USING FRACTIONAL PROGRAMMING FOR ZERO-NORM APPROXIMATION

Mostafa Medra⋆ *Andrew W. Eckford*† *Raviraj Adve*⋆

⋆Department of Electrical and Computer Engineering, University of Toronto, ON, Canada,
† Department of Electrical Engineering and Computer Science, York University, Toronto, ON, Canada.

ABSTRACT

This paper proposes the use of fractional programming (FP) to solve problems involving the zero-norm. FP is used to transform a non-convex ratio to a convex problem that can be solved iteratively, and guaranteed to converge to the global optimum under some constraints on the numerator and denominator. Recently the FP approach was extended to sums of ratios with proven convergence to a stationary point. In this paper, we reformulate the zero-norm as a ratio satisfying the FP conditions and transform the problem into iterative convex optimization. To assess the proposed tool, we investigate the power minimization problem under signal-to-interference-plus-noise ratio (SINR) constraints, when constraints on the transmitted and circuit power are accounted for. Specifically, the consumed circuit power depends on the number of active antennas, which can be modeled using zero-norm. Numerical simulations illustrate the validity of our proposed approach, demonstrating that significant performance gains over the state of the art can be obtained.

Index Terms— Fractional programming, zero-norm, power minimization, beamforming.

1. INTRODUCTION

Optimization of communication systems is important to achieve the required performance using the least possible resources. One class of challenging optimization problems are those requiring sparse outputs. While the sparsity of a signal can be determined by using the zero-norm (ℓ_0 norm), such a norm is not convex and the associated problems are NP hard; e.g., [1, 2]. The current approach in applications such as compressive sensing [3] and others is to replace the zero-norm with a (possibly weighted) one-norm.

An interesting optimization tool is fractional programming (FP) [4], where a non-convex ratio is transformed to a convex formulation with extra variable(s). While FP was used for a long time in other fields, it has been recently used for solving problems in wireless communication systems; e.g., [5–8]. Using iterative optimization, it was shown that FP converges to the global optimum when the numerator and denominator of the ratio satisfy some conditions (see below (2)) [4, 9, 10]. Recently, FP was extended to the sum of ratios using a quadratic transformation [11]. In this case, the resulting iterative solution converges to a stationary point.

In this paper, we extend the use of FP to solve problems involving the zero-norm. Specifically, we propose an approximation of the ℓ_0 norm that satisfies the necessary conditions for the FP, and, accordingly, we can apply the quadratic transform to obtain an iterative approach leading to a stationary point. Essentially, using the FP allows us to use a better approximation (than the ℓ_1 norm) to the ℓ_0 norm.

To show the applicability of our tool, we first test the approach on a simple "toy" scenario that heavily favors the ℓ_1 approach. We then investigate the problem of power minimization under certain signal-to-interference-plus-noise ratio (SINR) constraints. This problem is well-studied and closed-form solutions are available [12–14]. However, most of the formulations of this problem ignore the power consumed in the RF circuits, and focus only on the transmitted power. While the transmit power required to meet fixed SINR constraints is inversely proportional to the number of BS antennas [15], the RF circuit power is linear in this number. There is, therefore, an optimal number of antennas that requires the least possible total power [16, 17].

Introducing circuit power into the formulation is challenging because the consumed power depends on the number of active antennas, which is modelled by ℓ_0 norm. Using fewer antennas than available implies that our power allocation vector should be sparse. Using our proposed FP tool to model the sparseness, we will show that we can transform the problem into a convex iterative problem that can be efficiently solved. The problem can be viewed as jointly solving the antenna selection (e.g., [18, 19]) and beamforming design, when the objective is power minimization. This problem was solved for fixed beamforming directions (zero-forcing and maximum ratio transmission) when the number of antennas is large [20]. The analysis therein is based on asymptotic results, whereas our work is applicable to any number of available antennas.

In this paper, we first review the FP approach and then propose our suggested formulation to approximate the ℓ_0 norm. We then develop the problem of power minimization when we account for beamforming and circuit power, and show how we can formulate this problem as an iterative con-

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vex problem. To confirm the efficacy of our approach, we first show that the proposed FP approach provides gains over the ℓ_1 approach even in scenarios favoring this approach - where the matrix meets the restricted isometry property (RIP) constraint. We then illustrate the significant gains achievable in minimizing power consumption, in addition to those related to computations when the number of antennas is reduced.

