Primitive 2-factorizations of the complete graph

Giuseppe Mazzuoccolo

Dipartimento di Matematica, Università di Modena e Reggio Emilia, via Campi 213/B, 41100 Modena, Italy

Received 30 November 2004; received in revised form 15 October 2005; accepted 2 February 2006
Available online 2 June 2007

Abstract

Let \( \mathcal{F} \) be a 2-factorization of the complete graph \( K_v \) admitting an automorphism group \( G \) acting primitively on the set of vertices. If \( \mathcal{F} \) consists of Hamiltonian cycles, then \( \mathcal{F} \) is the unique, up to isomorphisms, 2-factorization of \( K_p^n \) admitting an automorphism group which acts 2-transitively on the vertex-set, see [A. Bonisoli, M. Buratti, G. Mazzuoccolo, Doubly transitive 2-factorizations, J. Combin. Designs 15 (2007) 120–132.]. In the non-Hamiltonian case we construct an infinite family of examples whose automorphism group does not contain a subgroup acting 2-transitively on vertices.

© 2007 Elsevier B.V. All rights reserved.

MSC: 05C70; 05C15; 05C25

Keywords: Factorization; Coloring of graphs and hypergraphs; Graphs and groups

1. Introduction

For an integer \( v \geq 3 \), let \( K_v \) be the complete (simple undirected) graph on \( v \) vertices with vertex-set \( V(K_v) \) and edge set \( E(K_v) \). For \( 3 \leq k \leq v \), a \( k \)-cycle \( C = (x_0, x_1, \ldots, x_{k-1}) \) is the subgraph of \( K_v \) whose edges are \([x_i, x_{i+1}]\), \( i = 0, \ldots, k-1 \), indices taken modulo \( k \). If \( k = v \), the cycle is called Hamiltonian.

A 2-factor \( F \) of \( K_v \) is a set of cycles whose vertices partition \( V(K_v) \). A 2-factorization of \( K_v \) is a set \( \mathcal{F} \) of edge disjoint 2-factors forming a cover of \( E(K_v) \). A 2-factorization in which all the 2-factors are isomorphic to a factor \( F \) is called an \( F \)-factorization. If each 2-factor of \( \mathcal{F} \) consists of a single Hamiltonian cycle, \( \mathcal{F} \) is called a Hamiltonian 2-factorization. The existence of a 2-factorization of \( K_v \) forces \( v \) to be odd.

The collection of cycles appearing in the factors of \( \mathcal{F} \) form a cycle decomposition of \( K_v \), which is called the underlying decomposition. We will denote it by \( D_{\mathcal{F}} \).

A permutation group \( G \) acting faithfully on \( V(K_v) \) and preserving the 2-factorization \( \mathcal{F} \) is called an automorphism group of \( \mathcal{F} \).

In some recent papers the possible structures and actions of \( G \) on vertices or factors have been investigated. In [3] the situation in which \( G \) acts regularly (i.e. sharply transitively) on vertices is studied in detail. In [2] a complete description of \( G \) and \( \mathcal{F} \) is given in case the action of \( G \) is doubly transitive on the vertex-set. In particular it is proved that if \( \mathcal{F} \) is Hamiltonian, then \( v \) is an odd prime \( p \), the group \( G \) is the affine general linear group \( AGL(1, p) \) and, if the vertices of

---

1 Research performed with the financial support of the Italian Ministry MIUR, project “Strutture Geometriche, Combinatoria e loro Applicazioni”.

E-mail address: mazzuoccolo@unimore.it.
Theorem 1. The full group of automorphisms of \( K_p \) are labelled by the elements of \( \mathbb{Z}_p \), then \( \mathcal{F} = \{ C_1, C_2, \ldots, C_{(p-1)/2} \} \), with \( C_i = (0, i, 2i, \ldots, (p-1)i) \) (subscripts mod \( p \)), \( i = 1, 2, \ldots, (p-1)/2 \). This factorization is the natural 2-factorization (also denoted by \( \mathcal{N}(\mathbb{Z}_p) \)) which arises from the cyclic group \( \mathbb{Z}_p \), see [3].

In this paper, we investigate primitive 2-factorizations, i.e., admitting an automorphism group \( G \) with primitive action on the vertex-set. Note that all 2-factorizations admitting an automorphism group doubly transitive on the vertex-set are also examples of primitive 2-factorizations.

