n .
\end{equation}

Note that the second term on the RHS of (86) can be rewritten as:

\begin{equation}
P_m(\text{DR}(t_2) > \epsilon_1 n, t_2 \leq n)
\end{equation}

\begin{equation}
P_m(\text{DR}(t_2) > \epsilon_1 n, t_2 \leq n, t_0 > 1)
\end{equation}

\begin{equation}
P_m(\text{DR}(t_2) > \epsilon_1 n, t_2 \leq n, t_0 < 1)
\end{equation}

Let $\mathcal{N} = \sum_{t=t_0}^{t_2} 1_j(t)$, $\overline{\mathcal{N}} = \sum_{t=t_0}^{t_2} 1_2(t)$.

First, we upper bound the first term on the RHS of (87). Note that for all $\tau_1 \leq t_0 < t \leq t_2$, if $1_j(t) = 1$ then $1_2(t) = 1$ since $S_m(t) > S_j(t)$ for all $t \geq \tau_1$ and the decision maker observes either the $K$ cells with the top $K$ highest sum LLRs or those with the second to the $(K+1)^{th}$ highest sum LLRs. Hence, $\overline{\mathcal{N}} \leq \mathcal{N}$. Thus,

\begin{equation}
\sum_{t=t_0}^{t_2} \tilde{\ell}_j 1_j(t) - \tilde{\ell}_2 1_2(t) = \sum_{t=t_0}^{t_2} [\tilde{\ell}_j 1_j(t) - \tilde{\ell}_2 1_2(t)] - D(f||g) (\mathcal{N} - \overline{\mathcal{N}})
\end{equation}

\begin{equation}
\leq \sum_{t=t_0}^{t_2} \tilde{\ell}_j 1_j(t) - \tilde{\ell}_2 1_2(t).
\end{equation}

Similar to (46), applying the Chernoﬀ bound completes the proof for this case.

Next, we upper bound the second term on the RHS of (87). Let $\epsilon_3 \equiv \frac{D(f||g)}{\ell^\ast(M, K)} > 0$. Note that

\begin{equation}
P_m(\text{DR}(t_2) > \epsilon_1 n, t_2 \leq n, t_0 < 1)
\end{equation}

\begin{equation}
P_m(\text{DR}(t_2) > \epsilon_1 n, t_2 \leq n, t_0 > 1)
\end{equation}

\begin{equation}
P_m(\text{DR}(t_2) > \epsilon_1 n, t_2 \leq n, t_0 < 1, t_1 \leq n)
\end{equation}

Thus, it remains to show that the second term on the RHS of (89) decreases exponentially with $n$. Note that $\text{DR}(t_2) > \epsilon_1 n$ implies

\begin{equation}
\sum_{t=t_0}^{t_2} [\tilde{\ell}_j 1_j(t) - \tilde{\ell}_2 1_2(t)] + D(f||g) (\mathcal{N} - \overline{\mathcal{N}})
\end{equation}

\begin{equation}
\leq \sum_{t=t_0}^{t_2} \tilde{\ell}_j 1_j(t) - \tilde{\ell}_2 1_2(t) + \epsilon_1 n
\end{equation}

for all $\tau_1 \leq \epsilon_3 n$.

As a result,

\begin{equation}
\sum_{t=t_0}^{t_2} [\tilde{\ell}_j 1_j(t) - \tilde{\ell}_2 1_2(t)] > \epsilon_1 n
\end{equation}

implies

\begin{equation}
\sum_{t=t_0}^{t_2} \tilde{\ell}_j 1_j(t) - \tilde{\ell}_2 1_2(t) > \epsilon_1 n
\end{equation}

for all $\tau_1 \leq \epsilon_3 n$.

Similar to (46), applying the Chernoﬀ bound completes the proof.

