

An Extension of the Modigliani–Miller Theorem to Stochastic Economies with Incomplete Markets and Interdependent Securities

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The Modigliani–Miller theorem is shown to hold in a general model of a multiperiod, stochastic economy with incomplete markets and perfect foresight. In the model, firms are allowed to trade all available securities; thus, share prices and dividends become fully interdependent. Also, several mechanisms of corporate control are considered, including value maximization and shareholder voting. It is demonstrated that firms have no incentive to trade securities in equilibrium; further, any such trading does not alter the set of equilibrium allocations. This is done by establishing that agents are able to adjust their own portfolios and neutralize corporate policy. *Journal of Economic Literature* Classification Numbers: 021, 022, 026, 313, 521. © 1988 Academic Press, Inc.

1. INTRODUCTION

This paper extends the Modigliani–Miller argument on the irrelevance of corporate financial policy to an economy with incomplete markets and fully interdependent securities. The approach taken is a general equilibrium model incorporating multiple periods and uncertain states. The market is idealized in that there are no transaction costs, short sales restrictions, or limited liability constraints. Also, agents are assumed to have fully rational expectations regarding future events. Within this context, it is shown that value maximizing firms have no incentive to change the composition of their portfolios, and that any such changes have no effect on equilibrium prices and allocations within the economy. Finally, the theory is shown to

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encompass other decision mechanisms within the firm, including direct shareholder control.

The intuition driving these results is precisely that suggested by Modigliani and Miller in their initial exposition [9]: with competitive, linear security markets, trading by firms and trading by individuals can be perfect substitutes. The framework adopted by Modigliani and Miller in their original presentation is limited to a partial equilibrium analysis of an economy in which firms choose their debt–equity ratio. In a more recent paper, Stiglitz [13] extends this result to a general equilibrium model of an economy with uncertainty. He shows that, given fixed production plans and an initial equilibrium with a specific debt–equity ratio, another equilibrium with a new debt–equity choice exists in which commodity prices and real allocations are unchanged, and investors have adjusted their portfolios to exactly offset the activities of firms.

An important issue which is not resolved by the existing literature concerns the objective of the firm in an incomplete markets setting. The standard approach, that firms act to maximize their initial share value, is not well specified: prices in existing markets no longer provide sufficient information to value alternative dividend streams unambiguously. Therefore, a model with value maximizing firms must incorporate beliefs regarding the “prices” that would prevail in the “missing” markets. Further, these conjectures may be different for each firm. Alternatively, Drèze [3, 4] and others [7, 2] have suggested alternative decision procedures for the firm which take into account the preferences and shareholdings of the owners. Since these considerations are important, in the model considered here the plans of the firm are not given a priori; they are endogenously determined as the outcome of an internal decision mechanism.

Another limitation of previous results is the restriction of a firm’s portfolio to debt and equity. In general, one would like to allow firms to be fully active participants in security markets, buying and selling stocks and bonds of other firms, trading in commodity futures, refinancing debt, and even issuing and repurchasing their own stock. The model should also be versatile enough to incorporate mutual funds, whose sole function is to engage in security transactions so as to produce specific dividend streams. The model described in this paper allows such generality.

Of course, incorporating such trading activity in a general model leads to new difficulties which do not arise in the simpler debt–equity framework. In particular, there is a simultaneity problem: if a share of a firm represents ownership of a portfolio containing stocks of other firms that in turn may hold shares of the first firm, it is not clear how share prices and dividends are jointly determined. This is important since firms must consider their effect on share prices and dividends when they contemplate changing their portfolios. This problem of interdependency has been addressed in a more

limited context by Duffie and Shafer [5]; however, their results require that firms share a common conjecture about prices in missing markets, an assumption which is not made here.

Thus, the task of validating the Modigliani–Miller theorem must be approached in several stages. First, firms will be assumed to maximize their initial share value, based on private beliefs about missing prices which are consistent with the information made available by the market. It is shown that in equilibrium, independent of their specific beliefs, firms have identical conjectures regarding the effects of their security trading on security prices and dividends. Further, they do not expect such trading to influence their initial share prices; hence, they have no incentive to trade. This verifies the first, partial equilibrium, form of the Modigliani–Miller theorem. The second form of the theorem is established by verifying that if firms do alter their portfolios, another general equilibrium exists in which commodity prices and real allocations are unchanged. Finally, the above results are extended to decision criteria other than value maximization which may be used by firms, including majority voting, veto power, and types of managerial control. This is done by establishing that in the new equilibrium after firms trade securities, the real shareholder composition of each firm is unaltered.

