STRESSES IN ANISOTROPIC CYLINDERS

Doron Gal (Galmudi) and Jack Dvorkin
Geophysics Department, Stanford University, Stanford, CA 94305-2215

Introduction

Axisymmetrical stresses in an infinitely long hollow isotropic circular cylinder (plane strain) quickly approach their asymptotic values as the external radius increases. This is not the case if the cylinder is even slightly anisotropic -- asymptotic solutions (for an infinitely large external radius) do not exist. We find the mechanical meaning of this disparity by using formulas for radial and hoop stresses in a cylindrically anisotropic cylinder with constant finite stresses at the boundaries: (a) The internal stresses increase infinitely with increasing external radius, if the cylinder is stiffer in the radial direction than in the tangential direction. (b) At any fixed point inside the cylinder, the stresses approach zero as the outer radius increases, if the cylinder is stiffer in the tangential direction than in the radial direction. We call the former effect "stress amplification", and the latter one "stress shielding." Both effects are closely related to the decay of boundary conditions and, in general, to the problem of applicability of Saint-Venant's principle to anisotropic solids.

Analysis

Stresses in an isotropic plane with a circular hole (plane strain) and uniform axisymmetrical far-field stresses can be calculated as asymptotes of those in a thick-walled cylinder with an external radius approaching infinity. However, such asymptotic values do not exist if the cylinder has even slight cylindrical anisotropy. Apparently, this qualitative observation is closely linked to the quantitative problem of decay of boundary conditions in anisotropic elasticity (e.g., [1] and [2]) and, more generally, to the issue of applicability of Saint-Venant's principle to anisotropic bodies. Comprehensive reviews of the latter problem can be found in [3], [4], and [5].

Consider an infinitely long linearly-elastic cylinder with cylindrical anisotropy (plain strain). Both the generator of the cylinder and the axis of anisotropy lie along the z-axis of a cylindrical coordinate system (r,θ,z). Normal stress component in the z direction is zero, the pressure inside the cylinder is zero, and the outside pressure P is constant.

Hooke's law that is appropriate for the problem is [6]:

$$e_{rr} = \frac{1}{E_r}\sigma_{rr} - \frac{\nu_{\theta r}}{E_\theta}\sigma_{\theta\theta}, \quad e_{\theta\theta} = \frac{1}{E_\theta}\sigma_{\theta\theta} - \frac{\nu_{r\theta}}{E_r}\sigma_{rr}, \tag{1}$$

where σ_{rr}, $\sigma_{\theta\theta}$, e_{rr}, and $e_{\theta\theta}$ are stress and strain components, E_r and E_θ are Young's moduli in the r and θ directions, respectively, and $\nu_{\theta r}$ and $\nu_{r\theta}$ are the appropriate Poisson's ratios.

By using equations (1) together with the equations of equilibrium, one can arrive at the following formulas for the radial (σ_{rr}) and hoop ($\sigma_{\theta\theta}$) stresses:

$$\sigma_{rr} = -\frac{Pb^{n+1}}{b^{2n} - a^{2n}}\left(r^{n-1} - a^{2n}r^{-n-1}\right), \quad \sigma_{\theta\theta} = -\frac{nPb^{n+1}}{b^{2n} - a^{2n}}\left(r^{n-1} + a^{2n}r^{-n-1}\right),$$
$$n = \sqrt{E_\theta / E_r}, \tag{2}$$

where a and b are the internal and external radii of the cylinder, respectively. These formulas are similar to those given in [6].

In the isotropic case where $n = 1$, equations (2) reduce to the standard solution

$$\sigma_{rr} = -\frac{Pb^2}{b^2 - a^2}\left(1 - \frac{a^2}{r^2}\right), \quad \sigma_{\theta\theta} = -\frac{Pb^2}{b^2 - a^2}\left(1 + \frac{a^2}{r^2}\right). \tag{3}$$

Equation (3) yields for $b >> a$:

$$\sigma_{rr} = -P\left(1 - \frac{a^2}{r^2}\right), \quad \sigma_{\theta\theta} = -P\left(1 + \frac{a^2}{r^2}\right). \tag{4}$$

By plotting the stresses as given by equation (3) versus the radial coordinate (Figure 1), we observe that stress distribution curves in a finite-thickness cylinder gradually converge to the asymptotic (infinite external radius) solution as given by equation (4).

