A polynomial-time algorithm for the paired-domination problem on permutation graphs

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Abstract

A set $S$ of vertices in a graph $H = (V, E)$ with no isolated vertices is a paired-dominating set of $H$ if every vertex of $H$ is adjacent to at least one vertex in $S$ and if the subgraph induced by $S$ contains a perfect matching. Let $G$ be a permutation graph and $\pi$ be its corresponding permutation. In this paper we present an $O(mn)$ time algorithm for finding a minimum cardinality paired-dominating set for a permutation graph $G$ with $n$ vertices and $m$ edges.

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1 Introduction

In this paper we in general follow [14] for notation and graph theory terminologies. Specifically, let $G = (V, E)$ be a graph with vertex set $V$ and edge set $E$, and let $v$ be a vertex in $V$. The order of $G$ is given by $n = |V|$ and its size by $m = |E|$. The open neighborhood of $v$ is defined as...
by $N(v) = \{u \in V \mid uv \in E\}$ and the closed neighborhood of $v$ is defined by $N[v] = N(v) \cup \{v\}$.

In general, let $N(S)$ and $N[S]$ denote, respectively, $\cup_{v \in S} N(v)$ and $\cup_{v \in S} N[v]$. For subsets $S,T \subseteq V$, the set $S$ dominates the set $T$ in $G$ if $N[T] \subseteq N[S]$. Each vertex $v$ of $G$ dominates itself and every vertex adjacent to $v$, i.e., all vertices in its closed neighborhood. For $S \subseteq V$, let $(S)$ denote the subgraph of $G$ induced by $S$.

A set $S \subseteq V$ is a dominating set of $G$ if every vertex not in $S$ is adjacent to at least a vertex in $S$. The domination number of $G$ is the minimum cardinality of a dominating set of $G$. A matching in a graph $G$ is a set of independent edges in $G$. A perfect matching $M$ in $G$ is a matching in $G$ such that every vertex of $G$ is incident to a vertex of $M$.

A paired-dominating set of a graph $G$ is a set $S$ of vertices of $G$ such that every vertex is adjacent to some vertex in $S$ and the subgraph induced by $S$ contains a perfect matching $M$ (not necessarily induced). Two vertices joined by an edge of $M$ are said to be paired and are also called partners in $S$. Every graph without isolated vertices has a paired-dominating set since the end-vertices of any maximal matching form such a set. The paired-domination number of $G$, denoted by $\gamma_{pr}(G)$, is the minimum cardinality of a paired-dominating set. The minimum paired-dominating set problem, abbreviated as MPDS, is to find a paired-dominating set $S$ of $G$ such that $|S|$ is minimized. Paired-domination was introduced by Haynes and Slater [14] as a model for assigning backups to guards for security purposes, and has been studied from the theoretic point of view, for example, in [2]–[4], [7, 8, 10, 11], [15]–[19], [21], [25]–[27], [29], among others.

The aim of this paper is to investigate the problem of determining $\gamma_{pr}(G)$ for a permutation graph $G$ from the algorithmic point of view. The decision problem to determine a minimum cardinality paired-dominating set of an arbitrary graph has been known to be NP-complete (see [13]). For the special case of trees, Qiao et al. [26] presented a linear time algorithm. Cheng et al. [8] proposed an $O(m + n)$ and $O(m(m + n))$ time algorithms to solve the MPDS problem for interval graphs and circular-arc graphs, respectively. The literature on algorithmic aspects of domination in graphs has been by surveyed and detailed by Chang [5].
Let $\pi = [\pi_1, \pi_2, \ldots, \pi_n]$ be a permutation on the set $V_n = \{1, 2, \ldots, n\}$. Then the permutation graph $G[\pi] = (V, E)$ is the undirected graph such that $V = V_n$ and $(i, j) \in E$ if and only if

$$(i - j)(\pi^{-1}(i) - \pi^{-1}(j)) < 0,$$

where $\pi^{-1}(i)$ is the position of $i$ in $\pi = [\pi_1, \pi_2, \ldots, \pi_n]$. Throughout the paper, we assume that the input is a permutation $\pi = [\pi_1, \pi_2, \ldots, \pi_n]$, and the given permutation graph $G$ contains no isolated vertices.

A permutation graph is an intersection graph based upon the permutation diagram [1], which is defined as follows: Write the number 1, 2, $\ldots$, $n$ horizontally from left to right. Under every $i$, write the number $\pi(i)$. Draw line segments connecting $i$ in the top row and $i$ in the bottom row, for each $i$. It is easy to see that two vertices $i$ and $j$ of $G[\pi]$ are adjacent if and only if the corresponding line segments of $i$ and $j$ intersect. Fig. 1 shows the permutation graph $G[\pi]$ where its corresponding permutation diagram of a permutation $\pi[3, 1, 5, 7, 4, 2, 6]$. The permutation graphs are known to have a variety of practical applications [12, 24] and for this reason, many algorithms for determining parameters in graph theory have been developed in the literature [6, 9, 20, 22, 23, 28, 30].

In this paper, we propose an efficient $O(mn)$ algorithm for solving the MPDS problem on permutation graphs. Our algorithm is based on a recursive formula by using the dynamic programming method. In Section 2, we describe our recursive formula of the dynamic programming. Our algorithm is described in Section 3. Section 5 contains some conclusions.

2 A dynamic programming approach

In this section we shall describe our basic approach based upon the dynamic programming approach. Essentially, we want to find an MPDS of $\{\pi_1, \pi_2, \ldots, \pi_n\}$ dominating $\{1, 2, \ldots, n\}$. In the following, we may assume that the permutation graph $G[\pi]$ discussed below is connected; otherwise we look at each (connected) component separately.

For convenience, we introduce more notation as follows:
(1) For any $1 \leq i, j \leq n$, and $V_i = \{\pi_1, \pi_2, \ldots, \pi_i\}$, denote $V_{i,j}$ as the subset of $V_i$ containing all elements smaller than or equal to $j$, i.e., $V_{i,j} = \{\pi_k \in V_i \mid \pi_k \leq j\}$. Clearly, $V_{i,j} \subseteq V_i$.

(2) For each $i, 1 \leq i \leq n$, denote $\pi_i^*$ as the minimum number over the suffix $\pi_i, \pi_{i+1}, \ldots, \pi_n$, i.e., $\pi_i^* = \min\{\pi_i, \pi_{i+1}, \ldots, \pi_n\}$, and set $V_i^* = V_i \cup \{\pi_i^*\}$.

(3) For any vertex set $S$, define $\max(S)$ as the maximum number in $S$.

(4) For a family $\mathcal{F}$ of sets of vertices, $\min(\mathcal{F})$ denotes a minimum cardinality set in $\mathcal{F}$ and $\max(S)$ is as large as possible if $\mathcal{F}$ is not the empty set; $\min(\mathcal{F})$ denotes a set of infinite cardinality otherwise. $\min(\mathcal{F})$ may not be unique. If there are more than one candidate for $\min(\mathcal{F})$, we select arbitrarily one of the candidates.

**Lemma 1** For a permutation graph $G[\pi]$ with no isolated vertices, $\langle V_i^* \rangle$ has no isolated vertices for each $i$, $1 \leq i \leq n$.