2. ℓ_0 NORM APPROXIMATION BY FP

For a single-ratio FP problem, Dinkelbach's approach [4] is typically applied to a problem of the form

$$\max_{\mathbf{x}} \quad A(\mathbf{x})/B(\mathbf{x}) \tag{1a}$$
$$\text{s.t.} \quad \text{Problem constraints.} \tag{1b}$$

This problem is generally non-convex. If $A(\mathbf{x})$ is non-negative and $B(\mathbf{x})$ is positive, Dinkelbach's approach replaces this problem with an iterative problem with an auxiliary variable, y, as follows:

$$\max_{\mathbf{x}} \quad A(\mathbf{x}) - yB(\mathbf{x}) \tag{2a}$$
$$\text{s.t.} \quad \text{Problem constraints.} \tag{2b}$$

where y is updated as $y = A(\mathbf{x})/B(\mathbf{x})$.

By using iterative optimization between the variables $\mathbf{x}$ and the auxiliary variable y, convergence is guaranteed if $A(\mathbf{x})$ is concave and $B(\mathbf{x})$ is convex, and the convergence point is the global optimum solution [4, 9, 10]. The problem involving a ratio satisfying those conditions is known as the concave-convex FP problem. Dinkelbach's transform [4] requires no extra constraints compared to other transforms; e.g., Charnes-Cooper transform [9,10], however, importantly, both are valid for a single-ratio only.

In [11], FP was extended to the case of multiple ratios by using a quadratic transform. That transform replaces $\sum_i A_i(\mathbf{x})/B_i(\mathbf{x})$ with

$$\sum_i 2y_i\sqrt{A_i(\mathbf{x})} - y_i^2 B_i(\mathbf{x}).$$

When each ratio is in the concave-convex format, and we iteratively update y_i using $y_i = \sqrt{A_i(\mathbf{x})}/B_i(\mathbf{x})$, then the iterative algorithm converges to a stationary point [11].

2.1. Proposed zero-norm approximation

Now we observe that, for a small enough δ, we can approximate the ℓ_0 norm of x_i as

$$\|x_i\|_0 \approx \|x_i\|_1/(\|x_i\|_1 + \delta) = 1 - \delta/(\|x_i\|_1 + \delta),$$

where $\|x_i\|_j$ is the jth-norm of x_i. A minimization over $\sum_i \|x_i\|_0$ can therefore be approximated as maximization over $\sum_i \delta/(\|x_i\|_1 + \delta)$. Since $\|x_i\|_1$ can be replaced by t_i with proper linear constraints ($t_i \geq \pm x_i$), the ratio $\delta/(\|x_i\|_1 + \delta) = \delta/(t_i + \delta)$ has a convex non-negative numerator, and concave positive denominator. Accordingly, this is in the required concave-convex format to which we can apply the quadratic transform. In this case, we have that $A_i = 1$ and $B_i(\mathbf{t}) = t_i + \delta$. Accordingly, the quadratic transformation reduces to

$$\delta \sum_i 2y_i - y_i^2(t_i + \delta).$$

To summarize the above discussion, if we have a problem with convex constraints that includes ℓ_0 norm as follows

$$\min_{\mathbf{x}} \quad \|\mathbf{x}\|_0 \tag{3a}$$
$$\text{s.t.} \quad \text{Problem constraints,} \tag{3b}$$

then, this problem can be approximated by the following convex problem

$$\max_{\mathbf{x},\mathbf{t}} \quad \sum_i 2y_i - y_i^2(t_i + \delta) \tag{4a}$$
$$\text{s.t.} \quad \text{Problem constraints.} \tag{4b}$$
$$t_i \geq x_i, \tag{4c}$$
$$t_i \geq -x_i. \tag{4d}$$

where we iteratively update $\mathbf{y}$ as $y_i = 1/(\|x_i\|_1 + \delta)$.