Definition 6: The dynamic range $\text{DR}_j(t)$ for $j \neq m$ at time $t$ is defined as follows:

\begin{equation}
\text{DR}_j(t) = \max_{k \neq m} S_k(t) - S_j(t) .
\end{equation}

Let

\begin{equation}
\eta \equiv \begin{cases}
\frac{D(f||g)}{\ell^\ast(M, K)} \log c , & \text{if } K = M \\
\log c , & \text{if } K < M \text{ and } I^\ast(M, K) = K \frac{D(f||g)}{M-1} \\
\frac{D^\ast(f||g)}{\ell^\ast(M, K)} \log c , & \text{if } K < M \text{ and } I^\ast(M, K) = D(g||f) + \frac{(K-1)D(f||g)}{M-1}
\end{cases}
\end{equation}

for all $\tau_1 \leq \epsilon_3 n$. 

Similar to (46), applying the Chernoﬀ bound completes the proof.
where $D'(f||g)$ is defined in (74).

**Definition 7:** $\tau_j^3$ denotes the smallest integer such that 
\[ \sum_{i=1}^{n} \ell_j(i) I_j(i) \leq \eta + DR_j(\tau_1) \] for $j \neq m$ for all $n \geq \tau_j^3 \geq \tau_2$.

**Remark 1:** Note that the total detection time is upper bounded by $\tau \leq \max_{j \neq m} \left( \tau_j^3 \right)$.

**Definition 8:** $n_3 \triangleq \tau - \tau_2$ denotes the total amount of time between $\tau_2$ until the search is terminated.

**Lemma 10:** Assume that the DGF policy is implemented. Then, for every fixed $\epsilon > 0$ there exist $C > 0$ and $\gamma > 0$ such that 
\[ P_m(n_3 > n) \leq Ce^{-\gamma n} \quad \forall n > -\epsilon \log c/I^*(M,K), \quad (95) \]
for all $m = 1, 2, \ldots, M$.

**Proof:** Let $N_j^3$ for $j \neq m$ denotes the total number of observations, taken from cell $j$ between $\tau_2$ and $\tau_j^3$. Note that $n_3 \leq \sum_{j \neq m} N_j^3$. Thus, it suffices to show that $P_m(\tau_j^3 > n)$ decreases exponentially with $n$. Note that 
\[ P_m(\tau_j^3 > n) \leq P_m(DR(\tau_2) > n \frac{D(f||g)}{2}) + P_m(\tau_j^3 > n | DR(\tau_2) \leq n \frac{D(f||g)}{2}). \quad (96) \]

By Lemma 9, the first term on the RHS of (96) decreases exponentially with $n$ for all $n > -\epsilon \log c/I^*(M,K)$. Thus, it remains to show that the second term on the RHS of (96) decreases exponentially with $n$.

Let $t_1, t_2, \ldots$ denote the time indices when cell $j$ is observed between $\tau_2$ and $\tau_j^3$. Since $\tau_2$ has occurred and $DR(\tau_2) \leq n \frac{D(f||g)}{2}$, $\tau_j^3$ occurs once $\sum_{i=1}^{r} -\ell_j(t_i) \geq n \frac{D(f||g)}{2}$ holds for all $r \geq N_j^3$. As a result, 
\[ P_m(\tau_j^3 > n | DR(\tau_2) \leq n \frac{D(f||g)}{2}) \leq P_m(\inf_{r > n} \sum_{i=1}^{r} -\ell_j(t_i) < n \frac{D(f||g)}{2}) \leq \sum_{r=n}^{\infty} P_m(\sum_{i=1}^{r} \tilde{\ell}_j(t_i) > r \frac{D(f||g)}{2}). \quad (97) \]

Thus, it suffices to show that there exists $\gamma > 0$ such that 
\[ P_m(\sum_{i=1}^{n} \tilde{\ell}_j(t_i) > n \frac{D(f||g)}{2}) \leq e^{-\gamma n}. \]
Applying the Chernoff bound and using the i.i.d. property of $\tilde{\ell}_j(t_i)$ completes the proof.

**Lemma 11:** Assume that the DGF policy is implemented. Then, the expected search time is upper bounded by: 
\[ E_m(\tau) \leq -(1 + o(1)) \frac{\log c}{I^*(M,K)}, \quad (98) \]
for $m = 1, \ldots, M$.