If \( \mathcal{F} \) is Hamiltonian, we prove that \( v \) is an odd prime \( p \) and \( \mathcal{F} = \mathcal{N}(\mathbb{Z}_p) \). Moreover, the group \( G \) is necessarily a subgroup of \( \text{AGL}(1, p) \) containing \( \mathbb{Z}_p \). In the non-Hamiltonian case, we give examples of primitive 2-factorizations whose full automorphism group does not contain a subgroup acting doubly transitively on the vertices. In the last section we also prove that a primitive 2-factorization of \( K_9 \) is necessarily 2-transitive, whence a 2-factorization arising from the affine line parallelism of \( \text{AG}(2, 3) \) in a suitable manner, see [2], and a primitive 2-factorization of \( K_{15} \) does not exist.

2. The Hamiltonian case

In this section we prove that \( \mathcal{N}(\mathbb{Z}_p) \) is the unique primitive Hamiltonian 2-factorization of a complete graph.

Lemma 1. Let \( \mathcal{F} \) be a 2-factorization of \( K_p \) with a transitive automorphism group \( G \). Then \( \mathcal{F} = \mathcal{N}(\mathbb{Z}_p) \) and \( G \leqslant \text{AGL}(1, p) \).

Proof. By transitivity of \( G \) on \( V(K_p) \), the integer \( p \) is a divisor of the order of \( G \), then an element \( g \in G \) of order \( p \) exists. The cyclic group generated by \( g \) acts regularly on the vertex-set. By proposition 2.8 of [3], it is \( \mathcal{F} = \mathcal{N}(\mathbb{Z}_p) \). The full group of automorphism of \( \mathcal{F} \) is \( \text{AGL}(1, p) \) (see [2], Section 1) and then \( G \leqslant \text{AGL}(1, p) \).

Theorem 1. Let \( \mathcal{F} \) be a Hamiltonian 2-factorization of \( K_v \) with primitive automorphism group \( G \). Then \( v = p \), \( \mathcal{F} = \mathcal{N}(\mathbb{Z}_p) \) and \( \mathbb{Z}_p \leqslant G \leqslant \text{AGL}(1, p) \).

Proof. Suppose \( G \) is of even order. We prove that \( G \) contains exactly \( v \) involutions. First of all observe that each involution of \( G \) fixes all the 2-factors of \( \mathcal{F} \). In fact let \( g \in G \) be an involution exchanging two vertices \( x_0 \) and \( x_1 \). Labelling the vertices such that \( C = (x_0, x_1, \ldots, x_{v-1}) \) is the unique cycle of \( \mathcal{F} \) containing \( [x_0, x_1] \), we obtain:

\[
g(x_i) = x_{v+1-i}, \quad g(x_{v+1-i}) = x_i \quad \text{for} \quad i = 1, \ldots, \frac{v-1}{2},
\]

where all indices are taken mod \( v \). Then \( g \) fixes the vertex \( x_{(v+1)/2} \) and each edge of the set \( E = \{ [x_i, x_{v+1-i}] / i = 1, \ldots, (v-1)/2 \} \). Suppose that at least two edges of \( E \) belong to the same 2-factor \( F \) of \( \mathcal{F} \). Then \( g \) should fix the unique cycle of \( F \) and two edges on it: a contradiction. We have proved that the elements in \( E \) are in different 2-factors. By the fact that the cardinality of \( E \) coincides with the number of 2-factors, we conclude that \( g \) fixes all the factors of \( \mathcal{F} \). Furthermore we can also observe that each involution in \( G \) fixes exactly one vertex of \( V(K_v) \). Let now \( x \in V(K_v) \), we have \( |G| = |G_x|v \), where \( G_x \) is the stabilizer of the vertex \( x \), then \( |G_x| \) is even and we have at least one involution in \( G_x \). Observe that \( G_x \) contains exactly one involution; in fact if we fix a 2-factor \( F \) and \( C \) is its cycle, the action of an involution of \( G_x \) is uniquely determined by its action on the vertices of \( C \) as above explained. We can conclude that the group \( G \) contains exactly \( v \) distinct involutions. In particular for each factor \( F \), the subgroup \( G_F \) contains \( v \) involutions: namely all the involutions of \( G \). It is well known that \( G_F \leqslant D_v \), the dihedral group on \( v \) vertices, and then \( G_F \cong D_v \) for each \( F \in \mathcal{F} \). Furthermore, for each factor \( F' \in \mathcal{F} - \{ F \} \), the dihedral groups \( G_F \) and \( G_{F'} \) contain exactly the same \( v \) involutions, therefore \( G_F \neq G_{F'} \). Label the vertices of \( K_v \) by the elements \( 0, 1, \ldots, v-1 \) in such a way that a 2-factor \( F \in \mathcal{F} \) contains the cycle \((0, 1, \ldots, v-1)\) and an element of \( G_F \) of order \( v \) maps the vertex \( i \) onto \( i+1 \), for each \( i \). Denote by \( g \in G_F \) this element. Suppose \( v \) is not a prime and let \( h \) be a proper divisor of \( v \). Let \( F' \in \mathcal{F} \) be the 2-factor containing the edge \([0, h] \). As \( G_{F'} = G_F \) we have \( g^h \in G_{F'} \) and then \( F' \) contains the cycle \((0, h, 2h, \ldots, v-h)\) of length less than \( v \): a contradiction. We conclude that \( v \) is a prime. By Lemma 1 the assertion follows in this case. We have proved that for a primitive Hamiltonian 2-factorization of \( K_v \) only two possibilities hold: either \( v \) is a prime or \( |G| \) is odd. By the O’Nan Scott Theorem (see [6]) and by the fact that a simple non-abelian group

