Minimal Null Designs and a Density Theorem of Posets

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Classically, null designs were defined on the poset of subsets of a given finite set (boolean algebra). A null design is defined as a collection of weighted $k$-subsets such that the sum of the weights of $k$-subsets containing a $t$-subset is 0 for every $t$-subset, where $0 \leq t < k \leq n$. Null designs are useful to understand designs or to construct new designs from a known one. They also deserve research as pure combinatorial objects. In particular, people have been interested in the minimum number of $k$-subsets of non-zero weight to make a non-zero null design, and the characterization of the null designs with the minimal number of $k$-subsets of non-zero weight, which we call minimal null designs. Minimal null designs were used to construct explicit bases of the space of null designs.

The definition of null designs can be extended to any poset which has graded structure (ranked poset) as the boolean algebra does. In this paper, we prove general theorems on the structure of the null designs of finite ranked posets, which also yield a density theorem of finite ranked posets. We apply the theorems to two special posets—the boolean algebra and the generalized $(q$-analogue of) boolean algebra—to characterize the minimal null $t$-designs.

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1. Introduction

We let $B_n$ denote the poset of all subsets of a given $n$-set with the order given as the set inclusion. We call $B_n$ the boolean algebra. Obviously, every maximal totally ordered set in $B_n$ has the same number, $n + 1$, of elements in $B_n$, so $B_n$ is a ranked poset of highest rank $n$ (the rank of $S \in B_n$ is the set size of $S$). For integers $0 \leq t < k \leq n$, a null $(t,k)$-design of $B_n$ is an $\mathbb{R}$-linear sum of $k$-subsets such that the sum of coefficients of $k$-subsets containing a $t$-subset is 0 for every $t$-subset. Note that null designs are also known as trades [6, 7]. We call a non-zero null design of $B_n$ minimal if it has the minimum possible number of $k$-subsets with non-zero coefficients. For example, $\{1, 3\} - \{1, 4\} - \{2, 3\} + \{2, 4\}$ is a minimal null $(1, 2)$-design of $B_4$.

Originally, null designs were defined to understand designs better [5]. Null designs, however, have also been studied in many different aspects. Beautiful bases of the space of null designs have been constructed [2, 4, 7], the dimension of the space has been determined and the minimal number of $k$-subsets needed to construct a non-zero null design has been revealed [3, 4]. Moreover, the minimal null $(t,k)$-designs were characterized as $S_n$ (permutation group on $n$-letters) images of those basis elements of Graham, Li and Li, or Frankl, when $k = t + 1$ [8]. Null designs also have been used to construct a new design from a known design and, here, minimal null designs play important roles [6].

In this paper, we extend the definition of null designs to any finite ranked poset, and prove some general theorems on the structure of null designs and a density theorem for posets. Moreover, we apply general theorems to two special ranked posets, the boolean algebra and the generalized boolean algebra (the poset of the Hamming scheme) to obtain the tight lower bound of the support size of null designs and to characterize the minimal null designs. Remember that a subset of an $n$-set (i.e., an element of $B_n$) can be realized as a $[0, 1]$-valued function on the set $[n] = \{1, 2, \ldots, n\}$. In this context, the elements of the generalized boolean algebra, $B_n^q$, are defined as $[0, 1, \ldots, q]$-valued functions on the set $[n]$ and the order is given as the function extension ($f \leq g \in B_n^q$ if $g$ is an extension of $f$).

For a given finite ranked poset $P$, with the highest rank $n$, we let $P_i$ be the set of elements of rank $i$ of $P$. $\mathbb{R}[X]$ denotes the vector space of $\mathbb{R}$-linear sums of elements in $X$ for any
finite set $X$. For integers $0 \leq t < k \leq n$, we define a linear map $d_{k,t}^P$ from $\mathbb{R}[P_k]$ to $\mathbb{R}[P_t]$ as $d_{k,t}^P(x) = \sum_{y \leq x} y$ for a basis element $x \in P_k$. $d_{k,t}^P$s are often called incidence maps between $P_k$ and $P_t$.

