A Unified Framework for Monetary Theory and Policy Analysis

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Introduction

- Reduced-form monetary macro models: not explicit about the role of money in overcoming spatial, temporal or informational frictions.
- Search models have explicit micro-foundations.
- Previous search models: ill-suited for the analysis of monetary policy due to the extreme restrictions on money holding.
- This model: no extreme restrictions on money holding.

Main feature of the model

- Previous models without restrictions on money holding are complicated by the endogenous distribution of money holding, $F(m)$.
- Assumption of quasi-linear preference makes $F(m)$ degenerate: No wealth effects in the demand for money.
- This framework is as easy to use as standard reduced-form models (e.g. study the cost of inflation)

Model: market structure and preferences

$$\text{Market structure} \begin{cases} \text{Day -- DM (search): special goods} \\ \text{Night -- CM (Walrasian): general goods} \end{cases}$$

- Preferences: $U(\underbrace{x, h}_{\text{day}}, \underbrace{X, H}_{\text{night}}) = u(x) - c(h) + U(X) - H.$
- x, X: consumption. h, H: labor supply.
- $\exists q^* \in (0, \infty) \;\; s.t. \;\; u'(q^*) = c'(q^*).$
  $\exists X^* \in (0, \infty) \;\; s.t. \;\; U'(X^*) = 1$ with $U(X^*) > X^*$

Model: DM

- DM: decentralized and anonymous $\rightarrow$ no credit.
  α: prob of meeting.
- special goods:
  prob(double coincidence of wants) $= \delta$.

  prob(single coincidence of wants) $= \sigma$.

  prob(neither wants the other produces) $= 1 - 2\sigma - \delta$.

Model: CM

- CM: All agents produce and consume a general good.
- Special goods and general goods are divisible and non-storable
  $\rightarrow$ no commodity money.

Model: distribution of money holdings

- money: perfectly divisible and storable in any non-negative quantity. M: total money stock
- $F_t(\tilde{m})$ ($G_t(\tilde{m})$): measure of agents starting the DM (CM) holding $m \leq \tilde{m}$, F_0, G_0 exogenously given.
- $\int m \; dF_t(m) = \int m \; dG_t(m) = M, \forall t$.
- ϕ_t: value of money in terms of general goods in CM.
- No uncertainty in the basic model except for random matching.
- Aggregate variables such as F_t, G_t and prices are taken as given, an agent's decisions depend only on his money holdings, m.

Value function: DM

An agent with m entering DM:

$$\begin{aligned} V_t(m) &= \alpha\sigma \int \{u[q_t(m,\tilde{m})] + W_t[m - d_t(m,\tilde{m})]\} dF_t(\tilde{m}) \\ &+ \alpha\sigma \int \{-c[q_t(\tilde{m},m)] + W_t[m + d_t(\tilde{m},m)]\} dF_t(\tilde{m}) \\ &+ \alpha\delta \int B_t(m,\tilde{m})\ dF_t(\tilde{m}) \\ &+ (1 - 2\alpha\sigma - \alpha\delta) W_t(m). \end{aligned} \quad (1)$$

Value function: CM

An agent with m entering CM:

$$W_t(m) = \max_{X,H,m'} \{U(X) - H + \beta V_{t+1}(m')\} \tag{2}$$

$$\text{s.t.} X = H + \phi_t m - \phi_t m'$$

$$X \geq 0, 0 \leq H \leq \bar{H}, m' \geq 0.$$

m': money taken out of the market.

- Assume interior solution for X, H, characterize equilibrium and then check $0 < H < \bar{H}$ is satisfied.

Bargaining: agents with m meets someone with $\tilde{m}$

- In a double-coincidence-of-wants meeting:
  symmetric Nash bargaining with the continuation value as the threat point:

$$B_t(m, \tilde{m}) = u(q^*) - c(q^*) + W_t(m).$$

- In a single-coincidence-of-wants meeting:
  Nash bargaining with the continuation value as the threat point, buyer's bargaining power θ:

$$\begin{aligned} \max_{q,d} \quad & [u(q_t) + W_t(m - d_t) - W_t(m)]^{\theta} \\ & [-c(q_t) + W_t(\tilde{m} + d_t) - W_t(\tilde{m})]^{1-\theta} \\ \text{s.t.} \quad & d \leq m, q \geq 0. \end{aligned}$$

- definition of equilibrium (p.468).

