On Characterization of Elementary Trapping Sets of Variable-Regular LDPC Codes

Mehdi Karimi and Amir H. Banihashemi
Dept. of Systems and Computer Engineering
Carleton University
Ottawa, Canada
Outline

- Error Floor and Trapping Sets
- Elementary Trapping Sets and Their Characterization
- Layered Superset (LSS) Property
- Samples of Results
- Concluding Remarks
Error Floor and Trapping Sets

- Finite-length LDPC codes under iterative decoding algorithms are prone to have an error floor.
Error Floor and Trapping Sets

- Finite-length LDPC codes under iterative decoding algorithms are prone to have an error floor.
Error Floor and Trapping Sets

- Finite-length LDPC codes under iterative decoding algorithms are prone to have an error floor.

- The error floor performance highly depends on the code’s graph and its “trapping sets”
Trapping Sets: Examples
Trapping Sets

- Classified as \((a, b)\)
Trapping Sets: Examples

(7,8)

(7,6)

(7,1)

(7,2)

(8,2)
Trapping Sets

• Classified as \((a,b)\)

• Information about TSs can be used for
  - error floor analysis
  - code/decoder design
Trapping Sets

- Classified as \((a,b)\)

- Information about TSs can be used for
  - error floor analysis
  - code/decoder design

- Existing search techniques are complex, and limited to short block lengths (~500), small values of \(a\) and \(b\) \((a < 11, b < 3)\), structured codes or codes with particular degree distributions (variable degree 3).
Trapping Sets

• Classified as \((a,b)\)

• Information about TSs can be used for
  - error floor analysis
  - code/decoder design

• Existing search techniques are complex, and limited to short block lengths (~500), small values of \(a\) and \(b\) \((a < 11, b < 3)\), structured codes or codes with particular degree distributions (variable degree 3).

• No general or simple characterization available!
Elementary Trapping Sets (ETS)

- Main cause of error floor in LDPC codes
Elementary Trapping Sets (ETS)

• Main cause of error floor in LDPC codes

• **Definition:** Induced subgraph contains check nodes of degree 1 or 2
Trapping Sets: Examples

(7,8)

(7,6)

(7,2)

(7,1)

(8,2)
Elementary Trapping Sets (ETS)

• Main cause of error floor in LDPC codes

• **Definition:** Induced subgraph contains check nodes of degree 1 or 2

• Is there a simple characterization of ETSs which lends itself well to a simple search algorithm?
Elementary Trapping Sets (ETS)

• Main cause of error floor in LDPC codes

• **Definition:** Induced subgraph contains check nodes of degree 1 or 2

• Is there a simple characterization of ETSs which lends itself well to a simple search algorithm?
Elementary Trapping Sets (ETS)

• Main cause of error floor in LDPC codes

• **Definition:** Induced subgraph contains check nodes of degree 1 or 2

• Is there a simple characterization of ETSs which lends itself well to a simple search algorithm?

  **Yes!**
Elementary Trapping Sets (ETS)

- Main cause of error floor in LDPC codes

- **Definition:** Induced subgraph contains check nodes of degree 1 or 2

- Is there a simple characterization of ETSs which lends itself well to a simple search algorithm?

  Yes! For variable-regular codes
Main Contributions

- Dominant ETSs of variable-regular LDPC codes are layered supersets (LSS) of short cycles
Main Contributions

• Dominant ETSs of variable-regular LDPC codes are layered supersets (LSS) of short cycles

• They can be characterized by a nested sequence of ETSs which starts from a short cycle and grows to the ETS one node at a time
Main Contributions

• Dominant ETSs of variable-regular LDPC codes are layered supersets (LSS) of short cycles.