**Proof.** Suppose to the contrary that there exists an $i_0$ ($1 \leq i_0 \leq n$) such that $\langle V_{i_0}^* \rangle$ has an isolated vertex $\pi_l$ ($l \leq i_0$). Then $\pi_l \leq \pi_{i_0}^*$, for otherwise $(\pi_l, \pi_{i_0}^*) \in E(G)$. If $\pi_l = \pi_{i_0}^*$ ($=\min\{\pi_{i_0}, \pi_{i_0+1}, \ldots, \pi_n\}$), then $\pi_l = \pi_{i_0}$. Hence, $\pi_{i_0}$ is an isolated vertex in $G$, contradicting the assumption of the lemma. If $\pi_l < \pi_{i_0}^*$, then $\pi_l = l$. Thus, for $1 \leq i < l$, $\pi_i < l$, and for $l < i \leq n$, $\pi_i > l$. This implies that $\pi_l$ is an isolated vertex in $G$, contradicting our assumption again. □

By Lemma 1, we see that $\langle V_i^* \rangle$ has no isolated vertices, so it is clear that for each $i$ and $j$, $1 \leq i, j \leq n$, there exists a subset $D$ of $V_i^*$ such that $D$ dominates all the vertices of $V_{i,j}$ and $\langle D \rangle$ has a perfect matching in $\langle V_i^* \rangle$.

Based on Lemma 1, for each $i$ and $j$, $1 \leq i, j \leq n$, we define $PD_{i,j}$ as follows:

(i). $PD_{i,j}$ is a minimum cardinality subset $S$ of $V_i^*$ such that $S$ is a dominating set of $\langle V_{i,j} \rangle$ and $\langle S \rangle$ has a perfect matching in $\langle V_i^* \rangle$;

(ii). $\max(PD_{i,j})$ is as large as possible.
In particular, we define $PD_{0,j} = \emptyset$ for $1 \leq j \leq n$. Clearly, $PD_{n,n}$ is a desired minimum cardinality paired-dominating set for $G[\pi]$.

We define $X = \{ S : S \subseteq V_i^* \text{ such that } S \text{ is a dominating set of } \langle V_i \rangle \text{ and } \langle S \rangle \text{ has a perfect matching in } \langle V_i^* \rangle \}$. and we further partition $X$ into three subsets: $X_1 = \{ S \in X : \pi_i^* \in S, \pi_i \in S \}$ and $X_3 = \{ S \in X : \pi_i^* \notin S, \pi_i \notin S \}$.

Following the above definitions, we have

$$PD_{i,j} = \begin{cases} \emptyset & \text{if } V_{i,j} = \emptyset, \\ \text{Min}(X) & \text{otherwise} \end{cases}$$

Consider the case $i = 1$. If $j < \pi_1$, then $V_{1,j} = \{ \pi_1 \} \cap \{ 1, 2, \ldots, j \} = \emptyset$, and so $PD_{1,j} = \emptyset$. Otherwise, $V_{1,j} = \{ \pi_1 \}$. According to our assumption that $G$ contains no isolated vertices, we have $\pi_1 \neq 1$. Then $\pi_i^* = 1$ and $V_i^* = \{ 1, \pi_1 \}$. Hence $PD_{1,j} = \{ 1, \pi_1 \}$. So we obtain

$$PD_{1,j} = \begin{cases} \emptyset & \text{if } j < \pi_1, \\ \{ 1, \pi_1 \} & \text{otherwise} \end{cases}$$

We first give several basic lemmas that will be useful for the proof of our recursive formula $PD_{i,j}$.

**Lemma 2** (Chao et al. [6]) For positive integers $i_1, i_2$ and $j$, if $1 \leq i_1 < i_2 \leq n$ and $1 \leq j \leq n$, then $V_{i_1,j} \subseteq V_{i_2,j}$ and $V_{i_1}^* \subseteq V_{i_2}^*$.

**Lemma 3** For $1 \leq i < j < k \leq n$ and $\pi_k < \pi_j < \pi_i$, if $w$ is adjacent to $\pi_j$, then $w$ is adjacent to at least one of $\pi_k$ and $\pi_i$.

**Proof.** The proof is straightforward and omitted. \(\Box\)

**Lemma 4** For $1 < l \leq i$, there exists a $PD_{l-1,\pi_i^*}$ such that $\pi_i^* \notin PD_{l-1,\pi_i^*}$.

**Proof.** Let $S$ be a $PD_{l-1,\pi_i^*}$. Thus $S \subseteq V_{l-1}^*$ is a dominating set of $(V_{l-1,\pi_i^*})$ and $(S)$ has a perfect matching in $(V_{l-1}^*)$. If $\pi_i^* \notin S$, then the desired result follows. If $\pi_i^* \in S$, then $\pi_i^* = \pi_i^*_{i-1}$
as \( S \subseteq V_{l-1}^* \). Hence, there exists a vertex \( \pi_{i'} \in S \) \((i' \leq l - 1)\) such that \( \pi_i^*, \pi_{i'} \) are paired in \( S \). So, we have \( \pi^{-1}(\pi_i^*) > i' \) and \( (\pi^{-1}(\pi_i^*) - i')(\pi_i^* - \pi_{i'}) < 0 \). Thus \( \pi_{i'} > \pi_i^* \). We claim that \( N(\pi_{i'}) \cap V_{l-1}^* - S \neq \emptyset \). If this is not so, then \( \pi_{i'} \) dominates no vertices of \( V_{l-1, \pi_i^*} \), and so does \( \pi_i^* \) as \( \pi_{i'} > \pi_i^* \). This means that \( S - \{\pi_i^*, \pi_i^*\} \subseteq V_{l-1}^* \) is a dominating set of \( V_{l-1, \pi_i^*} \) and \( \langle S - \{\pi_i^*, \pi_i^*\} \rangle \) has a perfect matching in \( \langle V_{l-1}^* \rangle \). Thus \( S - \{\pi_i^*, \pi_i^*\} \) is a \( PD_{l-1, \pi_i^*} \), which contradicts the minimality of \( S \). Let \( \pi_{i'} \in N(\pi_{i'}) \cap V_{l-1}^* - S \) and \( S' = S \cup \{\pi_{i'}\} - \{\pi_i^*\} \). Then \( S' \subseteq V_{l-1}^* \) is a dominating set of \( \langle V_{l-1, \pi_i^*} \rangle \) and \( \langle S' \rangle \) has a perfect matching in \( \langle V_{l-1}^* \rangle \) with \(|S'| = |S| \) and \( \max(S') \geq \max(S) \). So \( S' \) is a \( PD_{l-1, \pi_i^*} \), satisfying \( \pi_i^* \notin S' \), as required. \( \square \)

For \( 1 < i \leq n \), we define

\[
PD_{i} = \text{Min}(\{PD_{i-1, \pi_i^*} \cup \{\pi_i^*, \pi_i\} : \pi_i \in N(\pi_i^*), \pi_i^* \notin PD_{i-1, \pi_i^*}, l \leq i\})
\]

and

\[
PD_{\text{max}} = \begin{cases} 
PD_{i-1,j} \cup \{\pi_i, \max(V_i)\} & \text{if } \pi_i \neq \max(V_i), \\
V_i & \text{otherwise.}
\end{cases}
\]

By Lemma 4, \( PD_{i} \neq \emptyset \). The following Lemmas 5 and 6 assert that \( PD_{i} \) and \( PD_{\text{max}} \) (if \( \max(V_i) \neq \pi_i \) and \( \max(PD_{i-1,j}) < \pi_i \)) are candidates for computing \( PD_{i,j} \).

**Lemma 5** For any integers \( i \) and \( j \), \( 1 < i \leq n \) and \( 1 \leq j \leq n \), \( PD_{i} \in X_1 \subseteq X \).

**Proof.** By the definition of \( PD_{i}, \pi_i^* \notin PD_{i-1, \pi_i^*} \), while \( PD_{i-1, \pi_i^*} \) is a minimum dominating set of \( \langle V_{l-1, \pi_i^*} \rangle \). We claim \( \pi_l \notin PD_{i-1, \pi_i^*} \). If this is not the case, then it is easy to see that \( \pi_l = \pi_{l-1} \leq \pi_i^* \). On the other hand, since \( \pi_l \in N(\pi_i^*) \) \((l \leq i)\), \( \pi_l > \pi_i^* \), which is impossible.