How to find an equilibrium?

1. Derive some properties of the solution to the CM problem.
2. Solve the bargaining problem.
3. Simplify V_t and solve for individual's problem of choosing $m_t'(m)$: $m_t' = M$ for all agents regardless of m_t,
   $\Rightarrow F_{t+1}$ degenerate
4. Combine the solutions to CM and DM problems to reduce the model to a single difference equation.

Linearity of $W(m)$

- Substitute for H from the budget equation to write (2) as

$$W_t(m) = \phi_t m + \max_{X,m'}\{U(X) - X - \phi_t m' + \beta V_{t+1}(m')\}$$

where

$$\max_{X,m'}\{U(X) - X - \phi_t m' + \beta V_{t+1}(m')\} \equiv W(0).$$

- Notes:
  - $X(m) = X^*$ where $U'(X^*) = 1$.
  - $m'_t(m)$ does not depend on m. (quasi-linear utility rules out the wealth effect)
  - W_t is linear in m with slope ϕ_t: $W(m) = W(0) + \phi m$.

Bargaining problem

Given that $W(m) = W(0) + \phi m$, the bargaining problem,

$$\max_{q,d} \quad [u(q_t) + W_t(m - d_t) - W_t(m)]^{\theta}$$
$$[-c(q_t) + W_t(\tilde{m} + d_t) - W_t(\tilde{m})]^{1-\theta}$$
$$\text{s.t.} \quad d \leq m, q \geq 0,$$

becomes:

$$\max_{q,d} \quad [u(q) - \phi_t d]^{\theta}[-c(q) + \phi_t d]^{1-\theta}$$
$$\text{s.t.} \quad d \leq m, q \geq 0.$$

Bargaining solution

- Bargaining solution:

$$q_t(m,\tilde{m}) = \begin{cases} \hat{q}_t(m) & \text{if} \quad m < m_t^* \\ q^* & \text{if} \quad m \geq m_t^* \end{cases}$$

$$d_t(m,\tilde{m}) = \begin{cases} m & \text{if} \quad m < m_t^* \\ m^* & \text{if} \quad m \geq m_t^* \end{cases}$$

- $\hat{q}_t(m)$ solves $\phi_t m = z(q_t)$

$$z(q) \equiv \frac{\theta c(q)u'(q) + (1-\theta)u(q)c'(q)}{\theta u'(q) + (1-\theta)c'(q)}.$$

$$m_t^* = z(q^*)/\phi_t$$

Verifying the bargaining solution

- Ignoring the constraint $d \leq m$, the necessary and sufficient conditions for a solution are

$$\theta[-c(q_t) + \phi_t d_t]u'(q_t) = (1-\theta)[u(q_t) - \phi_t d_t]c'(q_t)$$
$$\theta[-c(q_t) + \phi_t d_t] = (1-\theta)[u(q_t) - \phi_t d_t]$$

- If the constraint is not binding ($m \geq m_t^*$): $q_t = q^*$, $d_t = m_t^* = [\theta c(q^*) + (1-\theta)u(q^*)]/\phi_t$.
(spend m_t^* dollars to get q^*)

- If the constraint is binding: q_t is given by $\hat{q}_t(m)$ with $d_t = m \Rightarrow \phi_t m = z(q_t)$.
(spend all his money to get $\hat{q}_t(m)$)

- Solutions do not depend on sellers' money holdings $\tilde{m}$!