• They can be characterized by a nested sequence of ETSs which starts from a short cycle and grows to the ETS one node at a time.
Main Contributions

- Dominant ETSs of variable-regular LDPC codes are layered superset (LSS) of short cycles

  \[ \equiv \]

- They can be characterized by a nested sequence of ETSs which starts from a short cycle and grows to the ETS one node at a time

  - LSS of the 8-cycle \( v_2v_3v_4v_5 \):

    \[
    \begin{align*}
    v_2v_3v_4v_5 & \\
    \downarrow & \\
    v_2v_3v_4v_5 + v_6 & \\
    \downarrow & \\
    v_2v_3v_4v_5 + v_6 + v_7 & \\
    \downarrow & \\
    v_2v_3v_4v_5 + v_6 + v_7 + v_1
    \end{align*}
    \]
Main Contributions

- LSS property lends itself to a simple search algorithm (that starts from short cycles of the code’s Tanner graph as input and finds all the ETSs with LSS property in a guaranteed fashion)
Main Contributions

• Comparison with existing results:
  
  - More general characterization (both random and structured codes, wide range of variable node degrees and TS classes)
Main Contributions

• Comparison with existing results:
  - More general characterization (both random and structured codes, wide range of variable node degrees and TS classes)
  - Significantly faster search algorithm
Cycles & ETSs

• Short cycles are problematic for iterative decoding

• Any ETS contains at least one cycle

• It is natural to study the relationship between short cycles and more complex ETSs

• Short cycles are easy to enumerate
Non-Isomorphic Structures of ETSs

- Variable-regular TG with variable degree $d_i$ and girth $g$:
  - Each $(a,b)$ class of ETSs has a number of non-isomorphic structures

Fig. 2. All the possible non-isomorphic topologies for $(6, 2)$ ETSs in left-regular LDPC codes with $d_t = 4$ and $g = 6.$
Non-Isomorphic Structures of ETSs

• For a large number of dominant classes of ETSs, using normal graph representation and combinatorial arguments, we provably find all the non-isomorphic structures
Non-Isomorphic Structures of ETSs

- For a large number of dominant classes of ETSs, using normal graph representation and combinatorial arguments, we provably find all the non-isomorphic structures.

**Proposition 1.** Any \((6,2)\) ETS of a left-regular LDPC code with \(d_t = 4\) and \(g = 6\) has one of the structures presented in Figure 2.
Normal Graphs

- An example of a (6,6) ETS of a code with $d_i=4$, $g=6$
Normal Graphs

• An example of a (6,6) ETS of a code with $d_i=4$, $g=6$

![Graph Image]

• Search complexity for non-isomorphic structures (nauty package)
  
  - Bipartite graph (21 nodes, 24 edges, girth at least 6, min and max degrees 1 and 4, respectively): 53,727,932 structures
  
  - Normal graph (6 nodes, 9 edges, min and max degrees 2 and 4, respectively): 11 structures (all valid)
Layered Superset (LSS) Property

• Definition:

Consider an \((a, b)\) ETS \(S\) in \(\mathcal{T}\). Let \(C \subset S\) be an ETS in \(\mathcal{T}\) of size \(\alpha < a\). We say that \(S\) is a \emph{layered superset (LSS)} of \(C\) if there exists a nested sequence of ETSs: \(C \triangleq S^{(0)} \subset S^{(1)} \subset \cdots \subset S^{(a-\alpha)} \triangleq S\), such that \(S^{(i)} \in \mathcal{T}\) has size \(\alpha + i\) for \(i = 0, \ldots, a - \alpha\).
Layered Superset (LSS) Property

• Definition:

Consider an \((a, b)\) ETS \(S\) in \(T\). Let \(C \subset S\) be an ETS in \(T\) of size \(\alpha < a\). We say that \(S\) is a layered superset (LSS) of \(C\) if there exists a nested sequence of ETSs: \(C \triangleq S^{(0)} \subset S^{(1)} \subset \cdots \subset S^{(a-\alpha)} \triangleq S\), such that \(S^{(i)} \in T\) has size \(\alpha + i\) for \(i = 0, \ldots, a - \alpha\).