\( \mathcal{F} = \mathcal{N}(\mathbb{Z}_p) \) and \( \mathbb{Z}_p \leqslant G \leqslant \text{AGL}(1, p) \).
has even order (Feit–Thomson Theorem, see [7]), the socle of \( G \) is a regular elementary abelian \( p \)-group for some prime \( p, v = p^n \) and \( G \) is isomorphic to a subgroup of the affine group \( \text{AGL}(m, p) \). But it is already known that a Hamiltonian 2-factorization of \( K_{p^m} \) with a group acting sharply transitively on vertices cannot exist if \( m > 1 \) (see [2], Proposition 4), then \( m = 1 \) and the assertion follows. \( \square \)

This theorem gives a complete classification of all primitive Hamiltonian 2-factorization of \( K_v \), while the non-Hamiltonian case remains an open problem. In the following paragraph we give an infinite family of examples and some non-existence results in this case.

### 3. The non-Hamiltonian case

Throughout this paragraph \( \mathcal{F} \) will be a non-Hamiltonian 2-factorization of \( K_v \) with primitive automorphism group \( G \). By Lemma 1 we have that \( v \) is not a prime in this case. When \( v \) is a genuine prime power an example for \( \mathcal{F} \) is the 2-transitive 2-factorization given in [2]. It has to be noticed that in this case the full automorphism group is 2-transitive, however, it could contain a proper subgroup which is primitive but not 2-transitive on the vertex-set.

3.1. Mixed translations over a finite field

Let \( F \) be a finite field of order \( p \) with \( p \neq 2 \) and, for \( n \geq 3 \), consider the \( n \)-dimensional vector space \( V = F^n \).

Identify the elements of \( V \) with the points of the affine space \( \text{AG}(n, p) \). Denote by \( t_a \), the translation of the affine space determined by the vector \( a \in V \) and by \( T = \{ t_a : a \in V \} \) the translation group on \( \text{AG}(n, p) \).

Let \( W_0 \) be an hyperplane of \( \text{AG}(n, p) \) through \((0, \ldots , 0)\), and denote by \( \overline{W_0} \) the vector subspace of \( V \) associated to \( W_0 \). Let \( W_0, \ldots , W_{p-1} \) be the affine hyperplanes obtained by \( W_0 \) as \( W_i = W_0 + i a \), for \( a \notin \overline{W_0} \) and \( i = 0, \ldots , p-1 \).

We will use mixed translations in what follows to obtain new examples of primitive 2-factorizations. For pairwise linearly independent vectors \( w_0, w_1, \ldots , w_{p-1} \) in \( \overline{W_0} \), these are transformations \( m_{w_0, w_1, \ldots , w_{p-1}} \) defined by

\[
m_{w_0, w_1, \ldots , w_{p-1}}(x) = \begin{cases} x + w_0 & \text{if } x \in W_0, \\ x + w_1 & \text{if } x \in W_1, \\ \vdots \\ x + w_{p-1} & \text{if } x \in W_{p-1}. \end{cases}
\]

Note that \( m_{w_0, w_1, \ldots , w_{p-1}} \) has a fixed-point-free action.