2.2. Basis pursuit

The well-known problem in compressive sensing [3], where we want to find the sparsest vector $\mathbf{x}$ within a certain error ϵ from the measured data ($\mathbf{y} = \mathbf{A}\mathbf{x}$), can be stated as an ℓ_0 norm minimization problem as follows

$$\min_{\mathbf{x}} \quad \|\mathbf{x}\|_0 \tag{5a}$$
$$\text{s.t.} \quad \|\mathbf{A}\mathbf{x} - \mathbf{b}\| < \epsilon, \tag{5b}$$

Since this problem is NP hard, the basis pursuit approach [3] is to use the one-norm as follows

$$\min_{\mathbf{x}} \quad \|\mathbf{x}\|_1 \tag{6a}$$
$$\text{s.t.} \quad \|\mathbf{A}\mathbf{x} - \mathbf{b}\| < \epsilon. \tag{6b}$$

The main draw back of this method is that, when some values of vector $\mathbf{x}$ are large, there can be many very small non-zero entries. That is true because the large values dominate the objective value of the optimization problem. Accordingly, a well-known approach is to penalize the resulting small values in order to find a sparser solution. The resulting sparseness can be enhanced by using an iterative weighted version of the one-norm as follows

$$\min_{\mathbf{x}} \quad \sum_i s_i \|x_i\|_1 \tag{7a}$$
$$\text{s.t.} \quad \|\mathbf{A}\mathbf{x} - \mathbf{b}\| < \epsilon, \tag{7b}$$

where s_i is updated after each iteration using $s_i = 1/(\|x_i\|_1 + \delta)$. In this case, when $\|x_i\|_1$ is small, the corresponding s_i will be large, encouraging the minimization problem to push $\|x_i\|_1$ further to zero.

Since the problem contains zero-norm minimization objective, and the constraints are convex, we can provide a formulation similar to (4) according to our proposed FP approach as follows:

$$\max_{\mathbf{x},\mathbf{t}} \quad \sum_i 2y_i - y_i^2(t_i + \delta) \tag{8a}$$
$$\text{s.t.} \quad \|\mathbf{A}\mathbf{x} - \mathbf{b}\| < \epsilon, \tag{8b}$$
$$t_i \geq x_i, \tag{8c}$$
$$t_i \geq -x_i. \tag{8d}$$

3. SYSTEM MODEL AND PROBLEM FORMULATION

We consider a downlink multiple-input single-output (MISO) system where a total of K users, each with a single antenna, are served by a base station (BS) with N_t antennas. We assume that the BS is provided with perfect channel state information of the users. We let $\mathbf{h}_k \in \mathbb{C}^{N_t}$ denote the channel between the BS and user k, and let s_k denote the intended normalized data symbol for that user. We design the precoding vector $\mathbf{w}_k$ for transmission from the BS to user k. When the BS transmits $\sum_{k=1}^{K} \mathbf{w}_k s_k$, we can write the received signal at user k as:

$$y_k = \mathbf{h}_k^H \mathbf{w}_k s_k + \textstyle\sum_{i \neq k} \mathbf{h}_k^H \mathbf{w}_i s_i + n_k, \tag{9}$$

in which n_k is zero-mean circular Gaussian noise of variance σ_k^2. We will express each user's quality-of-service (QoS) constraint by an SINR constraint: $\mathsf{SINR}_k \geq \gamma_k$. This SINR constraint can be written as

$$\mathsf{SINR}_k = \frac{\mathbf{h}_k^H \mathbf{w}_k \mathbf{w}_k^H \mathbf{h}_k}{\mathbf{h}_k^H (\sum_{i \neq k} \mathbf{w}_i \mathbf{w}_i^H) \mathbf{h}_k + \sigma_k^2} \geq \gamma_k. \tag{10}$$

Since the SINR expression does not change with the phase of $\mathbf{w}_k$, we can express the SINR constraint as

$$\mathbf{w}_k^H \mathbf{h}_k \sqrt{1 + 1/\gamma_k} \geq \sqrt{\sum_i |\mathbf{w}_i^H \mathbf{h}_i|^2 + \sigma_k^2}.$$

The conventional problem of minimizing the transmitted beamforming power under SINR constraints can, therefore, be written as

$$\min_{\mathbf{w}_k} \quad \sum_k \mathbf{w}_k^H \mathbf{w}_k \tag{11a}$$
$$\text{s.t.} \quad \mathbf{w}_k^H \mathbf{h}_k \sqrt{1 + 1/\gamma_k} \geq \sqrt{\sum_i |\mathbf{w}_i^H \mathbf{h}_i|^2 + \sigma_k^2}, \quad \forall k. \tag{11b}$$