From Lemma 2, \( V_{l-1}^* \subseteq V_i^* \) as \( l \leq i \). Hence, \( PD_{i-1, \pi_i^*} \cup \{\pi_i^*, \pi_l\} \subseteq V_i^* \). We next show that each vertex of \( V_{i,j} - V_{l-1, \pi_i^*} \) is dominated by \( \pi_i^* \) or \( \pi_l \). Let \( \pi_k \in V_{i,j} - V_{l-1, \pi_i^*} \). If \( \pi_k > \pi_i^* \), then

\[
(k - \pi_i^*)(l - \pi_k) < 0,
\]

and \( l \leq i \), \( \pi_i > \pi_i^* \), then \( \pi_i > \pi_i^* > \pi_k \). This implies that \( (\pi_k - \pi_l)(k - l) \leq 0 \), i.e., \( \pi_k = \pi_l \) or

\[
\pi_k, \pi_l \in E. \text{ Hence, all the vertices in } V_{i,j} \text{ are dominated by } PD_{i-1, \pi_i^*} \cup \{\pi_i^*, \pi_l\}. \text{ Therefore, } PD_{i-1, \pi_i^*} \cup \{\pi_i^*, \pi_l\} \in X_1. \text{ Note that } PD_{i} = \text{Min}(\{PD_{i-1, \pi_i^*} \cup \{\pi_i^*, \pi_l\} : \pi_l \in N(\pi_i^*), l \leq i\}), \text{ so } PD_{i} \in X_1 \text{, as desired. } \square
Lemma 6 For any integers $i$ and $j$, $1 < i \leq n$ and $1 \leq j \leq n$, if $\max(V_i) \neq \pi_i$ and $\max(PD_{i-1,j}) < \pi_i$, then $PD_{\max} \in X$.

Proof. Clearly, $PD_{\max} \subseteq V_i^*$. Since $\max(V_i) \neq \pi_i$ and $\max(PD_{i-1,j}) < \pi_i$, $\pi_i \not\in PD_{i-1,j}$ and $\pi_i < \max(V_i)$, and thus $\max(V_i) \notin PD_{i-1,j}$ and $(\max(V_i), \pi_i) \in E$. Note that $V_{i,j} - V_{i-1,j} \subseteq \{\pi_i\}$, and we have $PD_{\max} = PD_{i-1,j} \cup \{\pi_i, \max(V_i)\}$ as a dominating set of $\langle V_{i,j} \rangle$ and $\langle PD_{\max} \rangle$ has a perfect matching in $\langle V_i^* \rangle$, the desired result follows. □

In order to present the recursive formula of $PD_{i,j}$ for the case of $1 < i \leq n$, we further prove the following several lemmas.

Lemma 7 For each $S \in \Min(X_1)$, let $\pi_l = \max(S)$. Then $\pi_i^* < \pi_l$ and $\pi_l \in N(\pi_i^*)$.

Proof. By the definition of $X_1$, we have $\pi_i^* \in S$. Suppose $\pi_i^* \geq \pi_l$, then $\max(S) = \pi_i^*$. This implies that $\pi_i^*$ is an isolated vertex of $\langle S \rangle$, which contradicts the assumption that $\langle S \rangle$ has a perfect matching in $\langle V_i^* \rangle$. So $\pi_i^* < \pi_l$. Furthermore, since $(\pi_l - \pi_i^*)l - \pi^{-1}(\pi_i^*) < 0$, $(\pi_i^*, \pi_l) \in E$, and thus $\pi_l \in N(\pi_i^*)$. □

By the definition of $\Min(X_1)$, all the candidates $S$ for $\Min(X_1)$ have the same $\max(S)$. Let $S \in \Min(X_1)$, $\pi_l = \max(S)$ and let $M$ be a perfect matching in $\langle S \rangle$.

Lemma 8 For any integers $i$ and $j$, $1 < i \leq n$ and $1 \leq j \leq n$, if there exist $\pi_{i_1}$ $(i_1 < l)$ and $\pi_{i'}$ such that $(\pi_i^*, \pi_{i_1}) \in M$ and $(\pi_l, \pi_{i'}) \in M$, then $\Min(X_1 \cup \{PD_{\pi_i^*}\}) = PD_{\pi_i^*}$.

Proof. By Lemma 5, it suffices to show that there exits an $S^* \in PD_{\pi_i^*} \cap X_1$ such that $\max(S^*) \geq \max(S) = \pi_l$. Note that $\max(S) = \pi_l > \pi_{i'} \in S$ and $(\pi_l, \pi_{i'}) \in M$, so $l' > l$. We distinguish the following two cases depending on whether or not $\pi_{l-1}^*$ is equal to $\pi_i^*$.

Case 1. Suppose first $\pi_{l-1}^* = \pi_i^*$. In this case, we claim that $N(\pi_{i_1}) \cap V_l - S \neq \emptyset$. Otherwise, since $\pi_i^* < \pi_{i'} < \pi_l$ and $l < l' < \pi^{-1}(\pi_i^*)$, by Lemma 3, each vertex dominated by $\pi_{i'}$ in $G$ is adjacent to $\pi_l$ or $\pi_i^*$. Furthermore, for each $t > l$, $\pi_t \in V_{i,j}$, it is dominated by $\pi_i^*$ as $\pi_t > \pi_i^*$ ($= \pi_{l-1}^*$). This implies that $S - \{\pi_{i_1}, \pi_{i'}\}$ is a dominating set of $\langle V_{i,j} \rangle$ and $\langle S - \{\pi_{i_1}, \pi_{i'}\} \rangle$ has
a perfect matching \( M \cup \{(\pi^*_i, \pi_i)\} \) in \( V^*_i \) by making a pair of \( \pi_l \) and \( \pi^*_l \), contradicting the minimality of \( S \). Let \( \pi^*_{i'} \in N(\pi_{i'}) \cap V_l \setminus S \) and let \( S_1 = S \cup \{\pi^*_{i'}\} \setminus \{\pi_l\} \). Then \( S_1 \subseteq V^*_i \) is a dominating set of \( V_i,j \) and \( M_1 = (M \cup \{(\pi^*_{i'}, \pi_{i}), (\pi_i, \pi^*_l)\}) \setminus \{(\pi^*_l, \pi_{i}), (\pi_l, \pi^*_l)\} \) is a perfect matching in \( S_1 \). So \( S_1 \subseteq X_1 \) with \( |S_1| = |S| \) and \( \text{max}(S_1) \geq \text{max}(S) \) such that \( \pi_l \notin S_1 \) and \( \pi^*_l \notin S_1 \).

