Fuzzy differential equations and the extension principle

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Abstract

We study the Cauchy problem for differential equations, considering its parameters and/or initial conditions given by fuzzy sets. These fuzzy differential equations are approached in two different ways: (a) by using a family of differential inclusions; and (b) the Zadeh extension principle for the solution of the model. We conclude that the solutions of the Cauchy problem obtained by both are the same. We also provide some illustrative examples.

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1. Introduction

We begin by considering the initial value problem

\[
\begin{align*}
    x'(t) &= f(t, x(t), w), \\
    x(t_0) &= x_0
\end{align*}
\]

(1)

where \( x_0 \in \mathbb{R}^n \), \( w \in \mathbb{R}^k \) and \( f \) is continuous.

The importance of study of differential equations from the theoretical point of view as well as for their applications is well known [11,27]. However, in some cases, these equations are too restrictive for the description of phenomena. For example, in mathematical models that describe biological phenomena, the parameters values are subjected to inaccuracies caused by some variations to estimate the experimental data [3,16,19,26]. Also, in some quality control problems, the parameters are given by fuzzy subsets and this requires the use of fuzzy numbers in the calculations [17,24].
Supposing that \( x_0 \) and \( w \) are uncertain in the problem (1), we have the following problem

\[
\begin{align*}
\frac{d}{dt}x(t) &= \hat{f}(t, x(t), W), \\
x(t_0) &= x_0 \in X_0,
\end{align*}
\]

(2)

where \( \hat{f} \) is the Zadeh’s extension principle in \( w \) of function \( f \) given in (1), \( W \) and \( X_0 \) are fuzzy sets.

The following questions have to be answered in connection with the problem (2): what is an interpretation of problem (2)? What does a solution of (2) mean?

There are at least three possibilities for representing the solution of (2): the first involves the Hukuhara derivative [15]; the second, suggested by Hüllermeier [12] (see also [8]), is based on a family of differential inclusions and the last one obtained through Zadeh’s extension principle applied to the deterministic solution [19]. For some other recent and novel approaches, see, for example [1].

Each of these approaches has a particular feature: The first one, via Hukuhara derivative, uses the concept of derivative for fuzzy functions. At first, this approach had some shortcomings because its solution has the property that the diameter is non-decreasing as \( t \) increases, the fuzziness is non-decreasing in time (see for instance [7,8,10,14,18,20,23]). Nevertheless, Bede and Gal [5] have recently introduced the concept of weakly generalized differential of a fuzzy-number-valued function, thereby solving the problem of the Hukuhara derivative.

The second interpretation, by Hüllermeier, different from classical fuzzy differential equations involving the Hukuhara derivative, allows us to “characterize” the main properties of ordinary differential equations in a natural way, such as periodicity, stability, bifurcation, among others [8,23]. However, it is important to remark that this interpretation also has some its shortcomings. The main one is that there is a proper definition for the derivative of a fuzzy-number-valued function. When the fuzzy differential equation is interpreted with the help of differential inclusions, we need only the usual concept of differentiation. Consequently, it is interesting to study both approaches.

In [14], under certain conditions on \( f \) (for instance, \( f \) is an increasing function), it is shown that the two treatments described above are equivalent.

In [22] problem (2) is studied via differential inclusions (Hüllermeier interpretation), with an analysis of continuous dependence of fuzzy solutions on parameters and initial conditions concerning the hypograph metric.

Finally, a totally different approach is given in [19], where differential equations with fuzzy parameters and initial conditions (problem (2)) were studied. That work dealt with introducing the notion of fuzzy solutions by applying Zadeh’s extension principle to the deterministic solution and a presented numerical algorithm, based on monotonicity properties of \( f \). In [13] simulations of continuous fuzzy systems were done by using continuous simulation.

In this paper, we show that a solution for (2) can be obtained through Zadeh’s extension principle, similarly to [19]. We also show the existence of a fuzzy solution which is strongly dependent on the choice of both fuzzy initial condition and parameter. Moreover, we conclude that this fuzzy solution coincides with the solution obtained by using Hüllermeier’s interpretation, via differential inclusions.