2. THE MODEL

The stochastic economy is modeled using an event tree structure which is similar in spirit to that introduced by Debreu [1]. In this setting, goods at different times or states of the world are treated as distinct. The model of securities and the notion of a perfect foresight equilibrium is a descendant of the work of Radner [10]. Much of the notation is adapted from Duffie and Shafer [5], where the question of existence of equilibria for such economies is discussed.

2.1. *Event Tree*

The stochastic environment is modeled by an event tree consisting of a finite set of nodes E and directed arcs $A \subset E \times E$, such that (E, A) forms a tree with a distinguished root e_1 . The notation $e < e'$ is used to indicate that e precedes e' ; i.e., there is a unique path in the tree from e to e' . For example, one has $E = \{e | e \geq e_1\}$. Also, e^- denotes the unique immediate predecessor of node e ; thus, $\{n | n^- = e\}$ is the set of nodes which directly follow node e . Each node e in E represents a “state–date” pair to which the economy may evolve. There are l commodities at each such node available for consumption. A price process p in the space $L \equiv \{f: E \rightarrow \mathcal{R}^l\}$ specifies the vector of commodity prices $p(e)$ at each node e .

2.2. Agents

The economy is composed of some finite number m of agents. A consumption plan for an agent in the economy is a process $x \in L$, with $x(e)$ indicating consumption in state e . Each agent has a preference ordering over the space L represented by a utility function $u^i: L \rightarrow \mathcal{R}$ and is also given an exogenously specified endowment process $w^i \in L$. Hence, the consumption plan x for agent i has associated utility $u^i(x)$ and induces the net expenditure process $p \square (x^i - w^i) \in \{f: E \rightarrow \mathcal{R}\} \equiv D$, where the notation $(y \square z)(e) \equiv y(e) \cdot z(e)$ is used.

2.3. Production

Production in the economy is performed by n firms, each with a production set $Y_j \subset L$. Each production plan $y_j \in Y_j$ specifies the net commodity bundle $y_j(e)$ produced by firm j at each state e . Given a price process p , firm j generates the dividend stream $\Delta_j \equiv p \square y_j \in D \equiv \{f: E \rightarrow \mathcal{R}\}$, where $\Delta_j(e) = p(e) \cdot y_j(e)$. This ignores the effect of security trading strategies by firms, which will be considered later. Each agent i is initially endowed with a share θ_j^i of firm j , such that $\sum_{i=1}^m \theta_j^i = 1$.

2.4. Securities

Securities are distinct from firms in that they are not associated with any underlying production; they simply represent the allowable "contingent contracts" which agents may write and enforce. Thus they should include bonds, commodity futures, and complex combinations of the two.

Specifically, agents may also buy and sell k securities of the form $(\delta_h, d_h) \in D \times L$. Ownership of security h in state e represents a claim to the financial dividend $\delta_h(e)$, denominated in units of account, and the commodity bundle $d_h(e)$. Therefore, given prices, security h generates the financial dividend process $\Delta_h \equiv \delta_h + p \square d_h \in D$. Again, each agent i has an initial endowment θ_h^i of security h . As they are meant to represent holdings strictly "inside" the economy, it is assumed that $\sum_{i=1}^m \theta_h^i = 0$ for each security h .

Notice that it is possible, and in particular notationally convenient, to view firm shares as securities: given the production choice of firm j , shares of j may be thought of as a claim to the security (δ_j, d_j) , with $\delta_j = 0$ and $d_j = y_j$. Thus, given production choices, let (δ_j, d_j) , for $j = 1, \dots, n$, represent the common stocks, and (δ_h, d_h) , for $h = n + 1, \dots, n + k$, be the general securities available for trade. Finally, the notation $\Delta(e) \equiv (\Delta_1(e), \dots, \Delta_{n+k}(e))^T$ is used for the column vector of security dividend payments, which is determined once prices and production plans are known.

2.5. Security Prices and Trading

Since securities are tradeable, each must have a well-defined price for every state e in E . Therefore, the notation $\pi_j \in D$ shall represent the price process of security j . That is, $\pi_j(e)$ is the price in state e of a claim to the dividend process $\Delta_j \in D$. Also, the notation $\pi(e) \equiv (\pi_1(e), \dots, \pi_{n+k}(e))^T$ will be used for the column vector of security prices in state e .