For any \( \pi_k \in S_1 \), where \( l < k \leq i \), there exists \( \pi_{k'} \) such that \( (\pi_k, \pi_{k'}) \in M_1 \). We claim that \( k' < l \) and \( N(\pi_{k'}) \cap V_l - S_1 \neq \emptyset \). Indeed, if \( k' > l \), then for each vertex \( \pi_t \in N(\{\pi_k, \pi_{k'}\}) \cap V_l - S_1 \), we have \( \pi_t > \pi_k > \pi^*_l \), or \( \pi_t > \pi_{k'} > \pi^*_l \). Since \( \pi_t \) is dominated by \( \pi^*_l \), Moreover, note that for each vertex \( \pi_t \in V_l,j \), \( l < t \leq i \), it is also dominated by \( \pi^*_l \) as \( \pi_t \geq \pi^*_l \) (\( = \pi^*_t \)). This implies that \( S_1 - \{\pi_k, \pi_{k'}\} \) is a dominating set of \( V_i,j \) and \( S_i - \{\pi_k, \pi_{k'}\} \) still has a perfect matching in \( V^*_i \), which contradicts the minimality of \( S_1 \). So \( k' < l \). We further show that \( N(\pi_{k'}) \cap V_l - S_1 \neq \emptyset \). Otherwise, since \( k' < l < k \) and \( (\pi_k, \pi_{k'}) \in E \), \( \pi_{k'} > \pi_k > \pi^*_l = \pi^*_l \), then \( \pi_{k'} \) is dominated by \( \pi^*_l \). As above, we deduce that \( S_1 - \{\pi_k, \pi_{k'}\} \) is a dominating set of \( V_i,j \) and \( S_1 - \{\pi_k, \pi_{k'}\} \) has a perfect matching in \( V^*_i \), a contradiction. Let \( \pi_{k''} \in N(\pi_{k'}) \cap V_l - S_1 \) and let \( S_2 = S_1 \cup \{\pi_{k''}\} \setminus \{\pi_k\} \). Then \( S_2 \subseteq V^*_i \) is a dominating set of \( V_i,j \) with \( |S_2| = |S_1| \) and \( \{\pi_{k''}\} \) has a perfect matching in \( V^*_i \) and \( \text{max}(S_2) \geq \text{max}(S_1) \). For any \( \pi_s \in S_2 \), where \( l < k \leq i \), continuing the process as above, we can obtain after a finite number of steps a set \( S^* \subseteq V^*_i \) satisfying the following conditions:

(i). \( S^* \cap \{(\pi_{l+1}, \pi_{l+2}, \ldots, \pi_i) - \{\pi^*_l\}\} = \emptyset \);

(ii). \( S^* \subseteq V^*_i \) is a dominating set of \( V_i,j \) with \( |S^*| = |S| \) and \( \text{max}(S^*) \) in \( V^*_i \) has a perfect matching in which \( \pi^*_l \) and \( \pi_l \) are paired;

(iii). \( \text{max}(S^*) \geq \text{max}(S) \).

Then \( S^* \subseteq X_1 \). Since \( \pi^*_l < \pi_l \), it follows that no vertex in \( V_{l-1, \pi^*_l} \) is dominated by \( \pi^*_l \) or \( \pi_l \), so \( S^* - \{\pi^*_l, \pi_l\} \) is a dominating set of \( V_{l-1, \pi^*_l} \) and \( \text{max}(S^* - \{\pi^*_l, \pi_l\}) \subseteq V^*_i \) has a perfect matching. By the minimality of \( S^* \), we deduce that \( S^* - \{\pi^*_l, \pi_l\} \subseteq V^*_i \) is a minimum cardinality dominating set of \( V_{l-1, \pi^*_l} \) and contains a perfect matching. Then \( S^* - \{\pi^*_l, \pi_l\} \) is a \( PD_{l-1, \pi^*_l} \), and thus \( S^* \) is a \( PD_{\pi^*_l} \). Hence, \( |S| = |S^*| = |PD_{l-1, \pi^*_l}| + 2 \). Note that \( |PD_{\pi^*_l}| \leq \)
Case 2. Suppose \( \pi_{l-1}^* \neq \pi_i^* \). As in Case 1, we first find a set \( S_1 \in X_1 \) with \( |S_1| = |S| \) and max(\( S_1 \)) \( \geq \) max(\( S \)) such that \( \pi' \not\in S_1 \) and \( \pi_{l-1}^* \in S_1 \).

Suppose \( \pi_{l-1}^* \not\in S \). Since \( \pi_{l-1}^* < \pi_i^* < \pi_{i_1} \), \( (\pi^{-1}_{i_1} - \pi^{-1}_{l-1}((\pi_{i_1} - \pi_{l-1})) < 0 \), then \( (\pi_{i_1}, \pi_{l-1}^*) \in E \). Let \( S_1 = S \cup \{\pi_{l-1}^*, \pi_{i_1}\} \). Clearly, \( S_1 \subseteq V_i^* \). We further show that \( S_1 \) is a dominating set of \( (V_{i,j}) \). It suffices to show that all the vertices dominated by \( \pi' \) can be dominated by \( S_1 \). Indeed, let \( \pi_i \in N(\pi') \). If \( t > l \), it follows from \( \pi_i > \pi_i^* \) that \( \pi_i < \pi_i \) or \( \pi_i > \pi_i^* \). Observe that \( \pi_i < \pi_i \) and \( l < l' \leq i \leq \pi_i^* \), then \( \pi_i \) is dominated by \( \pi_i \) or \( \pi_i^* \). If \( t < l (l < l') \), then \( \pi_i > \pi_i^* \) or \( \pi_i \) as above. So \( S - \{\pi_i', \pi_i\} \) is a dominating set of \( (V_{i,j}) \).

Further, since \( \pi_{i_1} > \pi_i^* > \pi_{l-1}^*, (\pi_{l-1}^*, \pi_{i_1}) \in E \), then \( S - \{\pi_i', \pi_i\} \) has a perfect matching in \( (V_i^*) \) by making pairs of \( \pi_i \) and \( \pi_i^* \), \( \pi_{l-1}^* \) and \( \pi_{i_1} \), which contradicts the minimality of \( S \). Let \( \pi_i \in N(\pi_{i_1}) \cap V_i - S \) and let \( S_1 = S \cup \{\pi_i'\} \). Then \( S_1 \) is a dominating set of \( (V_{i,j}) \) and \( M_1 = M \cup \{(\pi_{i_1}, \pi_i'), (\pi_i, \pi_i^*), (\pi_{i_1}^*, \pi_i)\} \) is a perfect matching in \( (S_1) \). So \( S_1 \in X \) and max(\( S_1 \)) \( \geq \) max(\( S \)) such that \( \pi' \not\in S_1 \) and \( \pi_{l-1}^* \in S_1 \).

For any \( \pi_k \neq \pi_{l-1}^*, \pi_k \in S_1 \), where \( l < k \leq i \), there exists a \( \pi_k' \in S_1 \) such that \( (\pi_k, \pi_k') \in M_1 \).

We claim that \( k' < l \) and \( N(\pi_k') \cap V_i - S \) \( \neq \) \( \emptyset \). In fact, if \( k' > l \), then for each vertex \( \pi_i \in N(\{\pi_k, \pi_k'\}) \cap V_i - S \), we have \( \pi_i > \pi_k > \pi_{l-1}^* \) or \( \pi_i > \pi_k > \pi_{l-1}^* \), so \( \pi_i \) is dominated by \( \pi_{l-1}^* \). Moreover, for each vertex \( \pi_i \in V_{i,j}, l < t \leq i \), we have \( \pi_i > \pi_i \) or \( \pi_i > \pi_i > \pi_i^* \), so \( \pi_i \) is dominated by \( \pi_i^* \) or \( \pi_i \). This implies that \( S_1 - \{\pi_k, \pi_k'\} \) is a dominating set of \( (V_{i,j}) \) and \( (S_1 - \{\pi_k, \pi_k'\}) \) still has a perfect matching in \( (V_i^*) \), which contradicts the minimality of \( S_1 \).
So $k' < l$. Similar to the discussion in Case 1, we can deduce that $N(\pi_{k'}) \cap V_l - S_1 \neq \emptyset$.

Let $\pi_{k''} \in N(\pi_{k'}) \cap V_l - S'$ and let $S_2 = S_1 \cup \{\pi_{k''}\} - \{\pi_k\}$. Then $S_2 \subseteq V_{l'}^*$ is a dominating set of $(V_{i,j})'$ with $|S_2| = |S_1|$ and $(S_2)$ has a perfect matching in $(V_{l'}^*)$ and $\text{max}(S_2) \geq \text{max}(S_1)$.

