Limit distribution of the sum and maximum from multivariate Gaussian sequences

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Abstract

In this paper we study the asymptotic joint behavior of the maximum and the partial sum of a multivariate Gaussian sequence. The multivariate maximum is defined to be the coordinatewise maximum. Results extend univariate results of McCormick and Qi. We show that, under regularity conditions, if the maximum has a limiting distribution it is asymptotically independent of the partial sum. We also prove that the maximum of a stationary sequence, when normalized in a special sense which includes subtracting the sample mean, is asymptotically independent of the partial sum (again, under regularity conditions). The limiting distributions are also obtained.

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1. Introduction

In past decades, a number of papers have studied the asymptotic joint distribution of the maximum, \( M_n = \max_{1 \leq i \leq n} X_i \), and the partial sum, \( S_n = \sum_{i=1}^{n} X_i \), from a sequence of random variables \( \{X_i\} \). Such a study was motivated by the increasing volume of environmental data where the averages and extremes are available to researchers, as well as the theoretical interest in determining the influence of the extremes in the partial sums. An early influential work is [6], which deals with a sequence of independent and identically distributed random variables. Anderson and

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Recent papers, including Ho and Hsing [7], Ho and McCormick [8], McCormick and Qi [15], and Peng and Nadarajah [17], have studied the joint limit distributions of the maxima and sums from a stationary Gaussian sequence \( \{X_i\} \). Let \( EX_i = 0 \) and \( EX_i^2 = 1 \) and \( r(n) = EX_iX_{i+n} \). It is well-known from Berman [5] that the limit distribution of the maximum \( M_n \) from such a sequence behaves as if \( \{X_i\} \) were an independent Gaussian sequence if the correlation function \( r(n) \) satisfies

\[
\lim_{n \to \infty} r(n) \ln n = 0.
\]

(1.1)

Mittal and Ylvisaker [16] showed that if

\[
\lim_{n \to \infty} r(n) \ln n = \rho \in (0, \infty),
\]

(1.2)

the limit distribution of \( M_n \) turns out to be different, although the normalization constants for \( M_n \) are the same. Furthermore, McCormick and Mittal [14] proved that if

\[
\lim_{n \to \infty} r(n) \ln n = \infty,
\]

(1.3)

with some regularity conditions for \( \{r(n)\} \), then \( M_n \) has a different limit distribution than those in the two cases above. For the joint limit distribution of \( M_n \) and \( S_n \), Ho and Hsing [7] first showed that \( M_n \) and \( S_n \) have independent limit distributions if (1.1) and some additional conditions hold, and have dependent limit distributions if (1.2) holds. Later, Ho and McCormick [8] and McCormick and Qi [15] considered the problems in a more general setting. They showed that \( M_n - \bar{X}_n \) and \( S_n \) are asymptotically independent if

\[
\lim_{n \to \infty} \frac{\ln n}{n} \sum_{i=1}^{n} |r(i) - r(n)| = 0,
\]

(1.4)

where \( \bar{X}_n = S_n/n \). Condition (1.4) was introduced by McCormick [13], who studied the limit distribution of \( M_n - \bar{X}_n \). Under condition (1.4) he obtained

\[
\lim_{n \to \infty} P \left( a_n \left( \max_{1 \leq i \leq n} (X_i - \bar{X}_n) - b_n \right) \leq x \right) = \exp(-e^{-x}) \quad \text{for } x \in \mathbb{R},
\]

(1.5)

where

\[
a_n = \sqrt{2 \ln n} \quad \text{and} \quad b_n = a_n - (2a_n)^{-1} \ln(4 \pi \ln n).
\]

(1.6)

Condition (1.4) seems the weakest condition so far in the study of the limit distributions of the extremes for a stationary Gaussian sequence, in the sense that either (1.1) or (1.2) ensures (1.4) and so does condition (1.3) with additional regularity conditions, as used in McCormick and Mittal [14]:

(i) \( r(n) \to 0 \) and is monotonically nonincreasing for \( n \geq n_0 \) for some positive integer \( n_0 \) and

(ii) \( r(n) \ln n \) is monotonically nondecreasing for \( n \geq n_0 \).

See, e.g., Ho and McCormick [8] and McCormick and Qi [15].
The study of limit distributions of the extreme values $M_n$ for a stationary Gaussian sequence has drawn a lot of attention from statisticians in the past, but it is far from complete. Under any of the three conditions (1.1)–(1.3) (plus $C$), one can easily get the joint limit distribution of $M_n$ and $S_n$, and thus the limit distribution of $M_n$, as obtained in Berman [5], Mittal and Ylvisaker [16] and McCormick and Mittal [14]. Ho and McCormick [8] and McCormick and Qi [15] provided an alternative approach in the area. As a matter of fact, from the asymptotic independence of $M_n - \bar{X}_n$ and $S_n$, one can get the limit distribution of $M_n$, after suitable normalization, if $\lim_{n \to \infty} \sigma_n \ln n \in [0, \infty]$ exists, where $\sigma_n^2$ is the variance of $S_n$.

In this paper we will consider stationary sequences of multivariate Gaussian vectors and study the joint distribution of the maximum and partial sum from the sequences. To our knowledge, this is the first paper to discuss the topic in the multivariate setting. Our multivariate maximum is defined as the vector of coordinatewise maxima, which was used in earlier work on the limit distribution of extremes for a multivariate Gaussian sequence; e.g., Amram [1], Husler [10], Husler and Schupbach [11], Wisniewski [18,19], to mention a few. The existence of limit distributions of the maximum is shown in these references under the multivariate analogues of conditions (1.1)–(1.3).