Trading proceeds according to the following sequence: At each state e in E , agents first trade securities according to prices $\pi(e)$. Dividends are then paid. Agents subsequently exchange goods in spot markets for the l commodities with prices $p(e)$. Thus, in addition to a consumption plan, each agent has a security trading strategy γ which specifies the portfolio $\gamma(e)$ held after the completion of trading in state e . The space of possible trading strategies is denoted

$$\Gamma(\theta) \equiv \{f: E \cup \{e_0\} \rightarrow \mathcal{R}^{n+k} \mid f(e_0) = \theta\},$$

where $\theta \in \mathcal{R}^{n+k}$ is the initial endowment of securities held at the “pre-trade” state e_0 . A strategy for agent i is thus a pair $(x, \gamma) \in L \times \Gamma(\theta^i)$.

2.6. Budget Sets and Optimal Consumption

Agents must finance net consumption expenditures in each state through security trades and dividend payments. Given Δ and π , each security trading strategy γ induces a stream of financial revenues $\delta^\gamma[\Delta, \pi] \in D$ defined by

$$\delta^\gamma[\Delta, \pi](e) \equiv \gamma(e) \Delta(e) + [\gamma(e^-) - \gamma(e)] \pi(e), \quad (1)$$

where the notational convention $e_0 \equiv e_1^-$ is adopted. The first term represents the dividends received from securities held at the close of trading in state e , while the second is the net profit from security trades in that state. Clearly, the trading strategy (x, γ) is *budget feasible* for agent i if and only if $p(e) \cdot [x(e) - w^i(e)] \leq \delta^\gamma[\Delta, \pi](e)$ for all e in E . The agent’s problem thus becomes

$$\begin{aligned} \text{(P1)} \quad & \max_{x, \gamma} u^i(x) \\ & \text{s.t. } p \square (x - w^i) \leq \delta^\gamma[\Delta, \pi] \\ & x \in L, \gamma \in \Gamma(\theta^i), \end{aligned}$$

given p, Δ, π .

2.7. Optimal Production

The appropriate objective of the firm when asset markets are incomplete is rather problematic. It is well-known that for generic economies, given any production choice by the firm, some shareholders will disagree with

that choice and wish to alter it to improve their own utility (e.g., see Duffie and Shafer [5], Geanakoplos *et al.* [6], and DeMarzo [2]). For this reason, an alternative objective for the firm was initially proposed by Drèze [3], and later modified by Grossman and Hart [7]. More recent research by Drèze [4] and DeMarzo [2] has focused on actual decision mechanisms within the firm.

For now, however, firms shall be assumed to maximize their current share value. Later, it will be shown that this case is sufficiently general to encompass most other natural decision criteria.

In order to value maximize, a firm needs to discount the value of output in future states into today's share price. However, when markets are incomplete, they no longer provide sufficient information to price all contingent claims. Hence, in order to maximize its current share value, each firm j must conjecture a market valuation function $\Pi_j: D \rightarrow D$ which specifies the price process associated by the market to a given dividend stream. That is, $[\Pi_j(p \square y)](e)$ is the price firm j believes its shares will sell for in state e if it adopts production plan y . The production problem for firm j is therefore

$$(P2) \quad \max_y \Pi_j[p \square y](e_1) \\ \text{s.t. } y \in Y_j,$$

given p, Π_j .

2.8. Equilibrium

The stochastic, incomplete markets economy can now be described by

$$((E, A), (u^i, w^i, \theta^i), (Y_j), (\delta_h, d_h)),$$

for $i = 1, \dots, m, j = 1, \dots, n$, and $h = n + 1, \dots, n + k$.

The equilibrium concept for this economy adopts the standard assumption of perfect foresight on the part of agents, as developed by Radner [10]. In such an equilibrium, agents optimize on the basis of price expectations which are indeed correct.

DEFINITION 1. An *equilibrium* in this economy shall be a collection,

$$((x^i, \gamma^i), (y_j, \Pi_j), p, \pi),$$

for $i = 1, \dots, m$, and $j = 1, \dots, n$, such that

- (1) for each agent i , (x^i, γ^i) solves (P1) given p, A, π ,
- (2) for each firm j , y_j solves (P2) given p, Π_j ,
- (3) for each firm j , Π_j is consistent with π ; i.e., $\Pi_j(\Delta) = \pi$,

- (4) $\sum_{i=1}^m x^i - w^i = \sum_{j=1}^n y_j,$
 (5) $\sum_{i=1}^m \gamma^i(e) = \sum_{i=1}^m \theta^i$ for e in E .

Conditions 1 and 2 ensure that agents and firms are acting optimally given equilibrium prices and dividends. Condition 3 is a consequence of the perfect-foresight assumption as applied to firms: their conjectured market valuation function must be consistent with actual security prices. Finally, Conditions 4 and 5 represent clearing in commodity and asset markets, respectively.