Proceeding as above, we get a set $S^* \subseteq V_{l'}^*$ satisfying the following conditions:

(i). $S^* \cap \{\pi_{l+1}, \pi_{l+2}, \ldots, \pi_i\} - \{\pi_i^*\} = \pi^*_{l-1}$;

(ii). $S^*$ is a dominating set of $(V_{i,j})$ with $|S^*| = |S|$ and $(S^*)$ in $(V_{l'}^*)$ has a perfect matching in which $\pi_i^*$ and $\pi_l$ are paired;

(iii). $\text{max}(S^*) \geq \text{max}(S)$.

Then $S^* \in X_1$. As in Case 1, it can be verified that no vertex in $V_{l-1, \pi_i^*}$ is dominated by $\pi_i^*$ or $\pi_l$ since $\pi_i^* < \pi_l$, so $S^* - \{\pi_i^*, \pi_l\}$ is a dominating set of $(V_{l-1, \pi_i^*})$ and $(S^* - \{\pi_i^*, \pi_l\})$ in $(V_{l-1}^*)$ has a perfect matching. By the minimality of $S^*$, it follows that $S^* - \{\pi_i^*, \pi_l\} \subseteq V_{l-1}^*$ is a minimum cardinality dominating set of $(V_{l-1, \pi_i^*})$. Then $S^* - \{\pi_i^*, \pi_l\}$ is a $PD_{l-1, \pi_i^*}$, and thus $S^*$ is a $PD_{\pi_i^*}$.

Hence, $|S| = |S^*| = |PD_{l-1, \pi_i^*}| + 2$. Note that $|PD_{\pi_i^*}| \leq |PD_{l-1, \pi_i^*}| + 2 = |S|$ and if $|PD_{\pi_i^*}| = |PD_{l-1, \pi_i^*}| + 2$, then $\text{max}(PD_{\pi_i^*}) = \text{max}(S^*) \geq \text{max}(S)$. Therefore, $\text{Min}(X_1 \cup \{PD_{\pi_i^*}\}) = PD_{\pi_i^*}$.

\[ \square \]

**Lemma 9** For any integers $i$ and $j$, $1 < i \leq n$ and $1 \leq j \leq n$, if there exist $\pi_{i_1}$ ($i_1 > l$) and $\pi_{i_2}$ such that $(\pi_i^*, \pi_{i_1}) \in M$ and $(\pi_l, \pi_{i_2}) \in M$, then $\text{Min}(X_1 \cup \{PD_{\pi_i^*}\}) = PD_{\pi_i^*}$.

**Proof.** Similar to Lemma 8, we need to show that there exists an $S^* \in PD_{\pi_i^*} \cap X_1$ such that $\text{max}(S^*) \geq \text{max}(S)$. We claim that $\pi_l^* \neq \pi_i^*$, $\pi_{l-1}^* \notin S$, and $N(\pi_{l-1}^*) \cap \{\pi_1, \pi_2, \ldots, \pi_{l-1}\} \neq \emptyset$.

We first show that $\pi_l^* \neq \pi_i^*$. Suppose to the contrary that $\pi_l^* = \pi_i^*$, then it is easy to see that $\pi_i^* < \pi_l^* < \pi_l$ and $\pi_i^* < \pi_{i_1} < \pi_l$. Hence, by Lemma 3, $S - \{\pi_l^*, \pi_{i_1}\}$ is a dominating set of $(V_{i,j})$ and $(S - \{\pi_l^*, \pi_{i_1}\})$ has a perfect matching in $(V_{l'}^*)$ by pairing $\pi_i^*$ with $\pi_l$, which contradicts the minimality of $S$. So $\pi_l^* \neq \pi_i^*$. Second, we show that $\pi_l^* \notin S$. Suppose this is not the case, $\pi_l^* \in S$. For any vertex $\pi_t \in N[\pi_{i_1}]$, if $t < i_1$, then $\pi_t > \pi_{i_1}$. By our assumption that $(\pi_i^*, \pi_{i_1}) \in M$, we have $\pi_{i_1} > \pi_i^*$ as $i_1 < \pi^{-1}(\pi_i^*)$. Hence, $(\pi_t, \pi_i^*) \in E$. If $t \geq i_1$ (>$l$), then $\pi_t \leq \pi_{i_1} < \pi_l$, and thus $(\pi_t, \pi_l) \in E$. So $N[\pi_{i_1}] \subseteq N[\pi_l] \cup N[\pi_i^*]$. For any vertex
$\pi_t \in N[\pi_P]$, if $t \leq l - 1$, then $\pi_t > \pi_P \geq \pi_{l-1}^*$ and $t \leq l - 1 \leq \pi^-(\pi_{l-1}^*)$, so $(\pi_t, \pi_{l-1}^*) \in E$. If $l < t < l'$, then $\pi_t < \pi_t$ or $\pi_t > \pi_t^*$ and $l' \leq \pi^-(\pi_t^*)$, and thus $(\pi_t, \pi_t) \in E$ or $(\pi_t, \pi_t^*) \in E$.

If $t \geq l'$ (> $l$), then $\pi_l > \pi_P \geq \pi_t$, so $(\pi_l, \pi_t) \in E$. So $N[\pi_P] \subseteq N[\pi_l] \cup N[\pi_{l-1}] \cup N[\pi_t^*]$. Let $S' = S - \{\pi_P, \pi_t\}$. Then $S'$ is a dominating set of $(V_{i,j})$ and $M' = M \cup \{(\pi_i, \pi_t^*)\} - \{(\pi_l, \pi_P), (\pi_t^*, \pi_t)\}$ is a perfect matching in $(S')$. This contradicts the minimality of $S$. So $\pi_{l-1}^* \notin S$. Finally, we show that $N(\pi_{l-1}^*) \cap \{\pi_1, \pi_2, \ldots, \pi_{l-1}\} \neq \emptyset$. If $N(\pi_{l-1}^*) \cap \{\pi_1, \pi_2, \ldots, \pi_{l-1}\} = \emptyset$, then $N(\pi_{l'}) \cap \{\pi_1, \pi_2, \ldots, \pi_{l-1}\} = \emptyset$, so we have $N[\pi_P] \subseteq N[\pi_l] \cup N[\pi_t^*]$. Hence, $S - \{\pi_P, \pi_t\}$ is a dominating set of $(V_{i,j})$ and $\langle S - \{\pi_P, \pi_t\} \rangle$ has a perfect matching in $(V_{i,t}^*)$, contradicting the minimality of $S$.

Let $\pi_{l_1} \in N(\pi_{l-1}^*) \cap \{\pi_1, \pi_2, \ldots, \pi_{l-1}\}$ and $S_1 = S \cup \{\pi_{l-1}, \pi_{l_1}\} - \{\pi_P, \pi_{l_1}\}$. Since $N[\pi_{l_1}] \subseteq N[\pi_l] \cup N[\pi_t^*]$, and $N[\pi_P] \subseteq N[\pi_l] \cup N[\pi_{l-1}] \cup N[\pi_t^*]$, $S_1$ is a dominating set of $(V_{i,j})$ and $(S_1)$ has a perfect matching in $(V_{i,t}^*)$ by pairing $(\pi_{l_1}, \pi_{l_1}^*)$ and $(\pi_{l-1}, \pi_{l_1})$. So $S_1 \subseteq X_1$ with $|S_1| = |S|$ and $\max(S_1) \geq \max(S)$ such that $\pi_P \notin S_1$ and $\pi_{l-1}^* \in S_1$. Using analogous arguments as in Lemma 8, we can get a set $S' \subseteq X_1$ such that $S' \subseteq \{\pi_{l_1}^*, \pi_{l_1}\}$ is a $PD_{l_1-1, \pi_{l_1}^*}$ and $S'$ is a $PD_{\pi_{l_1}^*}$. Hence, $|S| = |S'| = |PD_{l_1-1, \pi_{l_1}^*}| + 2$. Note that $|PD_{\pi_{l_1}^*}| \leq |PD_{l_1-1, \pi_{l_1}^*}| + 2 = |S|$ and if $|PD_{\pi_{l_1}^*}| = |PD_{l_1-1, \pi_{l_1}^*}| + 2$, then $\max(PD_{\pi_{l_1}^*}) = \max(S) \geq \max(S)$. Therefore, $\min(X_1 \cup \{PD_{\pi_{l_1}^*}\}) = PD_{\pi_{l_1}^*}$.