• An example of results:

Proposition 2. All the \((6, 2)\) ETSs of a left-regular LDPC code with \(d_{l} = 4\) and \(g = 6\) are layered supersets of any one of their 6-cycle subsets.
Layered Superset (LSS) Property

• Definition:

Consider an \((a, b)\) ETS \(S\) in \(\mathcal{T}\). Let \(C \subset S\) be an ETS in \(\mathcal{T}\) of size \(\alpha < a\). We say that \(S\) is a layered superset (LSS) of \(C\) if there exists a nested sequence of ETSs: \(C \triangleq S^{(0)} \subset S^{(1)} \subset \cdots \subset S^{(a-\alpha)} \triangleq S\), such that \(S^{(i)} \in \mathcal{T}\) has size \(\alpha + i\) for \(i = 0, \ldots, a - \alpha\).

• An example of results:

**Proposition 2.** All the \((6, 2)\) ETSs of a left-regular LDPC code with \(d_1 = 4\) and \(g = 6\) are layered supersets of any one of their 6-cycle subsets.

• Efficient Algorithm:

**Lemma 1.** Consider an ETS \(S \in \mathcal{T}\) of size \(a + 1\). Suppose that \(S\) has an elementary trapping subset \(S' \in \mathcal{T}\) of size \(a\). Then, the variable node \(v \in S \setminus S'\) is only connected to unsatisfied check nodes of \(S'\) (at least two of them), i.e., there is no connection between \(v\) and the satisfied check nodes of \(S'\).
Characterization of ETSs of Variable-regular LDPC Codes

- Variable-regular LDPC code with variable degree $d_i$ and girth $g$:

  - Step 1: For each $(a,b)$ ETS class, find all the non-isomorphic structures

  - Step 2: Investigate the LSS property of each structure with respect to its cycles
Characterization of ETSs of Variable-regular LDPC Codes

- An example of results:

Theorem 1. For left-regular graphs with $d_l = 3$ and $g = 6$, the multiplicity of non-isomorphic LSS$_x$ structures for different values of $x$ are listed in Table I for different classes of ETSs in $T$.

<table>
<thead>
<tr>
<th>$a = 4$</th>
<th>$a = 5$</th>
<th>$a = 6$</th>
<th>$a = 7$</th>
<th>$a = 8$</th>
<th>$a = 9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b = 0$</td>
<td>${g}$</td>
<td>-</td>
<td>${g+2}$</td>
<td>-</td>
<td>${g+4}$</td>
</tr>
<tr>
<td>$b = 1$</td>
<td>-</td>
<td>${g}$</td>
<td>-</td>
<td>${g+2}$</td>
<td>${g+4}$</td>
</tr>
<tr>
<td>$b = 2$</td>
<td>${g}$</td>
<td>-</td>
<td>${g+2}$</td>
<td>${g+4}$</td>
<td>-</td>
</tr>
<tr>
<td>$b = 3$</td>
<td>-</td>
<td>${g+2}$</td>
<td>-</td>
<td>${g+4}$</td>
<td>${g+6}$</td>
</tr>
<tr>
<td>$b = 4$</td>
<td>${g+2}$</td>
<td>-</td>
<td>${g+4}$</td>
<td>${g+6}$</td>
<td><strong>NA</strong></td>
</tr>
<tr>
<td>$b = 5$</td>
<td>-</td>
<td>${g+4}$</td>
<td>-</td>
<td>${g+6}$</td>
<td>${g+8}$</td>
</tr>
<tr>
<td>$b = 6$</td>
<td>-</td>
<td>-</td>
<td>${g+6}$</td>
<td>-</td>
<td>${g+8}$</td>
</tr>
<tr>
<td>$b = 7$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>${g+8}$</td>
<td>-</td>
</tr>
<tr>
<td>$b = 8$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>${g+10}$</td>
</tr>
</tbody>
</table>

**Table I**

LSS properties of non-isomorphic structures of $(a,b)$ ETS classes for left-regular graphs with $d_l = 3$ and $g = 6$. 

NA: Not applicable.
Characterization of ETSs of Variable-regular LDPC Codes

- Another example:

**Theorem 2.** For left-regular graphs with $d_l = 3$ and $g = 8$, the multiplicity of non-isomorphic LSS$_x$ structures for different values of $x$ are listed in Table II for different classes of ETSs in $\mathcal{T}$.