2.9. Arbitrage Valuation

The equilibrium concept defined above places a consistency restriction on the market valuation functions firms may conjecture. A further "rationality" requirement seems natural: conjectured market valuation functions should not allow the possibility of profitable arbitrage. Specifically, any trading strategy generating non-negative dividends in all states, and non-zero dividends in some state, should require a positive initial investment. Formally, this is stated,

DEFINITION 2. A market valuation function Π is *arbitrage-free* if for any set of dividend streams Δ ,

$$\{\gamma \in \Gamma(0) \text{ such that } \delta^\gamma[\Delta, \Pi(\Delta)] \not\geq 0\} = \emptyset.$$

Clearly, since δ is linear in γ , this requirement must hold for any equilibrium in which at least one agent is non-satiated in each state. Thus it seems reasonable to impose this constraint on the rational conjectures of firms. As Ross [11] has shown in a one-period model, and as Duffie and Shafer [5] demonstrate for the multiperiod case discussed here, the elimination of profitable arbitrage has the following important consequence:

LEMMA 1. *The market valuation function Π is arbitrage-free, if and only if there exists a process $q \in D$ such that for all e in E , $q(e) > 0$ and*

$$\Pi[\Delta](e) = \Delta(e) + \sum_{n>e} \frac{q(n)}{q(e)} \Delta(n).$$

Proof. The result follows by linearity in γ of δ^γ and the application of an alternative theorem. ■

Thus, the process q can be interpreted as a set of "state" prices, determining the relative value of dividends paid in each state. Alternatively, q is the discount process used to value future profits. In any case, this result

allows the conjectured market valuation function Π_j of each firm to be associated with a discount process q_j . Therefore, an equilibrium may be redefined as a collection,

$$((x^i, \gamma^i), (y_j, q_j), p, \pi), \quad i = 1, \dots, m, j = 1, \dots, n,$$

with q_j replacing Π_j in Conditions 2 and 3, according to Lemma 1.

3. PARTIAL EQUILIBRIUM ANALYSIS

The incentives of the firm to engage in security trading are now examined. To do so, it is necessary to determine the effect of such trading on current share prices, *as conjectured by each firm*.

3.1. Firm Security Trading

The assumption is now relaxed that only the agents in the economy are free to trade securities. The firms may choose to hold securities or the common stock of other firms. Indeed, the securities themselves may represent claims against other stocks or securities; thus, the system may be fully interdependent. In general, such trading activity may not lead to a well-defined set of dividends and security prices. This could occur, for example, if one firm repurchased all of its outstanding shares. A regularity condition will be imposed to eliminate such cases.

Let $\beta_j \in \Gamma(0)$ denote the portfolio trading process adopted by firm (security) $j = 1, \dots, n + k$. Thus, holding one share of firm j is a claim to the portfolio β_j , in addition to the dividends of firm j . Let $\beta(e)$ represent the $(n + k) \times (n + k)$ matrix of portfolios held in state e , with row j of $\beta(e)$ equal to $\beta_j(e)$. Firms hold no securities initially; hence $\beta(e_0) \equiv 0$. As stated above, dividends and security prices must be simultaneously determined. For a unique determination, this will require a regularity condition:

DEFINITION 3. The trading process β is *proper* if $I - \beta(e)$ is non-singular for all e in E .

Given a proper trading strategy β , each firm j will use its conjectured market valuation function to determine the new dividend and security price processes which will result. Let $\hat{\Delta}^\beta, \hat{\pi}^\beta$ be the processes conjectured by a firm using an arbitrage-free valuation function identified by state prices q . Then consistency implies that $\hat{\pi}^\beta$ is determined from $\hat{\Delta}^\beta$ as follows:

$$\hat{\pi}^\beta(e) = \hat{\Delta}^\beta(e) + \sum_{n > e} \frac{q(n)}{q(e)} \hat{\Delta}^\beta(n). \quad (2)$$

Hence, by definition,

$$\begin{aligned}\hat{A}^\beta(e) &\equiv A(e) + \delta^\beta [\hat{A}^\beta, \hat{\pi}^\beta](e) \\ &= A(e) + \beta(e) \hat{A}^\beta(e) + [\beta(e^-) - \beta(e)] \hat{\pi}^\beta(e) \\ &= A(e) + \beta(e^-) \hat{A}^\beta(e) + [\beta(e^-) - \beta(e)] \sum_{n>e} \frac{q(n)}{q(e)} \hat{A}^\beta(n).\end{aligned}\quad (3)$$

Thus, since β is proper,

$$\hat{A}^\beta(e) = [I - \beta(e^-)]^{-1} \left\{ A(e) + [\beta(e^-) - \beta(e)] \sum_{n>e} \frac{q(n)}{q(e)} \hat{A}^\beta(n) \right\}, \quad (4)$$

which uniquely defines \hat{A}^β recursively.