This paper is organized as follows. In Section 2, we will consider an array of Gaussian vectors and study the asymptotic independence of maxima and partial sums under the assumption that the extreme vector has a limit distribution. In Section 3 we extend McCormick’s [13] result to the multivariate case under an analogue of condition (1.5) in order to apply the result in Section 2 to get the multivariate version for convergence of $(M_n - \bar{X}_n, S_n)$. Finally, in Section 4, we discuss some sufficient conditions for the existence of joint limit distributions for the multivariate maxima and sums.

2. Asymptotic independence of multivariate extremes and sums

For $n \geq 1$, let $\{X_{ni}, 1 \leq i \leq n\}$ be a Gaussian sequence of $d$-dimensional random vectors; i.e., all joint distributions are Gaussian. Set $X_{ni} = (X_{ni,1}, \ldots, X_{ni,d})$.

For $I \subset [1, n]$ set

$$M_n(I) = \max_{i \in I} X_{ni} \quad \text{and} \quad S_n(I) = \sum_{i \in I} X_{ni}.$$  

In the case $I = [1, n]$, we also put

$$M_n = M_n([1, n]) = \max_{1 \leq i \leq n} X_{ni} \quad \text{and} \quad S_n = S_n([1, n]) = \sum_{i=1}^{n} X_{ni}.$$  

Here maximization and summation for the vectors are componentwise. For example, $M_n = (M_{n1}, \ldots, M_{nd})$, where $M_{ns} = \max_{1 \leq j \leq n} X_{nj,s}$ for $1 \leq s \leq d$, and $S_n = (S_{n1}, \ldots, S_{nd}) = \sum_{j=1}^{n} X_{nj}$.

We assume that the variables are centered, i.e.,

$$EX_{ni,s} = 0, \quad 1 \leq s \leq d, \quad 1 \leq i \leq n, \quad n \geq 1. \quad (2.1)$$

Furthermore, setting $\sigma_n(s, t, i, j) = EX_{ni,s}X_{nj,t}$, $1 \leq s, t \leq d, 1 \leq i, j \leq n$, we shall assume that

$$\max_{1 \leq s \leq d} \max_{1 \leq i \leq n} |1 - \sigma_n(s, s, i, i)| = o\left(\frac{1}{\ln n}\right) \quad \text{as} \ n \to \infty. \quad (2.2)$$
In addition, we still need the following condition for the array:
\[
\lim_{n \to \infty} \frac{\ln n}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} |\sigma_n(s, t, i, j)| = 0 \quad \text{for } 1 \leq s, t \leq d.
\] (2.3)

Now we are ready to give the multivariate extension of Theorem 2.1 in McCormick and Qi [15]. The extension is, however, nontrivial in that asymptotic independence between two vectors is more complicated than that of univariate random variables. This can be seen from the proof that follows. The following two remarks are helpful in understanding the essence of Theorem 2.1 and its proof.

Remark 1. The idea in McCormick and Qi [15] is to produce an intermediate array sufficiently close to the \( \{X_{ni}\} \) array but independent of \( S_n \). For an array of random vectors, such an intermediate array, independent of the vector of partial sums, cannot be constructed in just one step. We have to construct an independent intermediate array for the first component of the vector of the sums, then based on this array, we carefully select an array so that it is independent of the second component of the partial sum vector. Of course, the new array is still independent of the first component.

Remark 2. One can normalize the partial sums componentwise so that the marginals of the vector of partial sums are always standard normal. However, the convergence of the joint distributions may still require a stronger condition. Therefore, in order to avoid imposing additional conditions that are needed only for the convergence of the joint distribution of the partial sums in this step, we will express our theorem in the form of the conditional distribution for the maxima.

Theorem 2.1. Assume (2.1)–(2.3) hold. If \( a_n(M_n - b_n) \) converges in distribution to some \( d \)-dimensional distribution function \( G \), then
\[
P(a_n(M_n - b_n) \leq \mathbf{x}|S_n) \to G(\mathbf{x}) \quad \text{in probability},
\] (2.4)
where \( \mathbf{x} = (x_1, \ldots, x_d) \) is any continuity point of \( G \), \( b_n \) is the vector each of whose components equals \( b_n \), and constants \( a_n \) and \( b_n \) are defined as in (1.6).

Proof. Without loss of generality, we prove the two-dimensional case \( (d = 2) \). From (2.3), there exists a sequence \( m = m(n) \) such that
\[
\lim_{n \to \infty} m(n) = \infty
\] (2.5)
and
\[
\lim_{n \to \infty} \frac{m(n) \ln n}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} |\sigma_n(s, t, i, j)| = 0 \quad \text{for } 1 \leq s, t \leq 2.
\] (2.6)

First, we construct an intermediate array that is independent of \( S_{n1} \) and very close to the array \( \{X_{ni}\} \). For that purpose, set for \( s = 1, 2 \),
\[
\delta_{ns}(i) = EX_{ni,s}S_{n1} = \sum_{j=1}^{n} \sigma_n(s, 1, i, j), \quad 1 \leq i \leq n,
\]
and let \( w_{ns}^+ \) and \( w_{ns}^- \) denote the sums, \( w_{ns}^+ = \sum_j \delta_{ns}^+(j) \) and \( w_{ns}^- = \sum_j \delta_{ns}^-(j) \), where \( \delta_{ns}^+(j) \) is the positive part of \( \delta_{ns}(j) \) and \( \delta_{ns}^-(j) \) is the negative part.
Let

\[ J_n = \left\{ j : 1 \leq j \leq n, -\frac{\ln m(n)}{n} w_n^+ \leq \delta_n(j) \leq \frac{\ln m(n)}{n} w_n^+ \text{ for } s = 1, 2 \right\}. \]  

(2.7)