Lemma 10 For any integers $i$ and $j$, $1 < i \leq n$ and $1 \leq j \leq n$, if $(\pi_i^*, \pi_l) \in M$, then $\min(X_1 \cup \{PD_{\pi_l^*}\}) = PD_{\pi_l^*}$.

Proof. Similar to Lemma 8, we again need to show that there exits an $S' \subseteq PD_{\pi_l^*} \cap X_1$ such that $\max(S') \geq \max(S)$. We consider the following two cases depending on whether or not $\pi_{l-1}^*$ is equal to $\pi_l^*$.

Case 1. Suppose $\pi_{l-1}^* = \pi_l^*$. Then, for any $\pi_k \in S$ for $l < k < i$, there exists $\pi_{k'} \in S$ such that $(\pi_k, \pi_{k'}) \in M$. Similar to the discussion for $S_1$ in Case 1 of Lemma 8, we can obtain a set $S^* \subseteq X_1$ satisfying the conditions (i)–(iii) in Case 1 of Lemma 8 and $S^*$ is a $PD_{\pi_l^*}$ with $\max(PD_{\pi_l^*}) \geq \max(S)$. Therefore, $\min(X_1 \cup \{PD_{\pi_l^*}\}) = PD_{\pi_l^*}$.
Case 2. Suppose $\pi_{l-1}^* \neq \pi_l^*$. If $\pi_{l-1}^* \in S$, then we deal with $S$ as in Case 2 of Lemma 8 for $S_1$. Finally, we can obtain a set $S^* \in X_1$ satisfying the conditions (i)–(iii) in Case 2 of Lemma 8 and $S^*$ is a $PD_{\pi_l^*}$ with $\max(PD_{\pi_l^*}) \geq \max(S)$. Hence, $\text{Min}(X_1 \cup \{PD_{\pi_l^*}\}) = PD_{\pi_l^*}$, thus the assertion holds. In what follows, we may assume that $\pi_{l-1}^* \notin S$. As in Case 1 of Lemma 8, we first find a set $S_1 \in X_1$ with $|S_1| = |S|$ and $\max(S_1) \geq \max(S)$ such that $\pi_{l-1}^* \in S_1$.

Suppose $S \cap (\{\pi_{l+1}, \ldots, \pi_i\} \setminus \{\pi_l^*\}) = \emptyset$. Since $\pi_l^* < \pi_l$, it follows that no vertex in $V_{l-1, \pi_l^*}$ is dominated by $\pi_l^*$ or $\pi_l$, so $S \setminus \{\pi_l^*, \pi_l\}$ is a dominating set of $\langle V_{l-1, \pi_l^*} \rangle$ and $\langle S \setminus \{\pi_l^*, \pi_l\} \rangle$ in $\langle V_{l-1} \rangle$ has a perfect matching. By minimality of $S$, we deduce that $S \setminus \{\pi_l^*, \pi_l\} \subseteq V_{l-1}$ is a minimum cardinality dominating set of $\langle V_{l-1, \pi_l^*} \rangle$ and contains a perfect matching. Then $S \setminus \{\pi_l^*, \pi_l\}$ is a $PD_{l-1, \pi_l^*}$, and thus $S$ is a $PD_{\pi_l^*}$. Hence, $|S| = |PD_{l-1, \pi_l^*}| + 2$. Note that $|PD_{\pi_l^*}| \leq |PD_{l-1, \pi_l^*}| + 2 = |S|$, it follows that $\text{Min}(X_1 \cup \{PD_{\pi_l^*}\}) = PD_{\pi_l^*}$.

Suppose $S \cap (\{\pi_{l+1}, \ldots, \pi_i\} \setminus \{\pi_l^*\}) \neq \emptyset$. Choosing a vertex $\pi_{k_0} \in S$ ($l < k_0 < i$), there exists $\pi_{k_0'}$ such that $(\pi_{k_0}, \pi_{k_0'}) \in M$. If $k_0' < l$, then $\pi_{k_0'} > \pi_{k_0} > \pi_{l-1}^*$, and so $(\pi_{k_0'}, \pi_{l-1}^*) \in E$. We claim that all the vertices in $N[\pi_{k_0}]$ are dominated by $\pi_{l-1}^*$, $\pi_l^*$ and $\pi_l$. Indeed, for any $\pi_t \in N[\pi_{k_0}]$, if $t < l$, then $\pi_t > \pi_{k_0} > \pi_{l-1}^*$, so $(\pi_t, \pi_{l-1}^*) \in E$; if $l \leq t \leq k_0$, then $\pi_t \leq \pi_l$ or $\pi_t > \pi_{l} > \pi_{l-1}^*$, so $\pi_t = \pi_{l}$, $(\pi_l, \pi_l) \in E$ or $(\pi_l, \pi_{l-1}^*) \in E$; if $t > k_0$, then $\pi_t < \pi_{k_0} < \pi_l$, so $(\pi_t, \pi_l) \in E$. The claim follows. Let $S_1 = S \cup \{\pi_{l-1}^*\} - \{\pi_{k_0'}\}$. Then $S_1$ is a dominating set of $\langle V_{i,j} \rangle$ and $\langle S_1 \rangle$ has a perfect matching in $\langle V_{i,j}^* \rangle$ by pairing $\pi_{k_0'}$ and $\pi_{l-1}^*$ and removing the edge $(\pi_{k_0}, \pi_{k_0'})$. We obtain a set $S_1 \in X_1$ with $|S_1| = |S|$ and $\max(S_1) \geq \max(S)$ such that $\pi_{l-1}^* \in S_1$. If $k_0' > l$, then there exists $\pi_{k_1}$ ($k_1 < l$) such that $(\pi_{k_1}, \pi_{k_0'}) \in E$ or $(\pi_{k_1}, \pi_{k_0}) \in E$. Otherwise, since all the vertices in $\{\pi_l, \ldots, \pi_l\}$ are dominated by $\pi_l^*$ and $\pi_l$, $S - \{\pi_{k_0}, \pi_{k_0'}\}$ is a dominating set of $\langle V_{i,j} \rangle$ and $\langle S - \{\pi_{k_0}, \pi_{k_0'}\} \rangle$ has a perfect matching in $\langle V_{i,j}^* \rangle$ by removing $(\pi_{k_0}, \pi_{k_0'})$, contradicting the minimality of $S$. Hence, $\pi_{k_1} > \pi_{k_0} > \pi_{l-1}^*$ or $\pi_{k_1} > \pi_{k_0'} > \pi_{l-1}^*$. This means that $(\pi_{k_1}, \pi_{l-1}^*) \in E$.