This demonstrates, as is done in Duffie and Shafer [5], the existence of new dividend and price processes consistent with a proper security trading strategy and a given discount process. This transformation is, however, non-trivial; it is far from obvious that firms cannot use such strategies to increase their current share value. Indeed, Eq. (4) makes it clear that the present dividend $\hat{A}_j^\beta(e)$ of any security may depend upon the past and future trading decisions of all other securities in the market. Also, both (2) and (4) seem to depend on the particular state prices q , which may be different for all firms in the economy.

3.2. Modigliani–Miller I

The issues just raised are resolved by the following result:

THEOREM 1. *Given an equilibrium $((x^i, \gamma^i), (y_j, q_j), p, \pi)$, if firms contemplate a proper trading strategy β and take their market valuation functions as given, the new dividend and price processes they will anticipate are*

$$\begin{aligned}\hat{\pi}^\beta(e) &= [I - \beta(e^-)]^{-1} \pi(e), \\ \hat{A}^\beta(e) &= [I - \beta(e)]^{-1} A(e) + \{ [I - \beta(e^-)]^{-1} - [I - \beta(e)]^{-1} \} \pi(e).\end{aligned}$$

Proof. First consider $\hat{\pi}^\beta$. The proposition can easily be checked to hold for all terminal states. Now, suppose it is true for all states $n > e$. Substituting (4) into (2) yields

$$\begin{aligned}\hat{\pi}^\beta(e) &= [I - \beta(e^-)]^{-1} \left\{ A(e) + [\beta(e^-) - \beta(e)] \sum_{n>e} \frac{q(n)}{q(e)} \hat{A}^\beta(n) \right\} \\ &\quad + \sum_{n>e} \frac{q(n)}{q(e)} \hat{A}^\beta(n) \\ &= [I - \beta(e^-)]^{-1} \left\{ A(e) + [I - \beta(e)] \sum_{n>e} \frac{q(n)}{q(e)} \hat{A}^\beta(n) \right\}.\end{aligned}$$

Also, (2) implies

$$\sum_{n>e} \frac{q(n)}{q(e)} \hat{\Delta}^\beta(n) = \sum_{n=e} \frac{q(n)}{q(e)} \hat{\pi}^\beta(n),$$

so that

$$\begin{aligned} \hat{\pi}^\beta(e) &= [I - \beta(e^-)]^{-1} \left\{ \Delta(e) + \sum_{n=e} \frac{q(n)}{q(e)} [I - \beta(e)] \hat{\pi}^\beta(n) \right\} \\ &= [I - \beta(e^-)]^{-1} \left\{ \Delta(e) + \sum_{n=e} \frac{q(n)}{q(e)} \pi(n) \right\} \\ &= [I - \beta(e^-)]^{-1} \pi(e), \end{aligned}$$

where the last two equations follow from the induction hypothesis and the consistency of the current state prices, respectively.

Finally, the formula for $\hat{\Delta}^\beta$ follows upon substituting for $\hat{\pi}^\beta$ in (3) and algebraic manipulation. ■

The above result has the following version of the Modigliani–Miller theorem, also shown in the model of Duffie and Shafer [5], as an immediate corollary:

COROLLARY 1. *Given an equilibrium, firms will not expect the adoption of a proper trading strategy β to affect current share prices. That is, $\hat{\pi}^\beta(e_1) = \pi(e_1)$.*

Proof. Since any security trading strategy which firms adopt in state e_1 must have $\beta(e_1^-) \equiv \beta(e_0) \equiv 0$, the theorem implies that $\hat{\pi}^\beta(e_1) = \pi(e_1)$. ■

Thus, firms will have no incentive to alter their financial structure. A firm's conjecture as to the new market value of its equity is exactly its current value at the state in which a new financial policy is adopted.