**Proof:** Note that $\tau \leq \tau_1 + n_2 + n_3$. Fix $\epsilon_1 > 0$ and let $N_1 = -\epsilon_1 \log c/I^*(M,K)$. Applying Lemma 7 yields: 
\[ E_m(\tau_1) \leq N_1 + \sum_{n_2=0}^{\infty} n_2 P_m(\tau_1) \]
\[ \rightarrow -c_1 \frac{\log c}{I^*(M,K)} \quad \text{as} \quad c \rightarrow 0. \]

Fix $\epsilon_2 > 0$ and let $N_2 = - (1 + \epsilon_2) \log c / I^*(M,K)$. Applying Lemma 8 yields: 
\[ E_m(n_2) \leq N_2 + \sum_{n_3=0}^{\infty} n_2 P_m(n_2) \]
\[ \rightarrow - (1 + \epsilon_2) \log c / I^*(M,K) \quad \text{as} \quad c \rightarrow 0. \]

Fix $\epsilon_3 > 0$ and let $N_3 = -\epsilon_3 \log c / I^*(M,K)$. Applying Lemma 10 yields: 
\[ E_m(n_1) \leq N_3 + \sum_{n_3=0}^{\infty} n_3 P_m(n_3) \]
\[ \rightarrow -\epsilon_3 \frac{\log c}{I^*(M,K)} \quad \text{as} \quad c \rightarrow 0. \]

Since (99), (100), (101) hold for any $\epsilon_1 > 0, \epsilon_2 > 0, \epsilon_3 > 0$, respectively, then (98) follows.

Combining Lemmas 4, 11 and Theorem 5 yields the asymptotic optimality property of the DGF policy, presented in Theorem 2.

**B. Proof of Theorem 1**

Following the same argument as in [4], it suffices to show that $P_m(\tau_1 > n)$ decreases polynomially with $n$ to prove the theorem. Since $D(f||g) > 0$, the KL divergence between the true hypothesis $m$ and any false hypothesis $j \neq m$ is strictly positive under any observed cell. In the case where $D(f||g) \geq D(f||g)/(M - 1)$, the Chernoff test selects $m^*(n)$ for all $n$. As a result, exponential decay of $P_m(\tau_1 > n)$ follows directly from Lemma 7. In the case where $D(f||g) < D(f||g)/(M - 1)$, the Chernoff test selects $m^*(n)$ for $j \neq 1$ randomly for all $n$. As a result, polynomial decay of $P_m(\tau_1 > n)$ follows by a similar argument as in [4] for the extended Chernoff test.

**C. Proof of Theorem 3**

The proof follows a similar line of arguments as in the proof of Theorem 2. Hence, we provide here only a sketch of the proof. First, similar to Lemma 4, it can be verified that declaring the target locations once $S_m(L) - S_m(L+1) \geq -\log c$ occurs achieves an error probability $O(c)$. Second, similar to Lemma 11, it can be verified that the detection time approaches $-\log c / I^*(M,K)$. For example, if $\frac{D(f||g)}{E} \geq \frac{D(f||g)}{M - L}$ and $K \geq L$ then all the $L$ targets and a fraction $r = \frac{K - L}{K}$ of the false hypotheses are observed at each given time in the asymptotic regime. Therefore, the detection time approaches $-\log c / D(f||g) / I^*(M,K)$. Similar arguments apply to the rest of the cases.
D. Proof of Theorem 4

The proof follows a similar line of arguments as in the proofs of Theorems 2 and 3. Hence, we provide only a sketch of the proof. With minor modifications to Theorem 3, it can be verified that (29) is the asymptotic lower bound on the Bayes risk when the number of targets \( \ell \) is known, \( K = 1 \) and (23) holds. Similar to Lemma 4, it can be verified that declaring a target once \( S_m(n) \geq -\log c \) occurs achieves an error probability \( O(c) \). Since the decision maker declares the target locations once \( S_m(n) \geq -\log c \) for any \( m \), the Bayes risk approaches (29) as \( c \rightarrow 0 \).

REFERENCES