Now put for \( s = 1, 2 \),

\[ S_{n,s}^+ = \sum_{\delta_n(j) \geq 0} X_{nj,s} \quad \text{and} \quad S_{n,s}^- = \sum_{\delta_n(j) < 0} X_{nj,s}, \]

and define the intermediate array \( \{Y_{nj,s}, \ j \in J_n, s = 1, 2\} \) by

\[ Y_{nj,s} = X_{nj,s} - I(\delta_n(j) > 0) \frac{\delta_n(j)}{w_n^+} S_{n,s}^+ + I(\delta_n(j) < 0) \frac{\delta_n(j)}{w_n^-} S_{n,s}^- . \]

Then

\[ \{Y_{nj,s}, \ j \in J_n, s = 1, 2\} \text{ and } S_{n1} \text{ are independent,} \]  

(2.8)

which follows from the fact that \( \{S_{n1}, Y_{nj,s}, \ j \in J_n, s = 1, 2\} \) are jointly Gaussian with zero means and \( EY_{nj,s}S_{n1} = 0 \) for \( j \in J_n \) and \( s = 1, 2 \).

By calculating the variances

\[ \text{Var}(S_{n,s}^+) = \sum_{\delta_n(i) \geq 0} \sum_{\delta_n(j) \geq 0} \sigma_n(s, s, i, j) \leq \sum_{i=1}^{n} \sum_{j=1}^{n} |\sigma_n(s, s, i, j)| \]

and

\[ \text{Var}(S_{n,s}^-) = \sum_{\delta_n(i) < 0} \sum_{\delta_n(j) < 0} \sigma_n(s, s, i, j) \leq \sum_{i=1}^{n} \sum_{j=1}^{n} |\sigma_n(s, s, i, j)| \]

we have from (2.6) that

\[ \frac{m(n) \ln n}{n^2} \text{Var}(S_{n,s}^+) \to 0, \]  

(2.9)

and therefore,

\[ \sqrt{\frac{m(n) \ln n}{n}} S_{n,s}^+ \xrightarrow{p} 0. \]  

(2.10)

Set \( \sigma'_n(s, t, i, j) = EY_{ni,s}Y_{nj,t}, 1 \leq s, t \leq 2, i, j \in J_n \). We have from (2.7) and (2.9) that

\[ \sigma'_n(s, t, i, j) = \sigma_n(s, t, i, j) + O\left(\frac{\ln m(n)^2}{mn \ln n}\right) \]

uniformly over \( i, j \in J_n \). This yields

\[ \lim_{n \to \infty} \frac{m(n) \ln n}{n^2(\ln m(n))^2} \sum_{i \in J_n} \sum_{j \in J_n} |\sigma'_n(s, t, i, j)| = 0 \quad \text{for } 1 \leq s, t \leq 2. \]  

(2.11)

Now we are ready to construct a new array \( \{Z_{nj,s}\} \) that is independent of \( S_{n2} \) while maintaining the independence of \( S_{n1} \). For \( i \in J_n \), define

\[ \delta'_n(i) = EY_{ni,s}S_{n2} = \sum_{j \in J_n} \sigma'_n(s, 2, i, j), \quad s = 1, 2 \]
and for \( s = 1, 2 \), let \( v_{ns}^+ \) and \( v_{ns}^- \) denote the sums, \( v_{ns}^+ = \sum_{j \in J_n} \delta_{n}^+(j) \) and \( v_{ns}^- = \sum_{j \in J_n} \delta_{n}^-(j) \). Set

\[
R_n = \left\{ j \in J_n : -\frac{\ln m(n)}{n} v_{ns}^- \leq \delta_{n}^'(j) \leq \frac{\ln m(n)}{n} v_{ns}^+ \text{ for } s = 1, 2 \right\}. \tag{2.12}
\]

Define for \( s = 1, 2 \)

\[
T_{ns}^+ = \sum_{j \in J_n : \delta_{n}^'(j) \geq 0} Y_{nj,s} \quad \text{and} \quad T_{ns}^- = \sum_{j \in J_n : \delta_{n}^'(j) < 0} Y_{nj,s}
\]

and define the intermediate array \( \{Z_{nj,s}, j \in R_n, s = 1, 2\} \) by

\[
Z_{nj,s} = Y_{nj,s} - I(\delta_{n}^'(j) > 0) \frac{\delta_{n}^'(j)}{v_{ns}^+} T_{ns}^+ + I(\delta_{n}^'(j) < 0) \frac{\delta_{n}^'(j)}{v_{ns}^-} T_{ns}^-
\]

for \( s = 1, 2 \). It is easily checked that \( E Z_{nj,s} S_{n2} = 0 \) for all \( j \in R_n \) and \( s = 1, 2 \), which together with (2.8), yields that

\[
\{Z_{nj,s}, j \in R_n, s = 1, 2\} \text{ and } (S_{n1}, S_{n2}) \text{ are independent.} \tag{2.13}
\]

Similarly, by calculating the variances of \( T_{ns}^\pm \) and using (2.11) we have

\[
\frac{m(n) \ln n}{n^2 (\ln m(n))^2} \text{Var}(T_{ns}^\pm) \to 0 \tag{2.14}
\]

and therefore,

\[
\frac{\sqrt{m(n) \ln n}}{n \ln m(n)} T_{ns}^\pm \xrightarrow{P} 0. \tag{2.15}
\]

For any set \( A \), denote its cardinality by \( \#(A) \). Then it is readily seen from (2.7) that

\[
\# \left( \left\{ j : \frac{|\delta_{n}^+(j)|}{w_{ns}^\pm} > \frac{\ln m(n)}{n} \right\} \right) \leq \frac{n}{\ln m(n)}
\]

and thus

\[
n - \frac{4n}{\ln m(n)} \leq \#(J_n) \leq n.
\]