For $0 \leq t < k \leq n$, the null $(t, k)$-design of $P$, $\omega$, is an element of $\mathbb{R}[P_k]$ satisfying $d_{k,t}^P(\omega) = 0$.

It is easy to check that the definition of null $(t, k)$-designs of $B_n$, given above, coincides with the general definition of null designs. Let $NP(t,k)$ be the set of all null $(t, k)$-designs of $P$, and let

$$NP(t) = \{ \omega \in \mathbb{R}[P] : d_i(\omega) = 0 \text{ for all } 0 \leq i \leq t \},$$

where $d_i : \mathbb{R}[P] \rightarrow \mathbb{R}[P_i]$ is defined by $d_i(x) = \sum_{y \leq x} y$, for $x \in P$. We say that $P$ satisfies the downmap condition for level $l$, $(D_l)$, if

$$d_{k,t}^P(\omega) = 0 \text{ implies } d_{k,t'}^P(\omega) = 0 \text{ for } t' \leq t, \quad \text{for all } t < k \leq l.$$

Note that $NP(t,k) \subseteq NP(t)$ if $P$ satisfies the condition $(D_l)$ for $k \leq l$. For given $\omega = \sum_{c_x \in \mathbb{R}} c_x x \in \mathbb{R}[P]$, we define the Support of $\omega$ by

$$\text{Supp}(\omega) \equiv \{ x : c_x \neq 0 \}$$

and for $y \in P$,

$$c_{\geq y} \equiv \sum_{x \geq y} c_x.$$

We call $\omega \in NP(t,k)$ minimal if the size of $\text{Supp}(\omega)$ attains the minimum possible value of support size of non-zero null designs. $\mu$ is the Möbius function defined on $P$ (see [10]).

In Section 2 we prove some general theorems on null designs of finite ranked posets and prove a density theorem of a finite ranked poset. In Sections 3 and 4, we consider the boolean algebra and the generalized boolean algebra, respectively, as special cases.

2. Preliminaries and a Density Theorem

In this section, we develop some basic tools which we will need in the following sections and prove a density theorem for finite ranked posets. Remember that the meet of $x, y \in P$, denoted by $x \wedge y$, is defined as the greatest lower bound of $x$ and $y$, if it exists. We call a poset $P$ a meet-semilattice if every pair of elements of $P$ has a meet (see [10]).

The following is the main tool we use in this paper. The proof for $NP(t,k)$ is given in [1].

**Proposition 1.** Let $P$ be a finite ranked meet-semilattice and let us assume that $\omega = \sum_{c_x \in \mathbb{R}} c_x x \in NP(t)$. Then for any fixed $y \in P_{t+1}$, and any $z \leq y$, the following holds

$$\sum_{x \in P} c_x = c_{\geq y} \mu(z,y). \quad (1)$$

**Proof.** For $z \leq y$, let us consider the sum

$$S \equiv \sum_{z \leq z' \leq y} c_{z'} \mu(z,z').$$


By the definition of $N_P(t)$, $c_{z,z'} = 0$ for all $z' < y$. So, $S = c_{z,y} \mu(z,y)$. On the other hand,
\[
S = \sum_{z \leq z' \leq y} \left( \sum_{z \leq x \in P} c_x \right) \mu(z, z')
\]
\[
= \sum_{x \in P} \left( \sum_{z \leq z' \leq x \wedge y} \mu(z, z') \right) c_x
\]
\[
= \sum_{x \in P, z \wedge y = \omega} c_x,
\]
since
\[
\sum_{z \leq z' \leq x \wedge y} \mu(z, z') = \begin{cases} 0 & \text{if } z \neq x \wedge y \\ 1 & \text{if } z = x \wedge y \end{cases}
\]
by the definition of the Möbius function. This completes the proof.