2. Basic concepts

We denote by \( \mathcal{K}^n \) the family of all the non-empty compact subsets of \( \mathbb{R}^n \). For \( A, B \in \mathcal{K}^n \) and \( \lambda \in \mathbb{R} \) the operations of addition and scalar multiplication are defined by

\[
A + B = \{a + b/a \in A, b \in B\}, \quad A = \{\lambda a/a \in A\}.
\]

A fuzzy set in universe set \( X \) is a mapping \( u : X \rightarrow [0, 1] \). We think of \( u \) as assigning to each element \( x \in X \) a degree of membership, \( 0 \leq u(x) \leq 1 \).

Let \( u \) be a fuzzy set in \( \mathbb{R}^n \), the \( n \)-dimensional Euclidian space, we define \( [u]^z = \{x \in \mathbb{R}^n/u(x) \geq z\} \) the \( z \)-level of \( u \), with \( 0 < z \leq 1 \). For \( z = 0 \) we have \( [u]^0 = \text{supp}(u) = \{x \in \mathbb{R}^n/u(x) > 0\} \), the support of \( u \).

A fuzzy set \( u \) is called compact if \( [u]^z \in \mathcal{K}^n \ \forall z \in [0, 1] \). We will denote by \( \mathcal{F}(\mathbb{R}^n) \) the space of all the compact fuzzy sets.
The operations of addition and scalar multiplication on $\mathcal{F}(\mathbb{R}^n)$ are defined by

$$(u + v)(x) = \sup_{y \in \mathbb{R}^n} \{u(y) \land v(x - y)\} \quad \text{and} \quad (\lambda u)(x) = \begin{cases} u(\lambda x) & \text{if } \lambda \neq 0, \\ \chi_{\{0\}}(x) & \text{if } \lambda = 0, \end{cases}$$

where $\chi_{\{0\}}$ is the characteristic function of $+0$. It is well known that the following properties are true for all $x$-levels

$$[u + v]^x = [u]^x + [v]^x; \quad \text{and} \quad [\lambda u]^x = \lambda[u]^x \quad \forall x \in [0, 1].$$

(3)

In [25] (see also [21]), Zadeh proposed a so-called extension principle, which became an important tool in fuzzy set theory and applications. The idea is that each function $f : X \to Y$ induces a corresponding function $\hat{f} : \mathcal{F}(X) \to \mathcal{F}(Y)$ (i.e., $\hat{f}$ is a function mapping fuzzy sets in $X$ to fuzzy sets in $Y$) defined for each fuzzy set $u$ in $X$ by

$$\hat{f}(u)(y) = \bigvee_{x \in X, f(x) = y} u(x),$$

if $\{x \in X, f(x) = y\} \neq \emptyset$ and $\hat{f}(u)(y) = 0$ if $\{x \in X, f(x) = y\} = \emptyset$. The function $\hat{f}$ is said to be obtained from $f$ by the extension principle.

An important result of extension principle is the characterization of the levels of the image of a fuzzy set through $\hat{f}$, where $f$ is a continuous function.

**Theorem 1.** [21] If $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous, then the Zadeh’s extension $\hat{f} : \mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$ is well defined, is continuous and

$$[\hat{f}(u)]^x = f([u]^x) \quad \forall x \in [0, 1].$$

(4)

Note that (4) is still valid if $f : U \to \mathbb{R}^n$, where $U$ is an open subset in $\mathbb{R}^n$.

### 2.1. Differential inclusions

Let us consider the following differential inclusion,

$$\begin{aligned}
& x'(t) \in F(t, x(t)), \\
& x(t_0) = x_0 \in X_0,
\end{aligned}$$

(5)

where $F : [t_0, T] \times \mathbb{R}^n \to \mathcal{K}^n$ is a set-valued function and $X_0 \in \mathcal{K}^n$.

A function $x(t, x_0)$ with the initial condition $x_0 \in X_0$ is a solution of (5) in interval $[t_0, T]$ if it is absolutely continuous and satisfies (5) for all $t \in [t_0, T]$, (for more details, see [2]). The attainable set in time $t \in [t_0, T]$, associates to problem (5) is the subset of $\mathbb{R}^n$ given by

$$\mathcal{A}_t(X_0) = \{x(t, x_0)/x(\cdot, x_0) \text{ is solution of (5) with } x_0 \in X_0\}.$$

The set-valued function $F$ allows the modelling of certain types of uncertainty [16], because for each pair $(t, x) \in [t_0, T] \times \mathbb{R}^n$, the derivative cannot be known precisely, but it is known to be an element of the set $F(t, x)$.