Furthermore, we have from (2.12) that

\[
n - \frac{8n}{\ln m(n)} \leq \#(R_n) \leq n.
\]
To help complete the proof of the theorem, we will show the following results:

\[ E|I(a_n(M_n(R_n) - b_n) \leq x) - I(a_n(M_n - b_n) \leq x)| \to 0 \]  
(2.16)

and

\[ a_n \left( \max_{j \in R_n} X_{nj,s} - \max_{j \in R_n} Z_{nj,s} \right) \xrightarrow{p} 0 \quad \text{for } s = 1, 2. \]  
(2.17)

Note that \( R_n \) is a subset of \([1, \ldots, n]\). Eq. (2.16) is immediate because the left-hand side of (2.16) is dominated by

\[ E \left( a_n \left( \max_{j \in R_n} X_{nj,1} - b_n \right) > x_1 \right) \]  

or

\[ a_n \left( \max_{j \in R_n} X_{nj,2} - b_n \right) > x_2 \)

\[ \leq (n - \#(R_n)) \left( 1 - \Phi \left( \frac{a_n^{-1} x_1 + b_n}{\sqrt{1 + \delta_n}} \right) + 1 - \Phi \left( \frac{a_n^{-1} x_2 + b_n}{\sqrt{1 + \delta_n}} \right) \right) \]

\[ = O \left( \frac{1}{\ln m(n)} \right), \]

where \( \delta_n = \max_{1 \leq i \leq n, 1 \leq s \leq 2} |1 - \sigma_n(s, s, i, i)| = o\left( \frac{1}{\ln n} \right) \). Eq. (2.17) can be verified by observing that

\[ a_n \left( \max_{j \in R_n} X_{nj,s} - \max_{j \in R_n} Z_{nj,s} \right) \leq \frac{a_n \ln m(n)}{n} (|S_{n3+}| + |S_{n3-}| + |T_{n3+}| + |T_{n3-}|), \]

which converges in probability to zero by virtue of (2.10) and (2.15).

Finally, to finish the proof of (2.4), it suffices to show that

\[ E|E(I(a_n(M_n - b_n) \leq x)|S_n) - G(x)| \to 0 \]

for \( x = (x_1, x_2) \), any continuity point of \( G \). For this, due to (2.16), we only need to show that

\[ E|E(I(a_n(M_n - b_n) \leq x)|S_n) - G(x)| \to 0. \]  
(2.18)

From (2.16) and (2.17),

\[ \left( a_n \left( \max_{j \in R_n} Z_{nj,1} - b_n \right), a_n \left( \max_{j \in R_n} Z_{nj,2} - b_n \right) \right) \overset{d}{\to} G. \]  
(2.19)

Let \( x = (x_1, x_2) \) be a fixed continuity point of \( G \). For any given \( \varepsilon > 0 \), select a \( \delta > 0 \) such that \( (x_1 + \delta, x_2 + \delta) \) and \( (x_1 - \delta, x_2 - \delta) \) are continuity points of \( G \) and \( G(x_1 + \delta, x_2 + \delta) - G(x_1 - \delta, x_2 - \delta) < \varepsilon \). Observing that

\[ a_n \left( \max_{j \in R_n} X_{nj,s} - b_n \right) = a_n \left( \max_{j \in R_n} Z_{nj,s} - b_n \right) + a_n \left( \max_{j \in R_n} X_{nj,s} - \max_{j \in R_n} Z_{nj,s} \right), \]
we get

\[
I\left( a_n \left( \max_{j \in \mathbb{R}^n} Z_{nj,1} - b_n \right) \leq x_1 - \delta, a_n \left( \max_{j \in \mathbb{R}^n} Z_{nj,2} - b_n \right) \leq x_2 - \delta \right)
- I\left( a_n \left( \max_{j \in \mathbb{R}^n} X_{nj,1} - \max_{j \in \mathbb{R}^n} Z_{nj,1} \right) > \delta \right) - I\left( a_n \left( \max_{j \in \mathbb{R}^n} X_{nj,2} - \max_{j \in \mathbb{R}^n} Z_{nj,2} \right) > \delta \right)
\leq I\left( a_n (M_n(R_n) - b_n) \leq x \right)
\leq I\left( a_n \left( \max_{j \in \mathbb{R}^n} Z_{nj,1} - b_n \right) \leq x_1 + \delta, a_n \left( \max_{j \in \mathbb{R}^n} Z_{nj,2} - b_n \right) \leq x_2 + \delta \right)
+ I\left( a_n \left( \max_{j \in \mathbb{R}^n} X_{nj,1} - \max_{j \in \mathbb{R}^n} Z_{nj,1} \right) < -\delta \right)
+ I\left( a_n \left( \max_{j \in \mathbb{R}^n} X_{nj,2} - \max_{j \in \mathbb{R}^n} Z_{nj,2} \right) < -\delta \right).
\]

By taking conditional expectations and using the independence of \( S_n = (S_{n1}, S_{n2}) \) and \( \{Z_{nj,s}\} \), we obtain the following:

\[
|E(I(a_n(M_n(R_n) - b_n) \leq x)|S_n) - G(x)|
\leq |P\left( a_n \left( \max_{j \in \mathbb{R}^n} Z_{nj,1} - b_n \right) \leq x_1 + \delta, a_n \left( \max_{j \in \mathbb{R}^n} Z_{nj,2} - b_n \right) \leq x_2 + \delta \right) - G(x_1, x_2)|
+ |P\left( a_n \left( \max_{j \in \mathbb{R}^n} Z_{nj,1} - b_n \right) \leq x_1 - \delta, a_n \left( \max_{j \in \mathbb{R}^n} Z_{nj,2} - b_n \right) \leq x_2 - \delta \right) - G(x_1, x_2)|
+ |P\left( a_n \left( \max_{j \in \mathbb{R}^n} X_{nj,1} - \max_{j \in \mathbb{R}^n} Z_{nj,1} \right) > \delta \right)|
+ |P\left( a_n \left( \max_{j \in \mathbb{R}^n} X_{nj,2} - \max_{j \in \mathbb{R}^n} Z_{nj,2} \right) > \delta \right)|
+ |P\left( a_n \left( \max_{j \in \mathbb{R}^n} X_{nj,1} - \max_{j \in \mathbb{R}^n} Z_{nj,1} \right) > \delta \right) - G(x_1, x_2)|
+ |P\left( a_n \left( \max_{j \in \mathbb{R}^n} X_{nj,2} - \max_{j \in \mathbb{R}^n} Z_{nj,2} \right) > \delta \right) - G(x_1, x_2)|
\leq \varepsilon,
\]