Property of the bargaining solution

For all $m < m_t^*$, $q_t'(m) = \dfrac{\phi_t}{z'(q_t)}$, where

$$z' = \frac{u'c'[\theta u' + (1-\theta)c'] + \theta(1-\theta)(u-c)(u'c'' - c'u'')}{[\theta u' + (1-\theta)c']^2} > 0.$$

- Note: $\hat{q}_t(m) \rightarrow q^*$ as $m \rightarrow m_t^*$. Hence, $q_t(m) = \hat{q}_t(m)$ is strictly increasing for $m < m_t^*$, is continuous at m_t^*, and is constant at $q_t(m) = q^*$ for all $m > m_t^*$.

Simplifying equation (1)

$$V_t(m) = \upsilon_t(m) + \phi_t m + \max_{m'}\{-\phi_t m' + \beta V_{t+1}(m')\}$$

where

$$\begin{aligned}\upsilon_t(m) &\equiv \alpha\sigma\{u[q_t(m)] - \phi_t d_t(m)\} \\ &+ \alpha\sigma \int \{\phi_t d_t(\tilde{m}) - c[q_t(\tilde{m})]\} dF_t(\tilde{m}) \\ &+ \alpha\delta[u(q^*) - c(q^*)] + U(X^*) - X^*\end{aligned}$$

By repeated substitution we have

$$V_t(m_t) = \upsilon_t(m_t) + \phi_t m_t$$
$$+ \sum_{j=t}^{\infty} \beta^{j-t} \max_{m_{j+1}}\{-\phi_j m_{j+1} + \beta[\upsilon_{j+1}(m_{j+1}) + \phi_{j+1} m_{j+1}]\} \quad (3)$$

Reduces the choice of $\{m_{t+1}\}$ to a sequence of problems defined in terms of primitives, since υ_{t+1} is a known function.

Equilibrium property

$$v'_{t+1}(m_{t+1}) = \alpha\sigma\{u'[q_{t+1}(m_{t+1})]q'_{t+1}(m_{t+1}) - \phi_{t+1}d'_{t+1}(m_{t+1})\}.$$

- $v'_{t+1}(m_{t+1}) = 0$ for all $m_{t+1} \geq m^*_{t+1}$ by the bargaining solution.
  $\Rightarrow$ $\phi_t < \beta\phi_{t+1}$ implies that the problem of choosing m_{t+1} in (3) has no solution, since the objective function is strictly increasing for all $m_{t+1} \geq m^*_{t+1}$.
  $\Rightarrow$ Any equilibrium must satisfy $\phi_t \geq \beta\phi_{t+1}$.
- Therefore, the minimum inflation rate consistent with equilibrium is $\frac{\phi_t}{\phi_{t+1}} = \beta$, which is the Friedman rule.

Equilibrium path: ϕ_t

- $\theta \approx 1$ or u' is log concave $(u'u''' \leq (u'')^2)$
  $\Rightarrow v''_{t+1} < 0$
  $\Rightarrow$ a unique choice of m_{t+1} in any equilibrium; i.e. F_{t+1} degenerate at $m_{t+1} = M$.

- Thus, $d_{t+1} = M$, the buyer exchanges all his money, and $q_{t+1} = \hat{q}_{t+1}(M)$.

- In any monetary equilibrium, FOC evaluated at $m_{t+1} = M$ is $\phi_t = \beta[v'_{t+1}(M) + \phi_{t+1}]$, or

$$\phi_t = \beta\{\alpha\sigma u'[q_{t+1}(M)]q'_{t+1}(M) + (1 - \alpha\sigma)\phi_{t+1}\} \quad (4)$$

Equilibrium path: q_t

- Inserting $\phi_t = z(q_t)/M$ and $q_t'(M) = \phi_t / z'(q_t)$ from the bargaining sol, (4) $\Rightarrow$

$$z(q_t) = \beta z(q_{t+1})[\alpha\sigma \frac{u'(q_{t+1})}{z'(q_{t+1})} + 1 - \alpha\sigma]. \tag{5}$$

  a difference equation in q_t.

- A monetary equilibrium is characterized by any path $\{q_t\}$ satisfying (5) that stays in $(0, q^*)$, since $q_t < q^*$ follows from $m_t < m_t^*$.