A generalization of problem (5), to model fuzzy dynamical systems, is obtained by replacing set $F(t, x)$ in (5) by a fuzzy set, that is by considering the fuzzy initial value problem (FIVP)

$$\begin{aligned}
& x'(t) = \tilde{F}(t, x(t)), \\
& x(t_0) = x_0 \in \tilde{X}_0,
\end{aligned}$$

(6)

where $\tilde{F} : [t_0, T] \times \mathbb{R}^n \to \mathcal{F}(\mathbb{R}^n)$ is a fuzzy set-valued function and $X_0 \in \mathcal{F}(\mathbb{R}^n)$.

In agreement with H"{u}llermeier [12], the (FIVP) (6), is interpreted as the family of differential inclusions

$$\begin{aligned}
& x'(t) \in [\tilde{F}(t, x(t))]^x, \\
& x(t_0) = x_0 \in [X_0]^x,
\end{aligned}$$

(7)

where $[\tilde{F}(t, x(t))]^x$ is the $x$-level of fuzzy set $\tilde{F}(t, x(t))$.

In order to simplify the notation, we will denote by $X_0$ the fuzzy set of the initial condition.
For each $a \in [0, 1]$, we say that $x : [t_0, T] \to \mathbb{R}^n$, is an $a$-solution of (6) if it is a solution of (7). We will denote by $\mathcal{A}_a([X_0]^*) := \mathcal{A}_a^T$, $t_0 \leq t < T$, the attainable set of the $a$-solutions, that is,

$$\mathcal{A}_a^T = \mathcal{A}_a([X_0]^*) = \{x(t, x_0) / x(., x_0) \text{ is solution of (7) with } x_0 \in [\hat{X}_0]^*\}.$$ 

Diamond, in [8], uses the Representation Theorem to prove that $\mathcal{A}_a^T$ are the $a$-levels of a fuzzy set $\mathcal{A}_a(X_0)$ in $\mathbb{R}^n$ for all $t_0 \leq t < T$. The fuzzy set $\mathcal{A}_a(X_0)$ will be said to be the attainable set of problem (6).

3. Differential equations with fuzzy initial condition

In what follows, we study the initial value problem (1) replacing only the initial condition by a fuzzy set, and keeping the parameter $w$ as a crisp constant. This is done both by applying the extension principle to the deterministic solution and also via H"ullermeier’s approach, i.e., using differential inclusions.

Our starting point is the initial value problem

$$\begin{cases}
  x'(t) = f(t, x(t)), \\
  x(t_0) = x_0,
\end{cases}$$

(8)

where $f$ is continuous and $x_0 \in \mathbb{R}^n$.

Supposing that the initial condition $x_0$ is uncertain and modelled by a fuzzy set, then problem (8) is replaced by

$$\begin{cases}
  x'(t) = f(t, x(t)), \\
  x(t_0) \in X_0,
\end{cases}$$

(9)

where $X_0$ is a fuzzy set.

3.1. Solution via extension principle

Let $U$ be an open subset in $\mathbb{R}^n$ such that there exists a solution $x(\cdot, x_0)$ of (8) with $x_0 \in U$ in the interval $[t_0, T]$, and for all $t \in [t_0, T]$, $x(t, \cdot)$ is continuous on $U$. Then, we can define the operator:

$$L_t : U \to \mathbb{R}^n,$$

by $L_t(x_0) = x(t, x_0)$, which is the unique solution of (8) and is continuous relative to $x_0$.

The application of the extension principle to $L_t$, leads to the extension

$$\hat{L}_t : \mathcal{F}(U) \to \mathcal{F}(\mathbb{R}^n),$$

which is the solution of problem (9), via extension principle, with initial condition $X_0 \in \mathcal{F}(U)$.

Remark 1. Note that the existence of $\hat{L}_t$ is guaranteed by Theorem 1, since the function $L_t$ is continuous (see [11]).

3.2. Solution via differential inclusions

According to Hüllemeier’s interpretation, we can write (9) as a family of differential inclusions

$$\begin{cases}
  x'(t) = f(t, x(t)), \\
  x(t_0) = x_0 \in [X_0]^*.
\end{cases}$$

(10)

Following Diamond [8], the attainable sets $\mathcal{A}_a^T$ are the $a$-levels of a fuzzy set $\mathcal{A}_a(X_0)$, where $\mathcal{A}_a^T \subset \mathbb{R}^n$, for each $a$ and for each $t \in [t_0, T]$.