which, coupled with (2.19) and (2.17), leads to

\[
\limsup_{n \to \infty} E|E(I(a_n(M_n(R_n) - b_n) \leq x)|S_n) - G(x)|
\leq G(x_1 + \delta, x_2 + \delta) - G(x_1, x_2) + G(x_1, x_2) - G(x_1 - \delta, x_2 - \delta)
+ \limsup_{n \to \infty} P\left( a_n \left( \max_{j \in \mathbb{R}^n} X_{nj,1} - \max_{j \in \mathbb{R}^n} Z_{nj,1} \right) > \delta \right)
+ \limsup_{n \to \infty} P\left( a_n \left( \max_{j \in \mathbb{R}^n} X_{nj,2} - \max_{j \in \mathbb{R}^n} Z_{nj,2} \right) > \delta \right)
\leq \varepsilon,
\]

proving (2.18). This finishes the proof of the theorem. \( \square \)
3. Limits of maxima from stationary Gaussian sequences

In this section and hereafter, we let \( \{X_n, n \geq 1\} \) be a stationary Gaussian sequence of \( d \)-dimensional random vectors and set \( X_n = (X_{n,1}, \ldots, X_{n,d}) \) for \( n \geq 1 \). Define \( M_n = \max_{1 \leq j \leq n} X_j, S_n = \sum_{j=1}^{n} X_j \) and \( \bar{X}_n = S_n/n \). We also write \( M_n = (M_{n1}, \ldots, M_{nd}), S_n = (S_1, \ldots, S_d) \) and \( \bar{X}_n = (\bar{X}_{n1}, \ldots, \bar{X}_{nd}) \).

Assume that
\[
EX_{n,s} = 0 \quad \text{and} \quad EX_{n,s}^2 = 1 \quad \text{for} \quad 1 \leq s \leq d, \quad n \geq 1.
\]
(3.1)

Denote the correlation function by
\[
\rho_{st}(j - i) = EX_i,sX_j,t.
\]
We introduce the following conditions:
\[
\lim_{n \to \infty} \frac{\ln n}{n} \sum_{k=1}^{n} |\rho_{st}(k) - \rho_{st}(n)| = 0 \quad \text{for} \quad 1 \leq s, \ t \leq d
\]
(3.2)
and
\[
\rho_{st}(n) - \rho_{ts}(n) = o \left( \frac{1}{\ln n} \right) \quad \text{for} \quad 1 \leq s \neq t \leq d.
\]
(3.3)

In the univariate case, McCormick [13] showed that
\[
\lim_{n \to \infty} P \left( a_n \left( \frac{M_{ns} - \bar{X}_{ns}}{\sqrt{1 - r_{ss}(n)}} - b_n \right) \leq x \right) = \exp(-e^{-x}) =: \Lambda(x) \quad \text{for} \quad x \in \mathbb{R}
\]
for any \( 1 \leq s \leq d \) such that (3.2) holds for \( t = s \) and \( |r_{ss}(1)| < 1 \). We shall prove that \( M_{ns} - \bar{X}_{ns}, \ 1 \leq s \leq d \) are asymptotically independent.

**Theorem 3.1.** For a stationary sequence of Gaussian vectors \( \{X_n, n \geq 1\} \), assume that conditions (3.1)–(3.3) hold. In addition, suppose
\[
\lim_{n \to \infty} \rho_{st}(n) = 0 \quad \text{for} \quad 1 \leq s \leq t \leq d
\]
(3.4)
and
\[
\sup_{n \geq 0} |\rho_{st}(n)| < 1 \quad \text{for all} \quad s \neq t.
\]
(3.5)

Let
\[
T_n = \left( \frac{M_{n1} - \bar{X}_{n1}}{\sqrt{1 - r_{11}(n)}}, \ldots, \frac{M_{nd} - \bar{X}_{nd}}{\sqrt{1 - r_{dd}(n)}} \right) =: (T_{n1}, \ldots, T_{nd}).
\]

Then as \( n \to \infty \)
\[
P \left( a_n(T_n - b_n) \leq x \right) = P \left( \bigcap_{s=1}^{d} \{a_n(T_{ns} - b_n) \leq x_s \} \right) \to \prod_{s=1}^{d} \Lambda(x_s), \quad (3.6)
\]
where \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \).
Proof. The proof relies on Slepian’s comparison lemma (see, e.g., Theorem 4.2.1 of Leadbetter et al. [12]) and an inequality developed by McCormick [13]. We summarize the following lemma from the proof of Theorem 2.1 in McCormick [13]:

**Lemma 3.1.** Assume that \(\{e_n\}\) is a bounded sequence of real numbers such that

\[
\lim_{n \to \infty} \frac{\ln n}{n} \sum_{j=1}^{n} |e_j - e_n| = 0
\]

and that \(\{h_n\}\) is a sequence of positive numbers such that

\[
\lim_{n \to \infty} h_n = h \in (0, \infty).
\]

Set \(\rho_n'(i, j) = \frac{a_{|j-i|} - e_n}{h_n}\). If

\[
\max_{1 \leq i < j \leq n} |\rho_n'(i, j)| \leq \delta < 1 \quad \text{for all large } n,
\]

then

\[
\sum_{1 \leq i < j \leq n} (1 - \rho_n^2(i, j))^{-1/2} |\rho_n(i, j)| \exp \left\{ - \frac{b_n^2}{1 + |\rho_n(i, j)|} \right\} \to 0 \quad \text{as } n \to \infty,
\]

where \(\rho_n(i, j)\) are constants and \(\max_{1 \leq i, j \leq n} |\rho_n'(i, j) - \rho_n(i, j)| = o(1/\ln n)\).