Although initial prices are unchanged, notice that security prices at future dates/events are affected. Yet, if one defines the "value" V_j^β of a security to be its share price minus the value of its portfolio—which corresponds, in a debt–equity model, to equity plus debt outstanding—the following is obtained:

COROLLARY 2. *Given an equilibrium, firms will not expect a proper trading strategy β to affect their value in any state. That is, $V^\beta = V$.*

Proof. From the theorem,

$$\begin{aligned} V^\beta(e) &\equiv \hat{\pi}^\beta(e) - \beta(e^-) \hat{\pi}^\beta(e) = [I - \beta(e^-)] \hat{\pi}^\beta(e) \\ &= \pi(e) = V(e), \end{aligned}$$

as stated. ■

4. GENERAL EQUILIBRIUM ANALYSIS

The previous section has demonstrated that the main implications of the Modigliani–Miller theory hold within the current model. Note, however, that the analysis thus far has been purely a partial equilibrium analysis: firms believe commodity prices are fixed and security prices are generated by a simple market valuation function, not the outcome of a full general equilibrium model. It has not yet been established that these original prices, allocations, and beliefs will be consistent with any new equilibrium in which firms do trade securities.

4.1. *Equilibrium with Firm Trading*

First, the definition of equilibrium must be extended to accommodate security trading by firms. Since the results of Section 3 indicated that firms do not have a market incentive to alter their trading strategies, an equilibrium is defined with respect to a particular proper trading strategy β , taken as given.

Note that, with firm trading, returns to an agent's trading strategy γ are given by $\delta^\gamma[\Delta^\beta, \pi]$, where Δ^β is the dividend process induced by the trading strategy β given initial dividends Δ and share prices π . Thus, Δ^β is defined by

$$\begin{aligned} \Delta^\beta(e) &\equiv \Delta(e) + \delta^\beta[\Delta^\beta, \pi](e) \\ &= \Delta(e) + \beta(e) \Delta^\beta(e) + [\beta(e^-) - \beta(e)] \pi(e). \end{aligned}$$

Since β is proper, this has a unique solution:

$$\Delta^\beta(e) = [I - \beta(e)]^{-1} \{ \Delta(e) + [\beta(e^-) - \beta(e)] \pi(e) \}. \quad (5)$$

The definition of an equilibrium can now be extended to include the participation by firms (and other securities) in the security markets:

DEFINITION 4. Given the stochastic, incomplete markets economy

$$((E, A), (u^i, w^i, \theta^i), (Y_j), (\delta_h, d_h), \beta),$$

for $i = 1, \dots, m, j = 1, \dots, n$, and $h = n + 1, \dots, n + k$, in which β is the proper trading strategy adopted by firms and securities, a collection

$$((x^i, \gamma^i), (y_j, q_j), p, \pi), \quad i = 1, \dots, m, \quad j = 1, \dots, n,$$

is said to be an *equilibrium* if

- (1) for each agent i , (x^i, γ^i) solves (P1) given p, Δ^β, π ,
- (2) for each firm j , y_j solves (P2) given p, q_j ,
- (3) for each firm j , $\pi(e) = \sum_{n \geq e} (q_j(n)/q_j(e)) \Delta^\beta(n)$,
- (4) $\sum_{i=1}^m x^i - w^i = \sum_{j=1}^n y_j$,
- (5) $\sum_{i=1}^m \gamma^i(e) + \sum_{j=1}^n \beta_j(e) = \sum_{i=1}^m \theta^i$ for e in E .

Notice that the firms' optimization problem is unchanged, since, by the analysis of Section 3, initial share prices are independent of security trades and may be calculated according to the same objective function. Note also that the security market clearing constraint has been adapted to include the effect of security trading by firms only. This is so since all other securities are in zero net supply and thus their portfolios have no aggregate effect.

4.2. Modigliani–Miller II

In a complete markets setting, agents can finance any net consumption stream with zero present value by appropriate security trading. Thus, in that context, agents care only about their initial wealth, and not about the dividend or price processes of particular firms; the results of the previous section would therefore be sufficient to demonstrate that corporate financial policy has no real effect on equilibria. However, with incomplete markets this may no longer be the case. Agents' budget sets are constrained by the set of dividend streams available; any transformation of the dividend processes may alter the feasibility of financing a given expenditure process. Hence, to be sure firm security trading has no real ramifications, it must be shown that the set of budget feasible consumption processes for each agent is unchanged. In effect, agents must be able to "undo" the trades made by firms. Fortunately, this can be verified, and the Modigliani–Miller theorem holds in the strong sense:

THEOREM 2. *Suppose the economy has an equilibrium*

$$((x^i, \gamma^i), (y_j, q_j), p, \pi), \quad i = 1, \dots, m, \quad j = 1, \dots, n.$$

If firms adopt a proper trading strategy β , then the economy has a new equilibrium

$$((x^i, \bar{\gamma}^i), (y_j, q_j), p, \bar{\pi}), \quad i = 1, \dots, m, \quad j = 1, \dots, n,$$

with portfolios $\bar{\gamma}^i(e) = \gamma^i(e)[I - \beta(e)]$, and security prices $\bar{\pi}(e) = [I - \beta(e^-)]^{-1}\pi(e)$ for all e in E .