Let $S_1 = S \cup \{\pi_{k_1}, \pi_{l-1}^*\} - \{\pi_{k_0}, \pi_{k_0'}\}$. Note that all the vertices in $N(\{\pi_{k_0}, \pi_{k_0'}\})$ are dominated by $\pi_l$, $\pi_l^*$ and $\pi_{l-1}^*$, so $S_1$ is a dominating set of $\langle V_{i,j} \rangle$ and $\langle S_1 \rangle$ has a perfect matching in $\langle V_{i,j}^* \rangle$ by pairing $\pi_{k_1}$, $\pi_{l-1}^*$, and removing the edge $(\pi_{k_0}, \pi_{k_0'})$. We again obtain a set $S_1 \in X_1$ with $|S_1| = |S|$ and $\max(S_1) \geq \max(S)$ such that $\pi_{l-1}^* \in S_1$. As before, by adding to $S_1$ the vertices in $\{\pi_l, \ldots, \pi_{l-1}\}$ and removing all the vertices of $S_1$ in $\{\pi_l, \ldots, \pi_l\} - \{\pi_{l-1}^*, \pi_l^*\}$, we can obtain
a set $S^* \in X_1$ satisfying the conditions (i)–(iii) in Case 2 of Lemma 8 and $S^*$ is a $PD_{\pi_i^*}$ with max$(PD_{\pi_i^*}) = \max(S^*) \geq \max(S)$. Hence, Min$(X_1 \cup \{PD_{\pi_i^*}\}) = PD_{\pi_i^*}$. □

By Lemmas 8–10, we obtain the following result.

**Lemma 11** For any integers $i$, $j$, if $1 < i \leq n$ and $1 \leq j \leq n$, Min$(X_1 \cup \{PD_{\pi_i^*}\}) = PD_{\pi_i^*}$.

**Lemma 12** For any integers $i$ and $j$, $1 < i \leq n$ and $\pi_i \leq j \leq n$, if max$(V_i) = \pi_i$, then $X_3 = \emptyset$.

**Proof.** Suppose to the contrary that $X_3 \neq \emptyset$. Let $S \in X_3$. Then $\pi_i, \pi_i^* \not\in S$ and $S (\subset V_i^*)$ is a dominating set of $\langle V_i,j \rangle$ and $\langle S \rangle$ has a perfect matching in $\langle V_i^* \rangle$. Since $\pi_i \leq j \leq n$, $\pi_i \in V_i,j$, so $\pi_i$ is dominated by a vertex $\pi_l$ ($l < i$) in $S$. Then $(\pi_i, \pi_l) \in E$, i.e., $(\pi_i - \pi_l)(i - l) < 0$. This implies that $\pi_l > \pi_i$, contradicting the assumption of max$(V_i) = \pi_i$. □

**Lemma 13** For any integers $i$ and $j$, $1 < i \leq n$ and $\pi_i \leq j \leq n$, if max$(PD_{i-1,j}) < \pi_i$, then Min$(X_3 \cup \{PD_{\max}\}) = PD_{\max}$.

**Proof.** If max$(V_i) = \pi_i$, by Lemma 12, $X_3 = \emptyset$. The result follows. So we may assume that max$(V_i) \neq \pi_i$. Let $Z$ denote the set $\{S : S \subseteq V_i^{*-1} \text{ and } S \text{ is a dominating set of } \langle V_i^{*-1} \rangle \}$ and $\langle S \rangle$ has a perfect matching in $\langle V_i^{*-1} \rangle$. Let $A$ be any set of $X_3$. Since $\pi_i \not\in A$ and $\pi_i^* \not\in A$, $A \subseteq V_i^{*-1}$. By Lemma 2, we have $V_i^{*-1} \subseteq V_i,j$, so $A \in Z$. Since $\pi_i \leq j$, $\pi_i \in V_i,j$, max$(A) > \pi_i$. Thus max$(A) > \pi_i > \max(PD_{i-1,j})$. Note that $PD_{i-1,j} = \min(Z)$ and, by our definition, max$(PD_{i-1,j})$ is as large as possible. Then it must be the case that $|A| > |PD_{i-1,j}|$. Hence, $|A| \geq |PD_{i-1,j}| + 2 = |PD_{i-1,j} \cup \{\max(V_i), \pi_i\}|$. Furthermore, $\max(A) \leq \max(V_i) = \max(PD_{i-1,j} \cup \{\max(V_i), \pi_i\})$. Therefore, Min$(X_3 \cup PD_{\max}) = PD_{\max}$. □

**Lemma 14** For any integers $i$ and $j$, if $1 < i \leq n$ and $1 \leq j \leq n$, then Min$(X_3 \cup \{PD_{i-1,j}\}) = PD_{i-1,j}$.

**Proof.** Define $Z$ as in Lemma 13. Let $A$ be any set of $X_3$. As in the proof of Lemma 13, we can verify that $A \in Z$. Note that $PD_{i-1,j} = \min(Z)$. So Min$(X_3 \cup \{PD_{i-1,j}\}) = PD_{i-1,j}$. □
Lemma 15 For any integers $i$ and $j$, if $1 < i \leq n$ and $1 \leq j \leq n$, then $\text{Min}\{X_1 \cup X_2\} = \text{Min}\{X_1\}$.

Proof. Let $S_1 = \text{Min}\{X_2\}$. According to the definition of $X_2$, $\pi_i^* \not\in X_2$, $\pi_i \in X_2$ and $\langle S_1 \rangle$ has a perfect matching $M$. So there exists a vertex $\pi_l \in X_2$ ($l < i$) such that $(\pi_i, \pi_l) \in M$. Then $(\pi_l - \pi_i)(l - i) < 0$, and thus $\pi_l > \pi_i$. Hence $\pi_i^* < \pi_i < \pi_l$ and $l < i < \pi^-(\pi_i^*)$. \hfill (1)

This means that $(\pi_i^* - \pi_i)(\pi^-(\pi_i^*) - l) < 0$, i.e., $(\pi_i, \pi_i^*) \in E$. Let $S_2 = (S_1 - \{\pi_i\}) \cup \{\pi_i^*\}$. From (1) and Lemma 3, it follows that $S_2 \subseteq V_i^*$ is a dominating set of $\langle V_{i,j} \rangle$ and $\langle S_2 \rangle$ has a perfect matching by pairing $\pi_l$ and $\pi_i^*$. So $S_2 \in X_1$, $|S_2| = |S_1|$ and $\max(S_2) \geq \max(S_1)$. Consequently, $\text{Min}\{X_1 \cup X_2\} = \text{Min}\{\text{Min}(X_1), \text{Min}(X_2)\} = \text{Min}\{\text{Min}(X_1), S_1\} = \text{Min}(X_1)$. \hfill \Box

In the following, we present the recursive formula of our dynamic programming.

Theorem 16 For any integers $i, j$, if $1 < i \leq n$ and $1 \leq j \leq n$, then the following recursive formula correctly computes $PD_{i,j}$,

$$PD_{i,j} = \begin{cases} \text{Min}(\{PD_{\pi_i^*}, PD_{\text{max}}\}) & \text{if } j \geq \pi_i \text{ and } \max(PD_{i-1,j}) < \pi_i, \\ \text{Min}(\{PD_{\pi_i^*}, PD_{i-1,j}\}) & \text{otherwise.} \end{cases}$$

Proof. According to our definitions, $X = X_1 \cup X_2 \cup X_3$. By Lemmas 5 and 6, we have $PD_{\pi_i^*} \in X_1 \subseteq X$, $PD_{\text{max}} \in X$. To complete our proof, we distinguish the following two cases.