As a corollary of Proposition 1, we can state a density theorem for families of elements of $P$. The idea of the proof is exactly the same as that for the proof for the case $P = B_n$ (see [3]).

**PROPOSITION 2 (DENSITY THEOREM).** Let $P$ be a finite ranked meet-semilattice such that $\mu(x_1, x_2) \neq 0$ for all $x_1 \leq x_2$. Suppose
\[
F \subseteq P, \quad |F| > \sum_{0 \leq t \leq s} |P_t|.
\]
Then there exists an $x \in P_{t+1}$, such that for every $z \leq x$ one can find $x_z \in F$ with $x_z \wedge x = z$.

**PROOF.** For every $y \in F$, define $\omega_y \in \mathbb{R}[P]$ by
\[
C_y(\omega_y) = \begin{cases} 1 & \text{if } y' \leq y, y' \in P_t, t \leq t, \\ 0 & \text{otherwise}, \end{cases}
\]
where $C_y(\omega_y)$ denotes the coefficient of $y'$ in $\omega_y$. Then $C_y$ is in $\mathbb{R}[\bigcup_{t=0}^s P_t]$ which is $(\sum_{0 \leq t \leq s} |P_t|)$-dimensional. Since $|F| > \sum_{0 \leq t \leq s} |P_t|$, $\{\omega_y : y \in F\}$ cannot be linearly independent. Let $\sum_{y \in F} c_y \omega_y = 0$ be a linear dependence among them. Define $\omega \in \mathbb{R}[P]$ by $\omega = \sum_{y \in F} c_y \omega_y$. Then $\omega \in N_P(t)$ by the above dependency relation. Moreover, there is an $s \geq t$ such that $\omega \in N_P(s)$ but $\omega \notin N_P(s+1)$. Let $y \in P_{t+1}$ such that $c_{y,y} \neq 0$ but $c_{x,y} = 0$ for all $z \leq y$. Then equation (1) ensures that for every $z \leq y$, there exists $y_z \in F$ satisfying $y_z \wedge y = z$. Now, for an arbitrary $x \leq y$ in $P_{t+1}$, the condition of the theorem is satisfied.

In the following two corollaries, $P$ is a finite ranked meet-semilattice with the condition $(D_n)$ for the highest rank $n$ of $P$. These are immediate from Proposition 1, since $N_P(t,k) \subseteq N_P(t)$ if $P$ satisfies $(D_n)$.

**COROLLARY 3.** Suppose that $\omega = \sum_{x \in P} c_x x \in N_P(t,k)$ but $\omega \notin N_P(t+1,k)$. Then there must be some $y \in P_{t+1}$ such that $c_{y,y} \neq 0$. For any such fixed $y$, $|\text{Supp}(\omega)| \geq |\{z : z \leq y, \mu(z, y) \neq 0\}|$.

**COROLLARY 4.** Let us assume that the lower bound in Corollary 3 is tight. Then the coefficients of minimal null designs in $N_P(t,k)$ are $\pm 1$ or 0, up to constant multiplication. Moreover, if $\omega = \sum_{x \in P} c_x x$ is a minimal null design, then for each $y \in P_{t+1}$ such that $c_{y,y} \neq 0$ and for each $z \leq y$, there is a unique $x_z \in \text{Supp}(\omega)$ such that $x_z \wedge y = z$ and $c_{x_z} = (-1)^{t+1+\text{rank}(z)} c_{y,y}$.

In Section 3, we consider $P = B_n$ the boolean algebra, and in Section 4, we consider $P = B_n^d$ the generalized boolean algebra. We use the results given in this section to find the minimal support size of non-zero null designs and to characterize the minimal null designs.
3. BOOLEAN ALGEBRA (POSET OF JOHNSON SCHEME)

In this section we consider the poset $P = B_n$. Note that each level of $B_n$ is a Johnson scheme, (in association schemes the boolean algebra is known as the poset of the Johnson scheme, see [11]).