However, equations (2) where $n \neq 1$, do not allow one to obtain an asymptotic formula for an infinite-thickness cylinder. Physically, this result means that both radial and hoop stresses in anisotropic cylinders do not converge to asymptotic values as the thickness of the cylinders increases:

If $n > 1$ (Figure 2) the influence of the stress at the external boundary decays much faster than in the isotropic case. We call this effect stress shielding: At any fixed point inside the cylinder, the stresses approach zero as the outer radius increases.

If $n < 1$, we have the opposite effect -- the external boundary condition affects the inside stresses stronger than it does in the isotropic case. This results in stress concentrations near the inner radius that are larger than in the isotropic case. These stresses will increase infinitely with the increasing outer radius (Figure 3). We call this effect stress amplification.

Radial and hoop stresses in cylinders of fixed radii are plotted in Figure 4 for isotropic and anisotropic cases. Again, if $n > 1$, stresses near the internal radius are smaller than in the isotropic case (shielding). If $n < 1$, these stresses are larger than in the isotropic case (amplification).

Both the stress shielding and the stress amplification effects can be clearly demonstrated on a simple mechanical system (Figure 5, left). This axisymmetrical system includes identical radial springs with stiffness E_r (per unit length). The radial springs are connected by two concentric hoops of tangential springs of stiffness E_θ. The radius of the external hoop is b and the radius of the internal hoop is a. The central angle between two adjacent radial springs is θ. A compressive radial force P is acting in the radial direction at every node of the external ring (Figure 5, right).

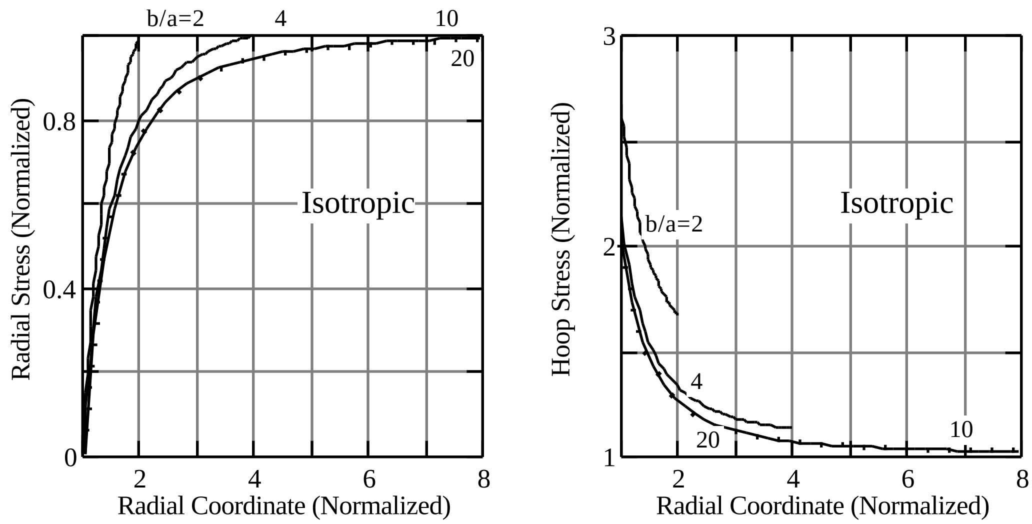

Figure 1. Radial (left) and hoop (right) stresses versus the radial coordinate in isotropic cylinders of varying thickness. The stresses are normalized by the external pressure, and the radial coordinate is normalized by the internal radius.