Taking into account that if \(i > j\), \(E(X_{i,s}X_{j,t}) = r_{ts}(i - j)\), straightforward calculations show that under conditions (3.2) and (3.3),

\[
E(X_{i,s} - \bar{X}_{ns})(X_{j,t} - \bar{X}_{nt}) - [r_{st}(|j - i|) - r_{st}(n)] = o \left( \frac{1}{\ln n} \right)
\]

uniformly over \(1 \leq i, j \leq n\). Now set

\[
\sigma_{ns}^2(i) = \text{Var}(X_{i,s} - \bar{X}_{ns}), \quad Y_{ni,s} = \frac{X_{i,s} - \bar{X}_{ns}}{\sigma_{ns}(i)} \quad \text{and} \quad \rho_{st}(i, j; n) = E(Y_{ni,s}Y_{nj,t}).
\]

From (3.7) it follows that

\[
\max_{1 \leq i \leq n} |\sigma_{ns}^2(i) - (1 - r_{ss}(n))| = o \left( \frac{1}{\ln n} \right)
\]

and

\[
\max_{1 \leq i \leq j \leq n} \left| \rho_{st}(i, j; n) - \frac{r_{st}(j - i) - r_{st}(n)}{\sqrt{(1 - r_{ss}(n))(1 - r_{tt}(n))}} \right| = o \left( \frac{1}{\ln n} \right).
\]

(3.8)

Since for any \(x \in R\),

\[
\{T_{ns} \leq x\} = \{a_n(Y_{nj,s} - b_n) \leq x + \theta_{ns}(j, x_s), 1 \leq j \leq n\},
\]
where \( \theta_{ns}(j, x_s) = (a_nb_n + x)(\frac{1-r_{ns}(n))}{\sigma_{ns}(j)} - 1) \) satisfies \( \max_{1 \leq j \leq n} |\theta_{ns}(j, x_s)| = o(1) \), to establish (3.6) it suffices to prove

\[
P(a_n(M^*_ns - b_n) \leq x_s, 1 \leq s \leq d) \to \prod_{s=1}^{d} \Lambda(x_s),
\]

(3.9)

where \( M^*_ns = \max_{1 \leq j \leq n} Y_{nj,s} \).

It is proved in McCormick [13] that

\[
P(a_n(M^*_ns - b_n) \leq x_s) \to (xs).
\]

For each \( n \geq 1 \) and \( 1 \leq s \leq d \), let \( \{W_{nj,s}, 1 \leq j \leq n\} \) be a vector distributed the same as \( \{Y_{nj,s}, 1 \leq j \leq n\} \), but with the vectors \( \{W_{nj,s}, 1 \leq j \leq n\}, 1 \leq s \leq d \) independent. Thus,

\[
P\left(a_n\left(\max_{1 \leq j \leq n} W_{nj,s} - b_n\right) \leq x_s, 1 \leq s \leq d\right) = \prod_{s=1}^{d} P(a_n(M^*_ns - b_n) \leq x_s) \to \prod_{s=1}^{d} \Lambda(x_s).
\]

Therefore, (3.9) follows if we are able to show that

\[
\left| P(a_n(M^*_ns - b_n) \leq x_s, 1 \leq s \leq d) - P\left(a_n\left(\max_{1 \leq j \leq n} W_{nj,s} - b_n\right) \leq x_s, 1 \leq s \leq d\right) \right| \to 0.
\]

(3.10)

By virtue of Slepian’s comparison lemma, the left-hand side of (3.10) is bounded from above by

\[
\frac{1}{2\pi} \sum_{1 \leq s \neq t \leq d} \sum_{1 \leq i, j \leq n} (1-\rho^2_{st}(i, j; n))^{-1/2} |\rho_{st}(i, j; n)| \exp \left\{ -\frac{(a_n^{-1}x_s + b_n)^2 + (a_n^{-1}x_t + b_n)^2}{2(1 + |\rho_{st}(i, j; n)|)} \right\}.
\]

Observe that \( \rho_{st}(i, j; n) = \rho_{ts}(j, i; n) \) and \( (a_n^{-1}x_s + b_n)^2 = b_n^2 + c_n \sim 2 \ln n \), where \( c_n = O(1) \). Then the sum above is dominated by

\[
c \sum_{1 \leq s \neq t \leq d} \sum_{1 \leq i, j \leq n} (1-\rho^2_{st}(i, j; n))^{-1/2} |\rho_{st}(i, j; n)| \exp \left\{ -\frac{b_n^2}{1 + |\rho_{st}(i, j; n)|} \right\}
\]

\[
\leq c \sum_{1 \leq s \neq t \leq d} \sum_{1 \leq i < j \leq n} (1-\rho^2_{st}(i, j; n))^{-1/2} |\rho_{st}(i, j; n)| \exp \left\{ -\frac{b_n^2}{1 + |\rho_{st}(i, j; n)|} \right\}
\]

\[
+c \sum_{1 \leq s \neq t \leq d} \sum_{1 \leq i \leq n} (1-\rho^2_{st}(i, i; n))^{-1/2} |\rho_{st}(i, i; n)| \exp \left\{ -\frac{b_n^2}{1 + |\rho_{st}(i, i; n)|} \right\}
\]