<table>
<thead>
<tr>
<th></th>
<th>$a = 4$</th>
<th>$a = 5$</th>
<th>$a = 6$</th>
<th>$a = 7$</th>
<th>$a = 8$</th>
<th>$a = 9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b = 0$</td>
<td>-</td>
<td>-</td>
<td>${g}$</td>
<td>-</td>
<td>${g+2, g+4}$</td>
<td>-</td>
</tr>
<tr>
<td>$b = 1$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>${g}$</td>
<td>-</td>
<td>${g+2, g+4}$</td>
</tr>
<tr>
<td>$b = 2$</td>
<td>-</td>
<td>-</td>
<td>${g}$</td>
<td>-</td>
<td>${g+2, g+4}$</td>
<td>-</td>
</tr>
<tr>
<td>$b = 3$</td>
<td>${g}$</td>
<td>-</td>
<td>${g+2, g+4}$</td>
<td>${g+2}$</td>
<td>${g+2, g+4}$</td>
<td>-</td>
</tr>
<tr>
<td>$b = 4$</td>
<td>${g}$</td>
<td>-</td>
<td>${g+2, g+4}$</td>
<td>-</td>
<td>${g+4, g+6, g+8}$</td>
<td>-</td>
</tr>
<tr>
<td>$b = 5$</td>
<td>${g+2}$</td>
<td>-</td>
<td>${g+4, g+6}$</td>
<td>${g+4}$</td>
<td>${g+6, g+8, g+10}$</td>
<td>$\text{NA}$</td>
</tr>
<tr>
<td>$b = 6$</td>
<td>-</td>
<td>-</td>
<td>${g+4}$</td>
<td>-</td>
<td>${g+6, g+8}$</td>
<td>$\text{NA}$</td>
</tr>
<tr>
<td>$b = 7$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>${g+6}$</td>
<td>-</td>
<td>${g+8, g+10}$ $\text{NA}$</td>
</tr>
<tr>
<td>$b = 8$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>${g+8}$</td>
<td>-</td>
</tr>
</tbody>
</table>

**Table II**

LSS properties of non-isomorphic structures of $(a, b)$ ETS classes for left-regular graphs with $d_l = 3$ and $g = 8$. 

16
Characterization of ETSs of Variable-regular LDPC Codes

- Yet another example:

<table>
<thead>
<tr>
<th>$a$</th>
<th>$b = 0$</th>
<th>$b = 1$</th>
<th>$b = 2$</th>
<th>$b = 3$</th>
<th>$b = 4$</th>
<th>$b = 5$</th>
<th>$b = 6$</th>
<th>$b = 7$</th>
<th>$b = 8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a = 4$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$a = 5$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$a = 6$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$a = 7$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$a = 8$</td>
<td>TS: ${1}$</td>
<td>AS: ${1}$</td>
<td>TS: ${2}$</td>
<td>AS: ${1}$</td>
<td>TS: ${7}$</td>
<td>AS: ${1}$</td>
<td>TS: ${5}$</td>
<td>AS: ${2}$</td>
<td>TS: ${\frac{g+2}{18}, 1}$</td>
</tr>
<tr>
<td>$a = 9$</td>
<td>TS: -</td>
<td>AS: -</td>
<td>TS: ${2}$</td>
<td>AS: ${1}$</td>
<td>TS: ${g}$</td>
<td>AS: ${1}$</td>
<td>TS: ${g+2, g+4}$</td>
<td>AS: ${1}$</td>
<td>TS: ${g, g+2, g+4}$</td>
</tr>
</tbody>
</table>

**TABLE VII**
LSS properties of non-isomorphic structures of $(a, b)$ ETS classes for left-regular graphs with $d_t = 4$ and $g = 8$. 