It is easy enough to check that $B_n$ is a meet-semilattice (actually a lattice) and that $B_n$ satisfies the condition $(D_n)$. We first prove a lemma which will allow us to assume that $n \geq k + t + 1$. This result was also proved in [6].

**Lemma 5.** If $n < k + t + 1$, then $N_{B_n}(t, k)$ is empty.

**Proof.** This is immediate from Proposition 1 since a disjoint pair of a $(t + 1)$-subset and a $k$-subset of $[n]$ cannot exist.

We identify a subset $\{i_1, \ldots, i_t\}$ of $[n]$, with the product $x_{i_1} \cdots x_{i_t}$ of commuting indeterminates. It is well known that the M"obius function on $B_n$ is given by

$$\mu(T, S) = (-1)^{|S-T|} \text{ where } T \leq S \in B_n.$$ 

Hence by Corollary 3, any non-zero null $(t, k)$-design has a support size of at least $2^{t+1}$. On the other hand, $(x_{i_1} - x_{i_2})(x_{i_3} - x_{i_4}) \cdots (x_{i_{2t+1}} - x_{i_{2t+2}}) x_{i_{2t+3}} \cdots x_{i_{k+t+1}}$ is a well known null $(t, k)$-design having support size $2^{t+1}$, which has been used to construct bases of $N_{B_n}(t, k)$. (See [2, 4, 6].)

When $k = t + 1$, $N_{B_n}(t, k)$ is a Specht module (an irreducible representation of the permutation group $S_n$) corresponding to two part partitions (see [8, 9]), and Liebler and Zimmermann did the following characterization of minimal null designs [8].

**Theorem 6.** If $\omega \in N_{B_n}(t, t + 1)$ and $|\text{Supp}(\omega)| = 2^{t+1}$, then $\omega$ is a multiple of $(x_{i_1} - x_{i_2})(x_{i_3} - x_{i_4}) \cdots (x_{i_{2t+1}} - x_{i_{2t+2}})$ for a $(2t + 2)$-subset $\{i_1, \ldots, i_{2t+2}\}$ of $[n]$.

We prove the main theorem in this section.

**Theorem 7.** Let $k > t + 1$ and $n = k + t + 1$. If $\omega \in N_{B_n}(t, k)$ and $|\text{Supp}(\omega)| = 2^{t+1}$, then $\omega$ is a multiple of $(x_{i_1} - x_{i_2})(x_{i_3} - x_{i_4}) \cdots (x_{i_{2t+1}} - x_{i_{2t+2}}) x_{i_{2t+3}} \cdots x_{i_{k+t+1}}$, where $\{i_1, \ldots, i_{k+t+1}\} = [n]$.

**Proof.** Let $\omega = \sum_{S \in N_{B_n}(t, k)} c_S S \in N_{B_n}(t, k)$. We first prove that every element of $\text{Supp}(\omega)$ contains a fixed $(k - t - 1)$-set, and then use Theorem 6 to finish the proof. If $\omega$ is also a null $(t + 1)$-design then $|\text{Supp}(\omega)|$ should be at least $2^{t+2}$, so $\omega$ cannot be a null $(t + 1)$-design. Hence, there must be $T_1 \in (B_n)_{t+1}$ such that $c_{\geq T_1} \neq 0$. Without loss of generality, we may assume that $T_1 = x_1 \cdots x_{t+1}$ and $c_{\geq T_1} = 1$. We prove some lemmas to prove the theorem.

The key idea of the proof is equation (1). We use equation (1) for two different $(t + 1)$-sets. We state Corollary 4 again for our context.

**Lemma 8.** For each $U \leq T_1$ there must be a unique $S_U \in \text{Supp}(\omega)$ such that $S_U \cap T_1 = U$ and $c_{S_U} = (-1)^{j+1-|U|}$.