This formulation is a convex conic formulation that can be efficiently solved, e.g., using CVX [21] accessible from MATLAB [14]. The KKT conditions of this problem allow for closed-form expressions for the optimal beamformers as well [12, 13]. However, the problem does not take into account the power consumed in the circuits driving the antennas. Such a number depends on how many antennas are active. If we let P_i denote the average power on the ith antenna, we can write $P_i = \left[\sum_k \mathbf{w}_k \mathbf{w}_k^H\right]_{i,i}$. Accordingly, the number of active antennas is then $\sum_i \|P_i\|_0$. If we assume that the the power consumed per antenna is denoted by c_1 and c_2 models the amplifier efficiency, then the more general problem of minimizing the *total consumed power* while satisfying the SINR constraints can be written as

$$\min_{\mathbf{w}_k, P_i} \quad c_1 \sum_i \|P_i\|_0 + c_2 \sum_k \mathbf{w}_k^H \mathbf{w}_k \tag{12a}$$
$$\text{s.t.} \quad \mathbf{w}_k^H \mathbf{h}_k \sqrt{1 + 1/\gamma_k} \geq \sqrt{\sum_i |\mathbf{w}_i^H \mathbf{h}_i|^2 + \sigma_k^2}, \quad \forall k. \tag{12b}$$
$$P_i \geq \left[\sum_k \mathbf{w}_k \mathbf{w}_k^H\right]_{i,i}, \quad \forall i. \tag{12c}$$

The weighted one-norm approach to solving (12), similar to (7), can be written as

$$\min_{\mathbf{w}_k, P_i} \quad c_1 \sum_i s_i P_i + c_2 \sum_k \mathbf{w}_k^H \mathbf{w}_k \tag{13a}$$
$$\text{s.t.} \quad \mathbf{w}_k^H \mathbf{h}_k \sqrt{1 + 1/\gamma_k} \geq \sqrt{\sum_i |\mathbf{w}_i^H \mathbf{h}_i|^2 + \sigma_k^2}, \quad \forall k. \tag{13b}$$
$$P_i \geq \left[\sum_k \mathbf{w}_k \mathbf{w}_k^H\right]_{i,i}, \quad \forall i. \tag{13c}$$

By applying the proposed FP approach to ℓ_0 norm, we approximate (12) by the following convex formulation

$$\min_{\mathbf{w}_k, P_i} \quad c_1 \sum_i 1 - (2\delta y_i - \delta y_i^2(P_i + \delta)) + c_2 \sum_k \mathbf{w}_k^H \mathbf{w}_k \tag{14a}$$
$$\text{s.t.} \quad \mathbf{w}_k^H \mathbf{h}_k \sqrt{1 + 1/\gamma_k} \geq \sqrt{\sum_i |\mathbf{w}_i^H \mathbf{h}_i|^2 + \sigma_k^2}, \quad \forall k. \tag{14b}$$
$$P_i \geq \left[\sum_k \mathbf{w}_k \mathbf{w}_k^H\right]_{i,i}, \quad \forall i. \tag{14c}$$

The auxiliary variable y_i is updated as $y_i = 1/(P_i + \delta)$.

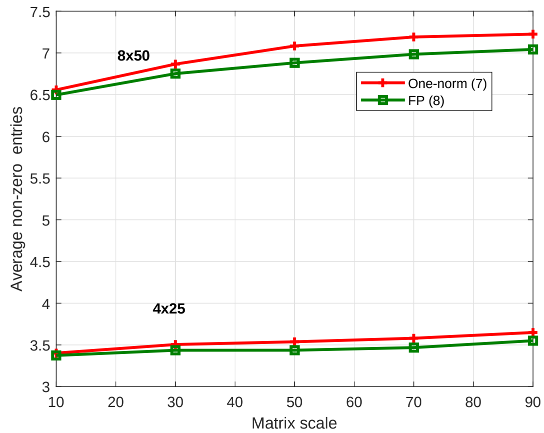

Fig. 1: The matrix scale versus the average number of the non-zero entries for $\epsilon = 0.1$.