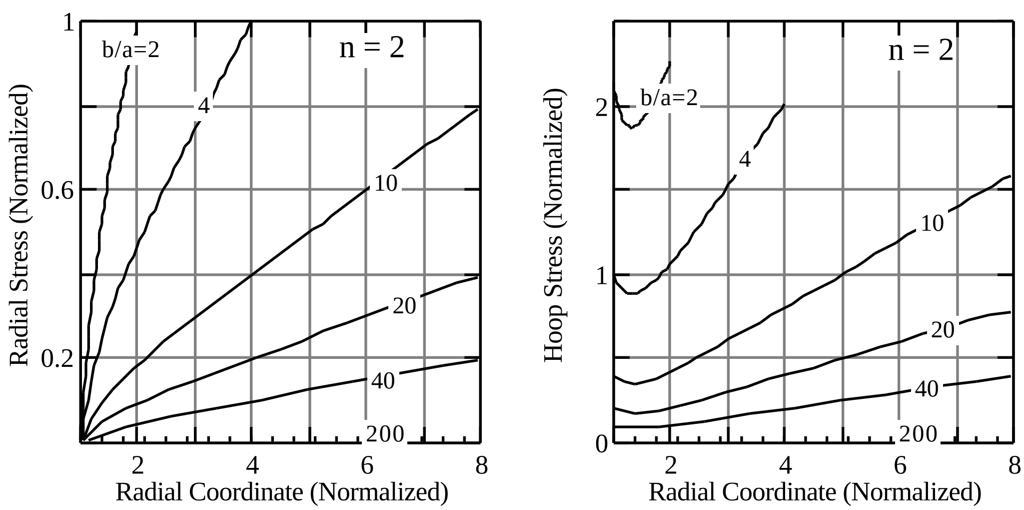

Figure 2. Radial (left) and hoop (right) stresses versus the radial coordinate in anisotropic ($n > 1$) cylinders of varying thickness. The stresses are normalized by the external pressure, and the radial coordinate is normalized by the internal radius.

By considering force balance at two nodes, A and B (Figure 5, right), we find that

$$\frac{\sigma_\theta^1}{\sigma_r} = \frac{q + n\sin\theta(1-q)}{\sin\theta}, \quad \frac{\sigma_\theta^1}{\sigma_\theta^2} = q + n\sin\theta(1-q), \tag{5}$$

where σ_r is the force in the radial springs, σ_θ^1 and σ_θ^2 are the forces in the external and internal hoops, respectively, $n = E_\theta / E_r$, and $q = a/b$.

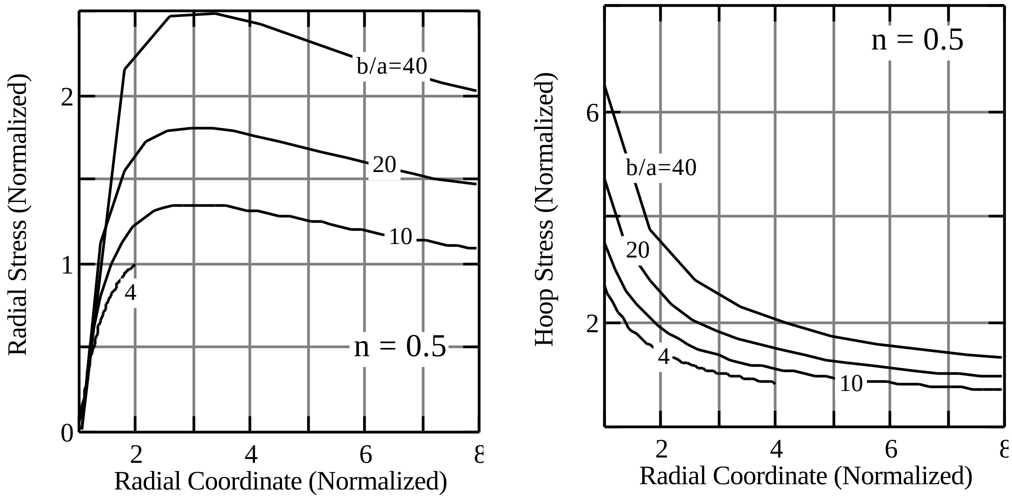

Figure 3. Radial and hoop stresses versus the radial coordinate in anisotropic ($n < 1$) cylinders of varying thickness. The stresses are normalized by the external pressure, and the radial coordinate is normalized by the internal radius.

Consider an extreme case $b >> a$. If $n << 1$ then from equations (5) $\sigma_r >> \sigma_\theta^1$ and $\sigma_\theta^2 >> \sigma_\theta^1$. Therefore, we have stress amplification near the center of the system. If $n >> 1$, equations (5) yield $\sigma_\theta^1 >> \sigma_r$ and $\sigma_\theta^1 >> \sigma_\theta^2$. In this case we have stresses near the center of the system shielded by the stiff external hoop.