For $U \leq T_1$, note that we use the notation $S_U$ for the unique element in $\text{Supp}(\omega)$ such that $S_U \cap T_1 = U$ and $c_{S_U} = (-1)^{j+1-|U|}$. Without loss of generality, we assume that $S_{T_1} = x_1 \cdots x_k$. Note that $c_{S_{T_1}} = (-1)^{j+1-|U|} = 1$. Let $U = \emptyset \leq T_1$ then $S_U = x_{t+2} \cdots x_{k+t+1}$ since $n = k + t + 1$ and $S_{\emptyset} \cap T_1 = \emptyset$. Moreover, $c_{S_{\emptyset}} = (-1)^{t+1}$. 

LEMMA 9. Let $T_2 = x_{k+1} \ldots x_{k+t+1} \in X_{t+1}^B$. Then $c_{\geq T_2} = (-1)^{t+1}$. Moreover, for each $U \leq T_2$, there must be a unique element in $\text{Supp}(\omega)$ so that the intersection with $T_2$ is $U$ and the coefficient is $(-1)^{t+1}(-1)^{t+1-|U|} = (-1)^{|U|}$.

PROOF. First, we prove that $c_{\geq T_2} \neq 0$. If $c_{\geq T_2} = 0$, then by (1) (by choosing $U = \emptyset \leq T_2$), \[ \sum_{S \in \mathcal{B}_n \mid S \cap T_2 = \emptyset} c_S = 0. \]
Note that the only $S \in \mathcal{B}_n$ such that $S \cap T_2 = \emptyset$ is $x_1 \ldots x_k = S_{t_1}$, since $n = k + t + 1$. So $c_{\mathcal{S}_{t_1}} = 0$ and this contradicts the fact that $c_{S_{t_1}} = 1$. Therefore, $c_{\geq T_2} \neq 0$. Now equation (1) for $T_2$ becomes, for each $U \leq T_2$,

\[ \sum_{S \in \mathcal{B}_n \mid S \cap T_2 = \emptyset} c_S = c_{\geq T_2} \mu(U, T_2) = c_{\geq T_2}(-1)^{t+1-|U|} \neq 0. \]

By choosing $U = T_2$, the equation becomes $\sum_{S \in \mathcal{B}_n \mid S \cap T_2 = \emptyset} c_S = c_{\geq T_2}$. Note that $T_2 \leq S_0$ and $S_0$ should be the unique element in $\text{Supp}(\omega)$ which contains $T_2$ by Corollary 4, because of the minimality of $|\text{Supp}(\omega)|$. Hence, $c_{\geq T_2} = c_{S_0} = (-1)^{t+1}$. The rest of the lemma is immediate. □

Let $T_3 = x_{t+1} \ldots x_k \in \mathcal{B}_n$. Then note that $T_1$, $T_2$, and $T_3$ are mutually disjoint and $T_1 \lor T_2 \lor T_3 = [n]$.

LEMMA 10. For each $U \leq T_1$, $S_U$ contains $T_3$, in other words, every element of $\text{Supp}(\omega)$ contains $T_3$.

PROOF. We use induction on the rank of $U$. The lemma is true for $\emptyset$ so let us assume that it is true for the $U \leq T_1$ of rank less than $i$. Let $U \leq T_1$ be an element of $\mathcal{B}_n$. Then $|S_U \cap T_1| = |U| = i$, and the maximum possible value of $|S_U \cap T_3|$ is $|T_3| = k - t - 1$ so $|S_U \cap T_2|$ is at least $t+1-i$. Note that $|S_U \cap T_2|$ is exactly $t+1-j$ for an element $U \leq T_1$ of $\mathcal{B}_n$, $j < i$, since $|S_U \cap T_3|$ is $k-t-1$ because of the induction hypothesis. This means that the $S_U$’s, where $T_1 \geq U \in \mathcal{B}_n$, are exactly the elements of $\text{Supp}(\omega)$, which satisfies Lemma 9 for $U'' \leq T_3$ of rank $t+1-j$, since $(t+1) = (t+1-(t+1-j)).$