Characterization of ETSs of Variable-regular LDPC Codes

- And another:

<table>
<thead>
<tr>
<th></th>
<th>$a \leq 7$</th>
<th>$a = 8$</th>
<th>$a = 9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b = 0$</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$b = 1$</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$b = 2$</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$b = 3$</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$b = 4$</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$b = 5$</td>
<td>-</td>
<td>-</td>
<td>TS: ${g_1}$</td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>-</td>
<td>AS: ${g_1}$</td>
</tr>
<tr>
<td>$b = 6$</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$b = 7$</td>
<td>-</td>
<td>-</td>
<td>TS: ${g_1}$</td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>-</td>
<td>AS: ${g_1}$</td>
</tr>
<tr>
<td>$b = 8$</td>
<td>-</td>
<td>TS: ${g_1}$</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>AS: ${g_1}$</td>
<td>-</td>
</tr>
<tr>
<td>$b = 9$</td>
<td>-</td>
<td>-</td>
<td>TS: ${g_3}$</td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>-</td>
<td>AS: ${g_2}$</td>
</tr>
</tbody>
</table>
Characterization of ETSs of Variable-regular LDPC Codes

Theorem 8. For left-regular graphs with $d_l = 6$ and $g > 6$, there does not exist any $(a, b)$ ETS in $T$ with $a < 10$ and $b \leq 10$. 
Characterization of ETSs of Variable-regular LDPC Codes

- Example: $d_i=3$, $g=8$

<table>
<thead>
<tr>
<th>FAS Class</th>
<th>LSS$_g$</th>
<th>LSS$_{g+2}$</th>
<th>LSS$_{g+4}$</th>
<th>LSS$_{g+6}$</th>
<th>LSS$_{g+8}$</th>
<th>Total Multiplicity (Algorithm 1)</th>
<th>Multiplicity (Exhaustive Search [23])</th>
</tr>
</thead>
<tbody>
<tr>
<td>(4, 4)</td>
<td>760</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>760</td>
<td>760</td>
</tr>
<tr>
<td>(5, 3)</td>
<td>14</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>14</td>
<td>14</td>
</tr>
<tr>
<td>(5, 5)</td>
<td>-</td>
<td>10156</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>10156</td>
<td>10156</td>
</tr>
<tr>
<td>(6, 4)</td>
<td>-</td>
<td>507</td>
<td>342</td>
<td>-</td>
<td>-</td>
<td>849</td>
<td>849</td>
</tr>
<tr>
<td>(6, 6)</td>
<td>-</td>
<td>-</td>
<td>66352</td>
<td>-</td>
<td>-</td>
<td>66352</td>
<td>66352</td>
</tr>
<tr>
<td>(7, 3)</td>
<td>-</td>
<td>39</td>
<td>8</td>
<td>-</td>
<td>-</td>
<td>47</td>
<td>47</td>
</tr>
<tr>
<td>(7, 5)</td>
<td>-</td>
<td>-</td>
<td>13121</td>
<td>9309</td>
<td>-</td>
<td>22430</td>
<td>22430</td>
</tr>
<tr>
<td>(8, 2)</td>
<td>-</td>
<td>4</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>(8, 4)</td>
<td>-</td>
<td>-</td>
<td>1742</td>
<td>350</td>
<td>178</td>
<td>2270</td>
<td>2270</td>
</tr>
<tr>
<td>(9, 1)</td>
<td>-</td>
<td>1</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(9, 3)</td>
<td>-</td>
<td>-</td>
<td>140</td>
<td>6</td>
<td>-</td>
<td>146</td>
<td>146</td>
</tr>
<tr>
<td>(10, 2)</td>
<td>-</td>
<td>-</td>
<td>5</td>
<td>1</td>
<td>-</td>
<td>6</td>
<td>6</td>
</tr>
</tbody>
</table>

7 hours
Concluding Remarks

• Investigated the ETS structure of variable-regular LDPC codes

• Developed an efficient approach to find all non-isomorphic structures of a given \((a,b)\) class of ETSs

• Proved that for any category of variable-regular LDPC codes with given \(d_l\) and \(g\), there exist integers \(A\) and \(B\) such that all the classes of \((a,b)\) ETSs with \(a < A\) and \(b < B\), are LSSs of short cycles

• LSS characterization of dominant ETSs is particularly important as it corresponds to a simple algorithm that can find all such ETSs in a guaranteed fashion starting from the short cycles of the graph
Dominant ETSs Are LSSs of Cycles!
Thank You!