Case 1. Suppose that $j \geq \pi_i$ and $\max(PD_{i,j}) < \pi_i$. If $\max(V_i) = \pi_i$, then, by Lemmas 11, 12 and 15, we have

$$\text{Min}(X) = \text{Min}(X_1 \cup X_2 \cup \{PD_{\pi_i^*}, PD_{\text{max}}\}) = \text{Min}(X_1 \cup \{PD_{\pi_i^*}, PD_{\text{max}}\}) = \text{Min}(\text{Min}(X_1 \cup \{PD_{\pi_i^*}\}), PD_{\text{max}}) = \text{Min}(PD_{\pi_i^*}, PD_{\text{max}}).$$

If $\max(V_i) \neq \pi_i$, then, by Lemmas 11, 13 and 15, we have

$$\text{Min}(X) = \text{Min}(X \cup \{PD_{\pi_i^*}, PD_{\text{max}}\})$$
\[
\begin{align*}
&= \min(X_1 \cup X_2 \cup X_3 \cup \{PD_{\pi^*_1}, PD_{\text{max}}\}) \\
&= \min(X_1 \cup X_3 \cup \{PD_{\pi^*_1}, PD_{\text{max}}\}) \\
&= \min(\min(X_1 \cup \{PD_{\pi^*_1}\}), \min(X_3 \cup \{PD_{\text{max}}\})) \\
&= \min(PD_{\pi^*_1}, PD_{\text{max}}).
\end{align*}
\]

**Case 2.** Suppose that \( j < \pi_i \) or \( \max(PD_{i-1,j}) \geq \pi_i \). We first show that \( PD_{i-1,j} \in X \). If \( j < \pi_i \), then \( V_{i,j} = V_{i-1,j} \), so \( PD_{i-1,j} \in X \). If \( \max(PD_{i,j}) \geq \pi_i \), then \( \pi_i \) is dominated by \( PD_{i-1,j} \), so \( PD_{i-1,j} \in X \). Note that \( PD_{i-1,j} \subseteq PD_{\text{max}} \). From Lemmas 11, 14 and 15, it follows that

\[
\min(X) = \min(X \cup \{PD_{\pi^*_1}, PD_{i-1,j}\})
\]

\[
= \min(X_1 \cup X_2 \cup X_3 \cup \{PD_{\pi^*_1}, PD_{i-1,j}\})
\]

\[
= \min(X_1 \cup X_3 \cup \{PD_{\pi^*_1}, PD_{i-1,j}\})
\]

\[
= \min(\min(X_1 \cup \{PD_{\pi^*_1}\}), \min(X_3 \cup \{PD_{i-1,j}\}))
\]

\[
= \min(PD_{\pi^*_1}, PD_{i-1,j}).
\]

\[
\Box
\]

3 **An algorithm for MPDS on permutation graphs**

Based on the recursive formula in Section 2, we next present the algorithmic steps to solve MPDS on permutation graphs. The overall structure of our algorithm is outlined as follows:

**Algorithm:** Finding an MPDS on a Permutation Graph.

- **Input:** A permutation \( \pi = [\pi_1, \pi_2, \ldots, \pi_n] \).
- **Output:** A minimum cardinality paired-dominating set of \( G[\pi] \).
- **Step 1.** Initialize \( PD_{0,j} = \emptyset \).
\[ PD_{1,j} = \begin{cases} \emptyset & \text{if } j < \pi_1, \\ \{1, \pi_1\} & \text{otherwise.} \end{cases} \]

for \( j = 1, 2, \ldots, n \).

**Step 2.** for \( i \leftarrow 2 \) to \( n \) do

**Step 3.** \( PD_{\pi^*_i} = \text{Min}\{PD_{l-1, \pi^*_i} \cup \{\pi^*_i, \pi_l\} : \pi_l \in N(\pi^*_i), \pi^*_i \notin PD_{l-1, \pi^*_i}, l \leq i\} \)

**Step 4.** for \( j \leftarrow 1 \) to \( n \) do

**Step 5.**

\[ PD_{\text{max}} = \begin{cases} PD_{l-1, j} \cup \{\pi_i, \text{max}(V_i)\} & \text{if } \pi_i \neq \text{max}(V_i), \\ V_i & \text{otherwise.} \end{cases} \]

**Step 6.**

\[ PD_{i,j} = \begin{cases} \text{Min}\{PD_{\pi^*_i}, PD_{\text{max}}\} & \text{if } j \geq \pi_i \text{ and max}(PD_{i-1,j}) < \pi_i, \\ \text{Min}\{PD_{\pi^*_i}, PD_{i-1,j}\} & \text{otherwise.} \end{cases} \]

**Step 7.** END

**Step 8.** END

**Step 9.** Output \( PD_{n,n} \).

The time complexity of the above algorithm can be analyzed as follows. The time required in Step 3 is at most \( d(\pi^*_i) \). The operations of Steps 5 and 6 can be performed in constant time. The time required in the loop from Step 4 to Step 7 is at most \( O(n) \). Consequently, the overall running time of the algorithm is \( O(mn) \) in an amortized sense.

**Theorem 17** Given any permutation \( \pi \), the algorithm finds a minimum cardinality paired-dominating set of the permutation graph \( G[\pi] \).

**Example.** To illustrate our algorithm, we compute the example shown in Fig. 1. as follows:

1. \( PD_{0,j} = \emptyset \);
2. \( PD_{\text{max}} = V_1, PD_{1,1} = PD_{1,2} = \emptyset, PD_{1,3} = \cdots = PD_{1,7} = \{1,3\} \);

3. \( \pi_2^* = 2, PD_{\pi_2^*} = \{3,2\}, PD_{\text{max}} = \{1,3\}, PD_{2,1} = \cdots = PD_{2,7} = \{3,2\} \) or \( \{1,3\} \);

4. \( \pi_3^* = 2, PD_{\pi_3^*} = \{3,2\}, PD_{\text{max}} = V_3, PD_{3,1} = \cdots = PD_{3,4} = \{3,2\} \) or \( \{1,3\}, PD_{3,5} = \cdots = PD_{3,7} = \{3,2\} \);

5. \( \pi_4^* = 2, PD_{\pi_4^*} = \{3,2\}, PD_{\text{max}} = V_4, PD_{4,1} = \cdots = PD_{4,4} = \{3,2\} \) or \( \{1,3\}, PD_{4,5} = \cdots = PD_{4,7} = \{3,2\} \);

6. \( \pi_5^* = 2, PD_{\pi_5^*} = \{3,2\}, PD_{\text{max}} = \{2,3,7,4\} \) or \( \{1,3,7,4\}, PD_{5,1} = \cdots = PD_{5,3} = \{3,2\} \) or \( \{1,3\}, PD_{5,4} = \cdots = PD_{5,7} = \{3,2\} \);

7. \( \pi_6^* = 2, PD_{\pi_6^*} = \{3,2\}, PD_{\text{max}} = \{1,3,2,7\}, PD_{6,1} = \cdots = PD_{6,3} = \{3,2\} \) or \( \{1,3\}, PD_{6,4} = \cdots = PD_{6,7} = \{3,2\} \);

8. \( \pi_7^* = 6, PD_{\pi_7^*} = \{3,2,7,6\}, PD_{\text{max}} = \{3,2,7,6\} \) or \( \{1,3,7,6\}, PD_{7,1} = \cdots = PD_{7,3} = \{3,2,7,6\} \) or \( \{1,3,7,6\}, PD_{7,4} = \cdots = PD_{7,7} = \{3,2,7,6\} \);

In light of our algorithm, \( PD_{7,7} = \{3,2,7,6\} \) is a minimum cardinality paired-dominating set of the graph.
4 Conclusions

In this paper we presented an $O(mn)$ algorithm for finding a minimum cardinality paired-dominating set for a permutation graph with order $n$ and size $m$. Our algorithm is based on a recursive formula in conjunction with applying the dynamic programming method. The idea was previously used by Chao et al [7] for finding the minimum cardinality dominating set on permutation graphs. We speculate that the time complexity of the MPDS problem on permutation graphs can be reduced to $O(n \log n)$ and we suggest that researchers investigate such a possibility. It is also interesting to determine whether there exist some other classes of graphs in which the minimum paired-domination problem is polynomially solvable.

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