The previous observation states that $|S_U \cap T_2|$ must be at most $t+1-i$, hence $|S_U \cap T_2| = t+1-i$. Furthermore, $|S_U \cap T_3|$ is exactly $k-t-1$, therefore $S_U \cap T_3 = T_3$. □

By Lemma 10, we have $\omega = x_{t+1} \ldots x_k \sum_{S \in \mathcal{B}_n \mid S \cap T_3 = \emptyset} c_S (S - (x_{t+2} \ldots x_k))$ where $(S - (x_{t+2} \ldots x_k))$ is the set difference. Now we prove that $\omega = \sum_{S \in \mathcal{B}_n \mid S \cap T_3 = \emptyset} c_S (S - (x_{t+2} \ldots x_k)) \in N_{B_n}(t, t+1)$. For $U \in \mathcal{B}_n$, if $U \cap T_3 = \emptyset$ then there is no $T \in \text{Supp}(\omega)$ which covers $U$. So let us assume that $U \cap T_3 = \emptyset$. Then $\sum_{U \leq T_3} c_S = \sum_{S \in \mathcal{B}_n \mid S \cap T_3 = \emptyset} c_S = 0.$

Since $\omega' \in N_{B_n}(t, t+1)$ and $|\text{Supp}(\omega')| = 2^{t+1}$, by Theorem 6, $\omega' = (x_{i_1} - x_{i_2}) (x_{i_3} - x_{i_4}) \ldots (x_{i_{2t+1}} - x_{i_{2t+2}}), \ldots \in [n] - T_3$. This completes the proof of Theorem 7. □

REMARK 1. In the paper by Hwang [6], minimal trades of $B_n$ are defined as the integral-valued trades (null designs) of support size $2^{t+1}$ such that the number of $1$-subsets contained in any $k$-subset in the support is $k + t + 1$. Moreover, the trades of support size $2^{t+1}$ are characterized as

\[ S_0(S_1 - S_2)(S_3 - S_4) \ldots (S_{2t+1} + S_{2t+2}), \]

with $S_i \subset [n]$, $S_i \cap S_j = \emptyset$ and $|S_{2i+1}| = |S_{2i+2}|$ for $i = 0, 1, \ldots, t$. Also, the minimal trades are characterized as

\[ (x_{i_1} - x_{i_2})(x_{i_3} - x_{i_4}) \ldots (x_{i_{2t+1}} - x_{i_{2t+2}})(x_{i_{2t+3}} - x_{i_{2t+4}} + \ldots x_{i_{2t+4+t}}). \]
If \( k = t + 1 \) or \( n = k + t + 1 \) then the minimal null designs in our definition are the minimal null trades of Hwang. Hence, Theorem 6 and Theorem 7 are restatements of Hwang’s result for the case of integral (integer-valued) null designs. However, our definition of null designs is broader in the sense that we allow \( \mathbb{R} \)-valued null designs and Corollary 4 covers the gap between the integer-valued and real-valued null designs.

**Example 1.** For \( t + 1 < k \), if \( n \neq k + t + 1 \), then the minimal elements of \( \mathcal{N}_B(t, k) \) do not have to be
\[
\omega = (x_{i_1} - x_{i_2}) (x_{i_3} - x_{i_4}) \ldots (x_{i_{2t+1}} - x_{i_{2t+2}}) x_{i_{2t+3}} \ldots x_{i_{k+1}},
\]
as in the case \( n = k + t + 1 \).
1. \( x_{1,2} = x_{3,4} \in \mathcal{N}_B(0, 2) \).
2. \( x_{1,2,3} = x_{1,4,5} = x_{2,3} x_6 + x_4 x_5 x_6 \in \mathcal{N}_B(1, 3) \), but the intersection of all sets in the support is \( \emptyset \).