\[
=: c \sum_{1 \leq s \neq t \leq d} A_{n,st} + c \sum_{1 \leq s \neq t \leq d} B_{n,st}.
\]
We shall demonstrate that $A_{n, st}$ and $B_{n, st}$ vanish as $n \to \infty$ for all $1 \leq s \neq t \leq d$. For any $1 \leq s \neq t \leq d$, from (3.4), (3.8) and (3.2) we see that the array $\{\rho_{st}(i, j; n), 1 \leq i < j \leq n\}$ satisfies the assumption in Lemma 3.1 and thus $A_{n, st} \to 0$. Finally, from (3.4), (3.5) and (3.8) we conclude that for some integer $n_0$, 

$$B_{n, st} \leq n \sum_{i=1}^{n} \left( 1 - \frac{b_n^2}{1 + \delta} \right) \leq (1 - \delta)^{-1/2} n \exp \left\{ -\frac{2}{1 + \delta} \ln n + \frac{2}{1 + \delta} \ln \ln n \right\},$$

which tends to zero as $n \to \infty$. This completes the proof. 

Define $s_{ni}^2 = \frac{1}{n} \sum_{j=1}^{n} (X_{nj} - \bar{X}_{ni})^2$ for $1 \leq i \leq d$. From Lemma 2.2 of McCormick [13], we have

$$\left( \ln n \right) \left[ s_{ni}^2 - (1 - r_{ii}(n)) \right] \overset{L^2}{\to} 0$$

if the conditions (3.1) and (3.2) hold. Immediately we get

**Corollary 3.1.** Under the conditions of Theorem 3.1, we have

$$P\left( \bigcap_{i=1}^{d} \left\{ a_n \left( \frac{M_{ni} - \bar{X}_{ni}}{s_{ni}} - b_n \right) \leq x_i \right\} \right) \to \prod_{i=1}^{d} \Lambda(x_i),$$

where $(x_1, \ldots, x_d) \in \mathbb{R}^d$.

4. **Joint distributions of maxima and sums from Gaussian sequences**

In this section we apply Theorem 2.1 to get the joint distribution for the maxima and sums from stationary Gaussian sequences.

**Theorem 4.1.** Under the conditions of Theorem 3.1,

$$P(a_n(T_n - b_n) \leq \mathbf{x}|S_n) \to \prod_{s=1}^{d} \Lambda(x_s) \text{ in probability},$$

where $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$.

**Proof.** Define $d$-dimensional vectors $\mathbf{r}(n) = (r_{11}(n), \ldots, r_{dd}(n))$, $\mathbf{1} = (1, \ldots, 1)$ and let $X_{ni} = \frac{1}{(1 - r_{ii}(n))^{1/2}} (X_i - \bar{X}_{ni})$ (the square root is taken coordinatewise). Then $\sum_{i=1}^{n} X_{ni} = \frac{1}{n(1 - r(n))^{1/2}} S_n$. We shall prove that conditions (2.1)–(2.3) hold for the array $\{X_{ni}\}$. Eq. (2.1) is trivial. As in
Section 3 set \( \tilde{X}_{ns} = \frac{1}{n} \sum_{j=1}^{n} X_{j,s} \) for \( 1 \leq s \leq d \). Then \( \text{Var}(\tilde{X}_{ns}) = O(1) \) and \( \sup_i |E(\tilde{X}_{ns}X_{i,t})| = O(1) \) for all \( 1 \leq s, t \leq d \) by virtue of Hölder’s inequality. Therefore we have

\[
\sigma_n(s,t,i,j) = EX_{ni,s}X_{nj,t} = \frac{1}{\sqrt{(1 - r_{ss}(n))(1 - r_{tt}(n))}} E \left( X_{i,s} - \frac{1}{n} \sum_{m=1}^{n} X_{m,s} \right) \left( X_{j,t} - \frac{1}{n} \sum_{m=1}^{n} X_{m,t} \right) + O \left( \frac{1}{n} \right)
\]

from (3.7), from which we conclude that \( \max_{1 \leq i \leq n} |1 - \sigma(s,s,i,j)| = o \left( \frac{1}{\ln n} \right) \) for all \( 1 \leq s \leq d \). That is, (2.2) holds.

From (4.2) and conditions (3.2) and (3.3) we can show (2.3). The details are omitted.

Let \( V_n = \max_{1 \leq j \leq n} (X_j - (1 - \frac{1}{n}) \bar{X}_n) \sqrt{1 - r(n)} \). Since

\[
a_n(T_n - b_n) - a_n(V_n - b_n) = -\frac{a_n \bar{X}_n}{n \sqrt{1 - r(n)}} \Rightarrow 0 \text{ with probability one,}
\]

application of Theorems 2.1 and 3.1 leads to

\[
G_n(x) = P(a_n(V_n - b_n) \leq x | S_n) \to \prod_{s=1}^{d} \Lambda(x_s) \text{ in probability.}
\]

Set \( G(x) = \prod_{s=1}^{d} \Lambda(x_s) \). Note that

\[
P(a_n(T_n - b_n) \leq x | S_n) = G_n \left( x - \frac{a_n \bar{X}_n}{n \sqrt{1 - r(n)}} \right) = G_n(x + o(1)) \to G(x)
\]

since \( G_n(x) \) is monotone in \( x \) and converges to the continuous function \( G(x) \) in probability at each point in \( \mathbb{R}^d \). This proves Theorem 4.1. \( \square \)

The following corollary is an immediate consequence of Theorem 4.1 under an analogue of condition (1.1).