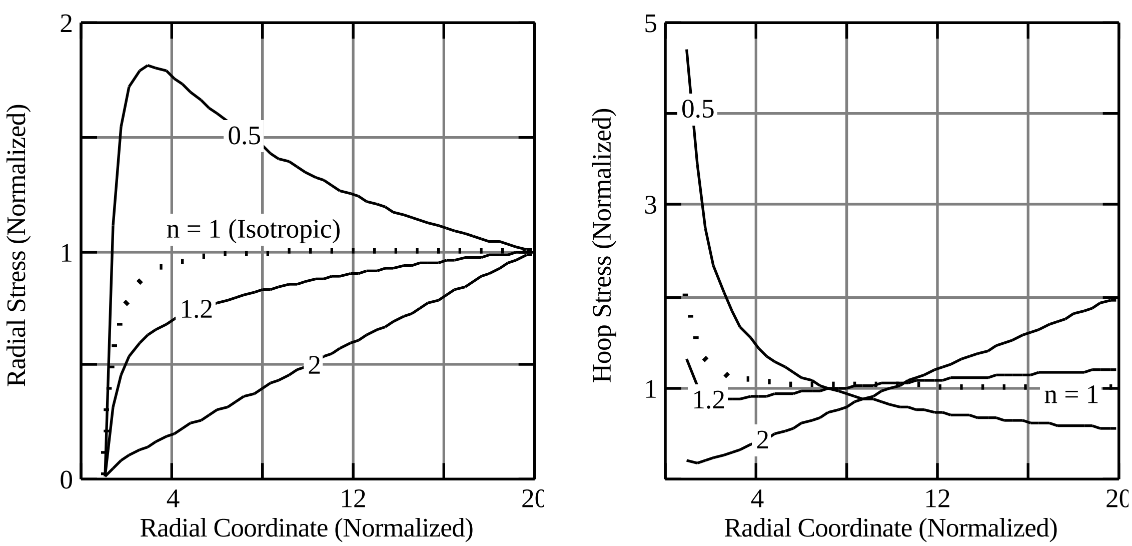

Figure 4. Radial (left) and hoop (right) stresses versus the radial coordinate in anisotropic cylinders of constant radius and varying parameter n. The stresses are normalized by the external pressure, and the radial coordinate is normalized by the internal radius.

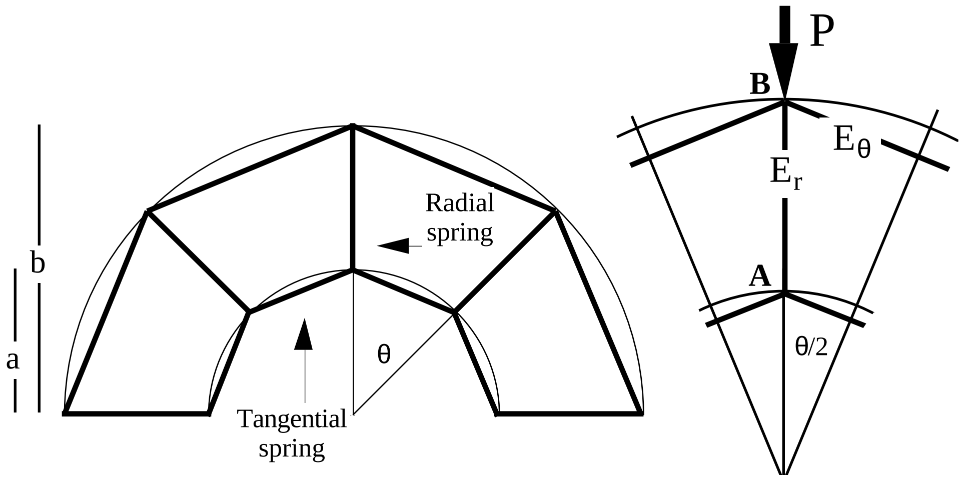

Figure 5. A radially-symmetric mechanical system of radial springs and two hoops of tangential springs. Left: upper half of the system. Right: a segment with one radial spring.

References

1. Knowles, J.K. and Horgan, C.O., Int. J. Solids Structures, **5**, 33 (1969).
2. Horgan, C.O., Int. J. Solids Structures, **10**, 837 (1974).
3. Horgan, C.O. and Knowles, J.K., Advances in Applied Mechanics, **23**, 179 (1983).
4. Horgan, C.O., Applied Mech. Reviews, **42**, 295 (1989).
5. Horgan, C.O. and Simmonds, J.G., Composites Engineering, **3**, 279 (1994).
6. Lekhnitskii, S. G., Theory of elasticity of an anisotropic elastic body, Mir, Moscow (1981).