Note, however, that \( x_{1,2} - x_{3,4} \) and \( x_{1,2,3} - x_{1,4,5} - x_{2,3,x_6} + x_4 x_5 x_6 = (x_1 - x_5)(x_2 x_3 - x_4 x_5) \) have the form of (2) as we mention in Remark 1.

### 4. Generalized Boolean Algebra (Poset of Hamming Scheme)

For given positive integers \( n, q \) \((q \geq 2)\), let \( B^q_n \) be the poset of functions \( f : A \rightarrow [q] \), where \( A \) is any subset of \([n]\) and \( g \) contains \( f \) if \( g \) is an extension of \( f \) (see [11]). We can identify \( f : A \rightarrow [q] \) with an \( n \)-tuple \((a_1, \ldots, a_n)\), where \( a_i = f(i) \), \( i \in A; a_i = 0 \), \( i \not\in A \).

It is clear that the Hamming scheme \( H(n, q) \) is the top level of this poset. Moreover, for \( f_1 = (a_1, \ldots, a_n) \) and \( f_2 = (b_1, \ldots, b_n) \) in \( B^q_n \), the meet \( f_1 \wedge f_2 \) is \((c_1, \ldots, c_n)\) where \( c_i = a_i \) if \( a_i = b_i \) and \( c_i = 0 \) if \( a_i \neq b_i \).

**Lemma 11.** For any \( f = (a_1, \ldots, a_n) \in (B^q_n), \ g \in (B^q_n : g \leq f) \cong B_i \).

**Proof.** Without loss of generality, we assume that \( f = (a_1, \ldots, a_i, 0, \ldots, 0) \), where \( a_j \neq 0 \) for \( j < i \). Note that \( g = (b_1, \ldots, b_n) \leq f \) if \( b_{i+1} = \ldots = b_n = 0 \) and for \( j < i \), \( b_j = a_j \) or \( b_j = 0 \). For each \( g = (b_1, \ldots, b_n) \leq f \), define \( \phi(g) \in B_i \) as \( \{ j : b_j \neq 0 \} \), then \( \phi \) is an isomorphism between two posets \( \{ g \in B^q_n : g \leq f \} \) and \( B_i \).

The following corollaries are immediate from the previous lemma.

**Corollary 12.** \( B^q_n \) satisfies the condition \((D_n)\).

**Corollary 13.** For \( g \in (B^q_n), f \in (B^q_n), \) such that \( g \leq f, \mu(g, f) = (-1)^{i-j} \).

By the above corollaries, Corollary 3 gives us a lower bound \( 2^{i+1} \) for the support size of non-zero elements in \( \mathcal{N}_{B^q_n}(t, k) \). Since \( B^q_n \) is a generalized boolean algebra (if \( q = 1 \) then \( B^q_n = B_n \)), it would be reasonable to have the same result for the boolean algebra.

**Theorem 14.** The minimum of the support size of non-zero elements of \( \mathcal{N}_{B^q_n}(t, k) \) is \( 2^{t+1} \).

**Proof.** For given \( t < k \), fix \( a_1 \neq a_2 \) in \([q]\), \( b_1, \ldots, b_{k-1} \in [q] \) and two disjoint subsets \( I = \{i_1, \ldots, i_{t+1}\}, J = \{j_1, \ldots, j_{k-1}\} \) of \([n]\). Let
\[
F = \{ f \in (B^q_n)_k : f(i_1) = a_1 \text{ or } a_2 \text{ on } I, \ f(j_m) = b_m \text{ on } J, \text{ and } f = 0 \text{ on } [n] - I - J \},
\]
and \( \omega = \sum_{f \in F} sign(f) f \) where \( sign(f) = (-1)^{i+1-j}, \ i = \text{the number of } i_j \text{ such that } f(i_j) = a_1 \). Then \( \omega \) is a null \((t, k)\)-design, whose support size is \( 2^{t+1} \).
We characterize the minimal null $t$-designs of $B_n^t$ for the case $n = k = t + 1$.