**Corollary 4.1.** Assume that \( r_{st}(n) = o \left( \frac{1}{\ln n} \right) \) for all \( 1 \leq s, t \leq d \). If \( \sup_{n \geq 0} |r_{st}(n)| < 1 \) for \( 1 \leq s \neq t \leq d \) and \( |r_{ss}(1)| < 1 \) for \( 1 \leq s \leq d \), then

\[
P(a_n(M_n - b_n) \leq x | S_n) \to \prod_{s=1}^{d} \Lambda(x_s) \text{ in probability,}
\]

for \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \).

Under the analogue of condition (1.2), we obtain the following corollary:
Corollary 4.2. Assume that \( r_{st}(n) \ln n \to \sigma_{st} \in [0, \infty) \) and \( \sigma_{st} = \sigma_{ts} \) for all \( 1 \leq s, t \leq d \). Define the \( d \times d \) matrix \( \Sigma = (\sigma_{st}) \). If \( \sup_{n \geq 0} |r_{st}(n)| < 1 \) for \( 1 \leq s \neq t \leq d \), then
\[
\left( a_n \left( \frac{M_n}{\sqrt{1 - r(n)}} - b_n \right), \frac{1}{n} S_n \right) \overset{d}{\to} (Z + S, S),
\]
where \( Z \) and \( S \) are two independent random vectors such that \( S \) is normal with mean zero and covariance matrix \( \Sigma \), and \( Z \) is a vector formed by \( d \) independent random variables with an identical marginal distribution \( \Lambda(x) = \exp(-e^{-x}) \).

**Proof.** It is easily verified that the covariance matrix of \( \frac{a_n}{n} S_n \) converges to \( \Sigma \). It is readily seen from Theorem 4.1 that
\[
\left( a_n (T_n - b_n), \frac{a_n}{n} S_n \right) \overset{d}{\to} (Z, S).
\]
Finally, note that
\[
a_n \left( \frac{M_n}{\sqrt{1 - r(n)}} - b_n \right) = a_n (T_n - b_n) + \frac{a_n}{n \sqrt{1 - r(n)}} S_n.
\]
Application of the continuity mapping theorem yields the corollary. \( \square \)

Of particular interest, we consider the situation when \( r_{ss}(n) \ln n \to \infty \).

**Theorem 4.2.** Under the conditions of Theorem 3.1, if \( r_{ss}(n) \ln n \to \infty \) for \( 1 \leq s \leq d \) and
\[
\frac{r_{st}(n)}{\sqrt{r_{ss}(n)r_{tt}(n)}} \to \sigma_{st} \in [-1, 1] \quad \text{for} \quad 1 \leq s, t \leq d,
\]
then
\[
\left( \frac{1}{\sqrt{r(n)}} (M_n - b_n \sqrt{1 - r(n)}), \frac{1}{n \sqrt{r(n)}} S_n \right) \overset{d}{\to} (S, S),
\]
where \( S \) is a normal random vector with mean zero and covariance matrix \( \Sigma \).

**Proof.** It is straightforward to show under the conditions of Theorem 4.2 that the covariance matrix of \( \frac{1}{n \sqrt{r(n)}} S_n \) converges to \( \Sigma \). Thus, \( \frac{1}{n \sqrt{r(n)}} S_n \overset{d}{\to} S \). Theorem 4.2 follows from the observation that
\[
\frac{1}{\sqrt{r(n)}} (M_n - b_n \sqrt{1 - r(n)}) = \frac{1}{n \sqrt{r(n)}} S_n + \frac{\sqrt{1 - r(n)}}{a_n \sqrt{r(n)}} a_n (T_n - b_n)
\]
\[
= \frac{1}{n \sqrt{r(n)}} S_n + o_p(1) \quad (4.3)
\]
since \( a_n (T_n - b_n) \overset{d}{\to} Z \) from Theorem 4.1, and \( a_n \sqrt{r_{ss}(n)} \to \infty \) for each \( s \). \( \square \)

In Theorem 4.2 some conditions are imposed to ensure \( S_n \) has an asymptotic joint distribution. The conditions in Theorem 4.2 can be greatly relaxed if we rephrase the theorem in a different way.
Theorem 4.3. Assume (3.1) holds and
\[
\lim_{n \to \infty} \frac{\ln n}{n} \sum_{k=1}^{n} |r_{ss}(k) - r_{ss}(n)| = 0 \quad \text{for } 1 \leq s \leq d.
\] (4.4)

If for each \(1 \leq s \leq d\), \(\lim_{n \to \infty} r_{ss}(n) \ln n = \infty\) and there exists a \(k \geq 1\) such that \(r_{ss}(k) < 1\), then
\[
\frac{1}{\sqrt{\mathbf{r}(n)}} \left( \mathbf{M}_n - b_n \sqrt{1 - \mathbf{r}(n)} \right) = \frac{1}{n\sqrt{\mathbf{r}(n)}} \mathbf{S}_n + o_p(1),
\]
and each coordinate of \(\frac{1}{n\sqrt{\mathbf{r}(n)}} \mathbf{S}_n\) converges in distribution to a standard normal distribution.

Proof. From Corollary 2.4 of McCormick and Qi [15] the conditions of the theorem ensure that each coordinate of \(a_n(\mathbf{T}_n - b_n)\) has a limit. Thus the theorem follows from (4.3).

Corollary 4.3. Assume that \(\lim_{n \to \infty} r_{ss}(n) \ln n = \infty\) for \(1 \leq s \leq d\). In addition, for each \(1 \leq s \leq d\), assume condition \(\mathbf{C}\) in Section 1 holds for \(r(n) = r_{ss}(n)\). Then Theorem 4.3 holds.

Proof. From Eq. (2.40) in McCormick and Qi [15] condition \(\mathbf{C}\) implies (4.4). Hence all conditions in Theorem 4.3 are fulfilled. This completes the proof of the corollary.

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