Proof. First, since the original allocation is an equilibrium, Conditions 2 and 4 must obviously hold in the new equilibrium. Condition 5 follows easily from the observation

$$\begin{aligned} \sum_{i=1}^m \bar{\gamma}^i(e) &= \sum_{i=1}^m \gamma^i(e)[I - \beta(e)] = \sum_{i=1}^m \theta^i [I - \beta(e)] \\ &= \sum_{i=1}^m \theta^i - \sum_{j=1}^n \beta_j(e). \end{aligned}$$

Next, the consistency Condition 3 must be verified. Since $\bar{\pi} = \hat{\pi}^\beta$ as defined in Theorem 1, then the definition of Δ^β implies $\Delta^\beta = \hat{\Delta}^\beta$. Thus, the results of Section 3 suffice to guarantee that $\bar{\pi}$ is indeed generated by Δ^β given state prices q_j .

Finally, it remains to be shown that Condition 1 is still satisfied. To demonstrate the optimality of the initial consumption choice, it is sufficient to show that the agents' budget sets are unaltered by the firms' trading strategy β . The following lemma thus completes the proof:

LEMMA 2. Let $\bar{\gamma}(e) = \gamma(e)[I - \beta(e)]$ and $\bar{\pi}(e) = [I - \beta(e^-)]^{-1}\pi(e)$. Then $\delta^{\bar{\gamma}}[\Delta^\beta, \bar{\pi}] = \delta^\gamma[\Delta, \pi]$.

Proof. From the definition,

$$\begin{aligned} \delta^{\bar{\gamma}}[\Delta^\beta, \bar{\pi}](e) &= \bar{\gamma}(e) \Delta^\beta(e) + [\bar{\gamma}(e^-) - \bar{\gamma}(e)] \bar{\pi}(e) \\ &= \bar{\gamma}(e^-) \bar{\pi}(e) + \bar{\gamma}(e) [\Delta^\beta(e) - \bar{\pi}(e)]. \end{aligned}$$

Now, $\bar{\gamma}(e^-) \bar{\pi}(e) = \gamma(e^-)[I - \beta(e^-)] \bar{\pi}(e) = \gamma(e^-) \pi(e)$ by hypothesis. Also, as noted above, $\bar{\pi} = \hat{\pi}^\beta$ implies $\hat{\Delta}^\beta = \Delta^\beta$, so from Theorem 1,

$$\Delta^\beta(e) - \bar{\pi}(e) = [I - \beta(e)]^{-1} \{ \Delta(e) - \pi(e) \}.$$

Therefore, $\bar{\gamma}(e) [\Delta^\beta(e) - \bar{\pi}(e)] = \gamma(e) [\Delta(e) - \pi(e)]$, and thus,

$$\delta^{\bar{\gamma}}[\Delta^\beta, \bar{\pi}](e) = \gamma(e^-) \pi(e) + \gamma(e) [\Delta(e) - \pi(e)] = \delta^\gamma[\Delta, \pi](e),$$

as required. ■

5. FIRM CONTROL AND PRODUCTION OBJECTIVES

As was discussed in Section 2, the appropriate objective for the firm when markets are incomplete is problematic. Thus, the assumption has

been maintained that firms are “value maximizers”: production choices are those which are profit maximizing relative to the firms conjectured “discount prices” for future revenue. Such an objective is open to the criticism that the production choice of the firm does not take into account the preferences of the firm’s shareholders, who presumably control the firm. Since it is well known that when markets are incomplete shareholders will generally disagree about the optimal production plan, value maximization appears both inefficient and unrealistic.

5.1. *Alternative Decision Mechanisms*

Two strains of literature have developed in response to this criticism. Addressing the issue of efficiency, Drèze [3] suggests that the firm should maximize a weighted average of its shareholders’ utilities, with the weights reflecting current shareholdings. Grossman and Hart [7] extend this idea to a multiperiod setting and argue that the weights should correspond to the initial shareholder composition of each firm. If the notation $\lambda^i \in D$ is used to represent the subjective discount rate of agent i , that is, assuming differentiable utility and that good 1 is numeraire,

$$\lambda^i(e) \equiv \frac{\partial u^i(x^i)}{\partial x_1(e)} \bigg/ \frac{\partial u^i(x^i)}{\partial x_1(e_1)},$$

then this approach suggests that the firm should discount future revenue according to the process

$$q_j(e) = \sum_{i=1}^m \theta_j^i \lambda^i(e).$$

Alternatively, Drèze [4] and DeMarzo [2] have considered a range of decision mechanisms within the firm which attempt to model directly actual shareholder control. These mechanisms include majority voting, veto power, and types of managerial control. These papers both conclude that the resulting production choice of the firm will be “value maximizing” with respect to a discount process q_j which is some weighted average of the subjective discount rates of its shareholders; that is,

$$q_j(e) = \sum_{i=1}^m \alpha_j^i \lambda^i(e), \quad (6)$$

for some weights $\alpha_j^i \geq 0$, $\sum_{i=1}^m \alpha_j^i = 1$, which depend on the actual shareholder composition and decision rule of firm j .