Identify the vertex-set of \( K_{p^n} \) with the point-set of the affine space \( \text{AG}(n, p) \). For each translation \( t_a \) of \( T \) we obtain a 2-factor whose cycles are obtained in the following way: take \( p^{n-1} \) points, \( x_1, \ldots , x_{p^{n-1}} \), lying on different lines in the parallelism class generated by the vector \( a \). For each of these points, construct the cycle:

\[(x_j, x_j + a, \ldots , x_j + (p-1)a),\]

where \( j = 1, \ldots , p^{n-1} \). A 2-factor is constructed in the same manner for each choice of the translation \( t_{ia}, i = 1, \ldots , p-1 \). Obviously \( t_{ia} \) and \( t_{-ia} \) give rise to the same 2-factor and then we obtain \((p-1)/2\) distinct 2-factors associated to the same class of parallel lines.

Analogously, \((p-1)/2\) distinct 2-factors are generated from each mixed translation \( m_{w_0, w_1, \ldots , w_{p-1}} \). For each hyperplane \( W_i \) \((i = 0, \ldots , p-1)\) we construct \( p^{n-2} \) \( p \)-cycles as follows: take \( p^{n-2} \) points, \( x_1, \ldots , x_{p^{n-2}} \), of \( W_i \) which belong to distinct lines in the parallelism class generated by \( w_i \) on \( W_i \), and construct the cycles:

\[(x_j, x_j + w_i, \ldots , x_j + (p-1)w_i),\]

where \( j = 1, \ldots , p^{n-2} \). As before \((p-1)/2\) distinct 2-factors are constructed from \( m_{w_0, k w_0, \ldots , k w_{p-1}}, k = 1, \ldots , p-1 \).
3.2. An example of a non-Hamiltonian primitive 2-factorization

Let \( d \) be a primitive divisor of \( p^n - 1 \), that is a divisor of \( p^n - 1 \) such that \( d \) does not divide \( p^m - 1 \) for \( m < n \). The existence of such a prime divisor is given by Zsigmondy’s Lemma, see for instance [9].

Let \( B \) be the subgroup of order \( d \) of the multiplicative group \( GF(p^n)^* \). Define \( G \) to be the group of all mappings \( g : V \to V \) of the form \( g(x) = b \cdot x + a \) for some element \( b \in B \) and some vector \( a \in V \). It is easy to prove that \( G \) acts primitively on \( V \) (see also [11]).

If we take \( p^n \) admitting a primitive divisor \( d \), with \( d \leq \lfloor (p^n - 1)/(p - 1) \rfloor \), then a \( G \)-invariant 2-factorization of \( K_{p^n} \), which is different from that arising from the standard line parallelism of \( AG(n, p) \) is constructed in the following proposition. There are infinitely many prime powers \( p^n \) which satisfy this requested condition on \( d \): for example, if we take \( 3^n \) with \( n \geq 3 \) odd and such that \( (3^n - 1)/2 \) is not a prime, then each prime \( d \) obtained by Zsigmondy’s Lemma satisfies the hypothesis.

**Proposition 1.** Let \( p^n, n \geq 3 \), be a prime power such that \( p^n - 1 \) admits a primitive divisor \( d \) with \( d \leq \lfloor (p^n - 1)/(p - 1) \rfloor \). There exists a primitive 2-factorization of \( K_{p^n} \), which is not 2-transitive.

**Proof.** Let \( G \) be the primitive group described above. This group has a primitive action on the points of \( AG(n, p) \). The action of \( G \) on the parallelism classes yields \( (p^n - 1)/d(p - 1) \) orbits of the same length. As \( d \leq \lfloor (p^n - 1)/(p - 1) \rfloor \) we have at least \( p \) distinct orbits. Let \( W_0 \) be a hyperplane of \( AG(n, p) \) through \((0, \ldots, 0)\) and let \( w_0, \ldots, w_{p-1} \in \overline{W_0} \) be \( p \) vectors which determine \( p \) parallelism classes in distinct orbits under the action of \( G \) on \( AG(n, p) \). Consider the mixed translation \( m_i = m_{W_0,i,w_0,\ldots,w_{p-1}}, i = 1, \ldots, (p - 1)/2 \), and let \( F_i \) be the 2-factor arising from \( m_i \). The group \( T_{W_0} = \{ t_a / a \in \overline{W_0} \} \) is the stabilizer of each \( F_i \) in \( G \) and \( |\text{orb}_G(F_i)| = p^d/(p^n-1) = pd \). The set \( \bigcup_i \text{orb}_G(F_i) \) contains \((p - 1)/2pd\) 2-factors which are associated to \( pd \) distinct classes of parallelism. For each of the remaining \((p^n - 1)/p - 1 - pd\) parallelism classes, take a vector \( a \) which determines the class, and construct the \((p - 1)/2pd\) 2-factors arising from the translations \( t_{ia}, i = 1, \ldots, (p - 1)/2 \) as described before. We obtain \((p^n - 1)/2 - (p - 1)/2pd\) further 2-factors which, together with the \((p - 1)/2pd\) 2-factors obtained from the mixed translations give rise to a 2-factorization \( \mathcal{F} \) of \( K_{p^n} \) admitting \( G \) as an automorphism group acting primitively on the vertex-set. The full automorphism group of \( \mathcal{F} \) does not contain a subgroup which is 2-transitive on the vertices: indeed in this case each 2-factor should arise from a translation of \( T \) in the way described in Section 3.1, as proved in [2]. □