Steady State

(5) $\Rightarrow$

$$\frac{u'(q)}{z'(q)} = 1 + \frac{1-\beta}{\alpha\sigma\beta}. \tag{6}$$

- $\theta = 1 : z(q) = c(q) \;\; \exists! \; q > 0$ solves (6) if, e.g. $u'(0) = \infty$.
- $\theta < 1 : \frac{u'(q)}{z'(q)}$ is monotone if, e.g. $\theta \approx 1$ or c is linear and u' log concave.
  If there is a unique solution, q, then $\frac{\partial q}{\partial \theta} > 0$, $\frac{\partial q}{\partial \sigma} > 0$, $\frac{\partial q}{\partial \alpha} > 0$ and $\frac{\partial q}{\partial \beta} > 0$.
- $\theta = 1 : q \to q^*$as $\beta \to 1$
  $\theta < 1 : q < q^*$as $\beta \to 1$
- Steady state is efficient iff $q = q^*$, which requires $\beta = 1$ and $\theta = 1$.

Changes in the money Supply

- New money is injected in CM: $M_{t+1} = (1+\tau)M_t$.
- Consider S-S where q and real balances $\phi M = z(q)$ are constant; i.e., $\phi_t/\phi_{t+1} = 1+\tau$.
- S-S condition:

$$\frac{u'(q)}{z'(q)} = 1 + \frac{1+\tau-\beta}{\alpha\sigma\beta}. \tag{7}$$

- $1+i = (1+r)(1+\pi)$; $\pi = \tau$: equilibrium inflation rate $r = \frac{1-\beta}{\beta}$ equilibrium real interest rate.

  (7)$\Rightarrow$

$$\frac{u'(q)}{z'(q)} = 1 + \frac{i}{\alpha\sigma}$$

- Assume a unique monetary S.S: $\frac{\partial q}{\partial \tau} < 0$; $\frac{\partial q}{\partial i} < 0$.

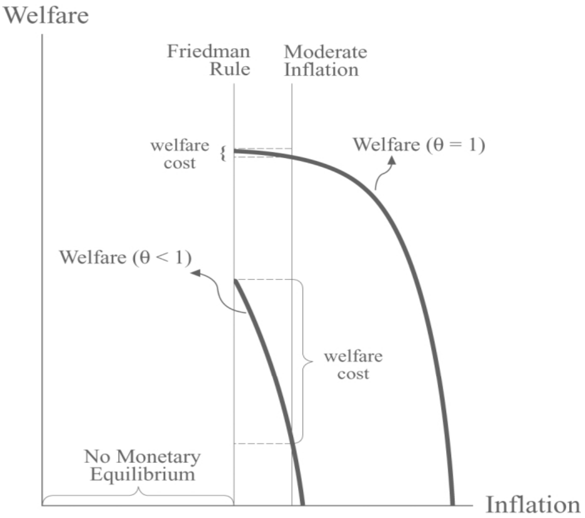
Results

- $\theta = 1 : z(q) = c(q)$, get q^* iff $\tau = \tau^F$ $(i = 0)$
- $\theta < 1 : q < q^*$ at τ^F since a necessary condition for monetary equilibrium is $\tau \geq \tau^F$ $(i \geq 0)$.
  The Friedman rule is optimal here but does not achieve the efficient outcomes q^*.
- Why?

Two types of inefficiencies

- due to $\beta < 1$: $q < q^*$
- due to $\theta < 1$: holdup problem.
- Hosios (1990) condition for efficiency:
  The bargaining solution should split the surplus so that each party is compensated for his contribution to the surplus in a match.
- The surplus in a single-match is all due to the buyer, since the outcome depends on m but not on $\tilde{m}$. Hence, efficiency requires $\theta = 1$ here.
- The wedge due to $\theta < 1$ is important for issues such as the welfare cost of inflation.

Welfare cost of moderate inflation



Welfare cost of inflation

- Calibrate the model to standard observations and use it to measure the cost of inflation.
- Going from 10 percent to 0 percent inflation is worth between 3 and 5 percent of consumption – much higher than previous estimates.
- The empirical relevance of the holdup problem is important to assessing the welfare cost of inflation.