4. SIMULATION RESULTS

In this section, we will first show the effectiveness of our proposed approach approximating the zero-norm using a simple setup that favors the one-norm approach. Then, we will show the performance gains in the total power minimization problem.

Basis pursuit, as described in (7), is the accepted approach to solving the original problem in (5) and is especially effective when the matrix $\mathbf{A}$ meets the RIP property [3]. To assess the performance of our proposed approach in (8) we run the following simulation. First, we generate a random matrix $\mathbf{A}$ and vector $\mathbf{b}$ with Gaussian entries of zero-mean and unit-variance. We will then scale the entries of the matrix $\mathbf{A}$. The scale causes the eigenvalues of the matrix to spread changing the probability of satisfying the RIP, accordingly, affecting the efficiency of the one-norm approach; e.g., [1–3]. For the scaled matrix $\mathbf{A}$ and vector $\mathbf{b}$, we solve (7) and (8) with $\epsilon = 0.1$. In Fig. 1, we plot the average number of the non-zero entries versus the scale value, once for matrix ($\mathbf{A}$) size of 8×50, and then for 4×25.

As the figure shows, the FP approach does provide some (small) benefit over the ℓ_1 approach. Given that the simulation is the ideal case for this approach, the fact there is some gain is, in fact, remarkable.

We now illustrate the performance of the proposed FP approach in solving the total power minimization problem. We consider a system consisting of a BS with N_t antennas, serving a total of $K = 4$ users. Fading is modelled using the standard Rayleigh model. We assume an SINR target of $\gamma = 3$dB for all users and that each user has normalized noise power. In Fig. 2, we plot the total transmission power versus number of BS antennas. We set $c_1 = 0.3$ W/antenna and the amplifier efficiency to 30%; i.e., $c_2 = 1/0.3$. For (11), we have that $\sum_i \|P_i\|_0 = N_t$, which causes the total power to grow almost linearly. We observe that our proposed FP-based approach in (14) provides significant gains compared to the weighted one-norm approach in (13) or the naive approach in (11).

In table 1, we list the average number of active antennas versus the total number of antennas. We observe that while the one-norm approach manages to lower the number of average active antennas, our proposed method provides significant reductions, and the number of active antennas is only decreasing as the total number of antennas increases. Note that when the number of active antennas is small, this results in far fewer computations to obtain the beamforming vectors.

Table 1: Average number of active antennas

N_t	16	20	24	28	32
One-norm (13)	11.4	12.3	15	16.8	19.5
FP (14)	10.8	10	10	9.9	9.7

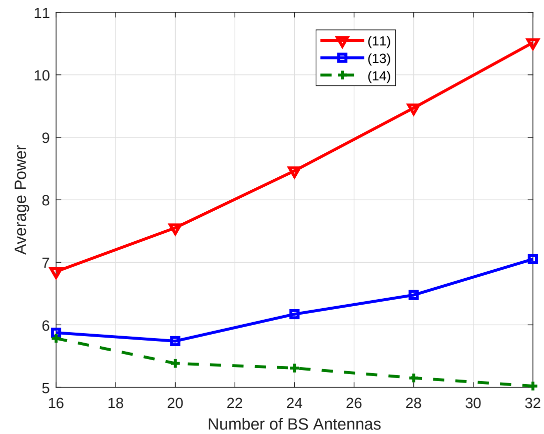

Fig. 2: The number of antennas versus the average total power for $K = 4$ users, γ=3dB, and σ^2=1

5. CONCLUSION

In this paper, we propose a new approach to the zero-norm by using fractional programming (FP); approximating the zero-norm by a fraction that satisfies the concave-convex properties of FP, allows us to transform the zero-norm to iterative convex optimization problem. We show the effectiveness of our approach in comparison to the basis pursuit weighted one-norm problem. To show the validity of our proposed tool for solving problems containing zero-norm, we formulated the power minimization SINR-constrained problem such that it includes the power dissipated in the RF circuits powering the antennas. This circuit power depends on the number of active antennas, which can be modeled using zero-norm. We show significant power gains compared to the conventional naive approach that only minimizes the transmit power, and a variant using a weighted one-norm approach.

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