If the number of vertices \( v \) is not a prime power, no examples of primitive 2-factorizations of \( K_v \) are available.

3.3. Non-existence results

Now we give some necessary conditions for the existence of a primitive 2-factorization and then we will use these conditions to prove that \( K_9 \) has no primitive 2-factorizations different from the 2-transitive one and that \( K_{15} \) does not admit a primitive 2-factorization.

**Lemma 2.** \( G \) does not fix a non-Hamiltonian factor of \( \mathcal{F} \).

**Proof.** The cycles of a non-Hamiltonian factor fixed by \( G \) form a system of imprimitivity for \( G \). □

**Lemma 3.** If \( v \) is not a prime, \( G \) does not fix a Hamiltonian factor of \( \mathcal{F} \).

**Proof.** Suppose \( G \) fixes a Hamiltonian factor \( F \), then \( G \leq D_v \). By the transitivity of \( G \) on the vertex-set and by the fact that \( v \) is odd, we also have \( Z_v \leq G \). The transversals of a proper subgroup of \( Z_v \) form a system of imprimitivity for \( G \). □

**Lemma 4.** If \( G \) fixes a 2-factor, then \( v \) is a prime \( p \), \( \mathcal{F} = \mathcal{N}(\mathbb{Z}_p) \) and either \( G \cong \mathbb{Z}_p \) or \( G \cong D_p \).

**Proof.** This follows from Lemmas 1–3. □
Proposition 2. A primitive 2-factorization of $K_9$ is isomorphic to the only 2-factorization $\mathcal{F}$ with $\text{Aut}(\mathcal{F}) \cong \text{AGL}(2, 3)$, and $G$ is a subgroup of $\text{AGL}(2, 3)$ acting primitively on the vertex-set.

Proof. All 2-factorizations of $K_9$ and the order of the corresponding automorphism groups are enumerated in [8, Table 2, p. 437]. By the transitivity of $G$ on the vertex-set, 9 must be a divisor of $|G|$ and only 4 possibilities remain. In the three cases where $\text{Aut}(\mathcal{F})$ is different from $\text{AGL}(2, 3)$ there is a factor fixed by $\text{Aut}(\mathcal{F})$, therefore by Lemma 4 these cases are ruled out. The assertion follows. □

Proposition 3. A primitive 2-factorization $\mathcal{F}$ of $K_{15}$ does not exist.

Proof. There are 6 groups admitting a primitive representation of degree 15: $S_6$, $A_6$, $A_7$, $\text{PSL}(4, 2)$, $S_{15}$ and $A_{15}$. The last three groups are at least 2-transitive on the vertex-set and then these cases are ruled out by the main theorem in [2]. The group $A_6$ is simple, its proper subgroups have index greater than 5, then the orbit-lengths under its action are equal to 1 or greater than 5. The first possibility is excluded by Lemma 4, then $A_6$ is transitive on the factors of $\mathcal{F}$, but the number of factors is not a divisor of the order of $A_6$: a contradiction. This implies that also $S_6$ is ruled out because it contains $A_6$. The only remaining possibility is $A_7$. Using the computer package GAP [10], we have checked that the pointwise stabilizer of 2 vertices is isomorphic to $A_4$ and fixes a further vertex. We can conclude that the 3-cycle, say $T$, with these 3 vertices belongs to $\mathcal{DF}$. Now, we have generated the only possible decomposition of $K_{15}$ preserved by $A_7$ as the orbit of $T$ under the action of the group $A_7$. The decomposition obtained is a Steiner triple system on 15 vertices and it is isomorphic to $PG(3, 2)$, see [11]. There are only two non-isomorphic parallelisms of $PG(3, 2)$, each having a group of order 168, and both intransitive (see for instance [5]). □

References