**Theorem 15.** Let $n = k = t + 1$, and let $\omega$ be a non-zero element of $N_{B_n^t}(1, k)$ with $|\text{Supp}(\omega)| = 2^{t+1}$, then $\omega$ is a multiple of $\sum_{f \in F} \text{sign}(f) f$, where the set $F$ is $F = \{ f : f(i) \in \{a_1, a_2\}, a_1 \neq a_2 \in \{q\}, i = 1, \ldots, n \}$, and for $f \in F$, $\text{sign}(f) = (-1)^{\text{number of } i \text{ such that } f(i) = a_0\}.

**Proof.** We use induction on $t$. If $t = 0$, then $n = k = 1$ so $\omega = (a) - (b)$, where $a \neq b \in \{q\}$. So, we now assume that the theorem is true for values less than $t$. Let $\omega = \sum f c f f$.

**Lemma 16.** For $a \in \{q\}$, let $\omega_a^t \equiv \sum_{f \in \text{Supp}(\omega)} c f f$ be a partial sum of $\omega$. If $\omega_a^t$ is not an empty sum, then $\omega_a^t \equiv \sum_{f \in \text{Supp}(\omega)} c f f$, where $f(i) = 0$ and $f(j) = f(j)$ if $j \neq i$, is an element of $N_{B_n^{t-1}}(t - 1, k - 1)$. Moreover, $|\text{Supp}(\omega_a^t)| = |\text{Supp}(\omega_a^t)| = 2^t$.

**Proof.** For $g \in X_{B_n^{t-1}}$, if $g(i) \neq 0$ then there is no element in $\text{Supp}(\omega_a^t)$ that contains $g$, so we assume that $g(i) = 0$. Now, the element $h$, defined as $h(j) = g(j)$ if $j \neq i$ and $h(i) = a$, has rank $t$, so $\sum_{h \leq f \in \text{Supp}(\omega)} c f f = 0$. Note that if $h \leq f$ then $f(i) = a$, so $\sum_{g \leq f \in \text{Supp}(\omega_a^t)} c f f = \sum_{h \leq f \in \text{Supp}(\omega)} c f f = 0$. This proves $\omega_a^t \in N_{B_n^{t-1}}(t - 1, k - 1)$ but the elements of $\text{Supp}(\omega_a^t)$ have fixed $i$th entry 0 so we can think $\omega_a^t$ as an element of $N_{B_n^{t-1}}(t - 1, k - 1)$, $|\text{Supp}(\omega_a^t)| \geq 2^t$ by the above observation and Theorem 14, and $|\text{Supp}(\omega_a^t)| \leq 2^t$ since $n = k = t + 1$ and $f(i) = 0$ for all $f \in \text{Supp}(\omega_a^t)$). Therefore, we have $|\text{Supp}(\omega_a^t)| = |\text{Supp}(\omega_a^t)| = 2^t$.

By the above lemma, for each $i$, $|[a \in \{q\} : f(i) = a$ for some $f \in \text{Supp}(\omega)]| = 2$. Let those two numbers $a_{i_1}$ and $a_{i_2}$ for each $i$. We only have two choices for each coordinate of $f \in \text{Supp}(\omega)$ and in this way the maximum number of functions we can make is $2^{t+1}$. Since $|\text{Supp}(\omega)| = 2^{t+1}$, $\text{Supp}(\omega) = \{ f : f(i) = a_{i_1} or a_{i_2} \}$. Now, by Corollary 4, the coefficient must work as expected in the theorem.

**Example 2.** The analogous result to Theorem 15 does not have to hold for $n > k$ or $k > t + 1$.
1. $(1, 1, 0) - (0, 1, 1) - (1, 0, 2) + (1, 0, 1) \in N_{B_n^3}(1, 2)$.
2. $(1, 1) - (2, 2) \in N_{B_n^2}(0, 2)$.

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**References**


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