5.2. Firm Incentives and Partial Equilibrium

Thus, these various formulations of the firm's objective all lead to the restriction (6) on the rate at which the firm discounts future revenue, which determines its optimal production plan. Furthermore, it is easily verified that the equilibrium condition for each agent guarantees that the subjective discount processes λ^i must be consistent with current security prices π (see DeMarzo [2]). Hence, any q_j satisfying (6) will also be consistent with π in equilibrium.

Since consistency is the only property required of the market valuation function q_j in deriving the results of Section 3, it is clear that those results will still hold for any discount process satisfying (6). Thus, for any of the alternative decision procedures discussed above, the firm will still have no incentive in equilibrium to initiate security trading. The only difference is the interpretation of q_j ; no longer an arbitrary belief or conjecture by the firm, it now corresponds to the preferences of the shareholders.

5.3. Firm Control and General Equilibrium

In addition to consistency, the results of Section 4 crucially depend upon the fact that the discount process q_j for each firm is unchanged in the new equilibrium. Under the initial assumption that these processes represent arbitrary beliefs constrained only by consistency, this requirement is innocuous. However, using the more reasonable decision mechanisms considered above, the discount process used by the firm may depend fundamentally on the composition of the firm's shareholders. Thus, if different agents control the firm in the new equilibrium of Theorem 2, the assumption that q_j is unchanged may no longer be valid.

To verify the general equilibrium version of the Modigliani–Miller theorem under these alternative decision criteria, it is therefore necessary to establish that the real shareholder composition of each firm is unaltered. Let the notation $\varphi_j^i(e)$ represent the total number of shares of security j to which agent i has a claim in state e . Clearly, if firms are not engaged in security trading (i.e., $\beta = 0$), then the process φ^i is identically γ^i , the agent's portfolio. However, this relationship no longer holds if firms are actively trading securities. Since it is assumed that the claim to a firm's portfolio is held directly by its shareholders, owning a fraction of firm j also entitles the agent to that fraction of firm j 's portfolio. Thus, the real ownership fraction φ^i is defined via the relationship

$$\varphi^i(e) \equiv \gamma^i(e) + \varphi^i(e) \beta(e),$$

which implies,

$$\varphi^i(e) = \gamma^i(e) [I - \beta(e)]^{-1}. \quad (7)$$

This result makes the following corollary to Theorem 2 straightforward:

COROLLARY 3. *In the equilibria discussed in Theorem 2, the real shareholder composition of each firm is unchanged; that is, $\bar{\varphi}^i = \varphi^i$ for each agent i .*

Proof. From (7),

$$\begin{aligned}\bar{\varphi}^i(e) &= \bar{\gamma}^i(e)[I - \beta(e)]^{-1} = \gamma^i(e)[I - \beta(e)][I - \beta(e)]^{-1} \\ &= \gamma^i(e) = \varphi^i(e),\end{aligned}$$

as required. ■

Thus, it has been established that security trading by firms need not change the real shareholder composition of each firm. Since control of all firms is unaltered, their objectives, discount processes, and optimal production plans will remain fixed as well.

6. CONCLUSION

Hence, the full implications of the Modigliani–Miller argument hold in this general model of an incomplete markets economy: firms have no incentive to engage in security trading, and any such trading will not affect equilibrium commodity prices, allocations, and the real ownership distribution within the economy. The theory also retains the intuitive content of Modigliani and Miller’s original insight: individuals are able to undo the financial activities of firms within their own portfolios. Notice that the above result still holds even if the set of admissible trading strategies available to agents is restricted. The only requirement is that the set of strategies available to the agents contains the linear subspace spanned by those available to the firms. This analysis could be extended in a straightforward manner to incorporate general incomplete preference orders for the agents and countably many states. In the latter case, regularity conditions would be required to guarantee that the arbitrage pricing relationship is well-defined (see Sethi *et al.* [12]). Also, limited liability and margin contracts could be added to the model. Hellwig [8] demonstrates that in the absence of short sales restrictions, this will not affect the results.

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