At this point we are able to state a result relating Hüllemeier’s solution to the one obtained through the extension principle.
Theorem 2. Let \( U \) be an open set in \( \mathbb{R}^n \) and \( X_0 \in \mathcal{F}(U) \). Suppose that \( f \) is continuous, that for each \( x_0 \in U \) there exists one unique solution \( x(\cdot, x_0) \) of the problem (8) and that \( x(t, \cdot) \) is continuous in \( U \). Then, there exists \( \hat{L}_t(X_0) \) and \[
abla_t(X_0) = \mathcal{A}_t(X_0)
\]
for all \( t_0 \leq t \leq T \). In other words, the attainable sets for problem (9) can be obtained as the image of the fuzzy initial condition by the Zadeh extension of the deterministic solution.

Proof 1. In order to prove this result we must show that \[
[\hat{L}_t(X_0)]^z = \mathcal{A}_t([X_0]^z) \quad \forall z \in [0, 1].
\]
By the hypotheses we have that \( f \) is continuous and for each \( x_0 \in U \) there exists a unique solution for (8) in the interval \([t_0, T]\).

Thus, for each \( t \in [t_0, T] \), we have that \( L_t : U \to \mathbb{R}^n \), given by \( L_t(x_0) = x(t, x_0) \), is well defined and is continuous. Then, by Theorem 1, \( \hat{L}_t : \mathcal{F}(U) \to \mathcal{F}(\mathbb{R}^n) \) is a continuous function, is well defined and \[
[\hat{L}_t(X_0)]^z = L_t([X_0]^z) = \{L_t(x_0)/x_0 \in [X_0]^z\} = \{x(t, x_0)/x_0 \in [X_0]^z\}. \]

Therefore, given \( z \in [0, 1] \), we have \[
\left[\hat{L}_t(X_0)\right]^z = L_t([X_0]^z) = \{L_t(x_0)/x_0 \in [X_0]^z\} = \{x(t, x_0)/x_0 \in [X_0]^z\}. \quad (11)
\]

On the other hand, the \( z \)-levels of the attainable set for problem (10) are given by \[
\mathcal{A}_t([X_0]^z) = \bigcup_{x_0 \in [X_0]^z} \mathcal{A}_t(x_0) = \{x(t, x_0)/x_0 \in [X_0]^z\}. \quad (12)
\]

From Eqs. (11) and (12) follows the result. \( \Box \)

Example 1. Let us consider the malthusian problem
\[
\begin{cases}
  x'(t) = -\lambda x(t), \\
  x(0) = x_0,
\end{cases}
\]
where \( \lambda > 0 \). As in Example 1.1 in [8], we consider that only the initial condition in (13) is fuzzy and given by \( X_0 \), where \( X_0 \) is a symmetric triangular fuzzy number with support \([-a, a]\). That is, \( [X_0]^z = [-a(1-z), a(1-z)] = (1-z)[-a, a] \).

The deterministic solution of (13) is
\[
L_t(x_0) = x_0 e^{-\lambda t}
\]
which is continuous and non-decreasing with respect to \( x_0 \in \mathbb{R} \). Then, there exists \( \hat{L}_t(X_0) \) and taking (4) into account we have
\[
[\hat{L}_t(X_0)]^z = L_t([-a(1-z), a(1-z)]) = [L_t(-a(1-z)), L_t(a(1-z))] = (1-z)e^{-\lambda t}[-a, a]
\]
for all \( t > 0 \) and \( z \in [0, 1] \).

Now, in [8] (also, in [10] for \( \lambda = 1 \) and \( a = 1 \)) the authors, using differential inclusions, obtained the following
\[
[\mathcal{A}_t(X_0)]^z = (1-z)e^{-\lambda t}[-a, a]
\]
and, consequently, \( \hat{L}_t(X_0) = \mathcal{A}_t(X_0) \), which agrees with Theorem 2.

In [9], problem (13) was considered with \( \lambda = 2 \) and with initial condition \( x(0) = X_0 \), a symmetric triangular fuzzy number, whose support is interval \([0, 1]\). Then, by using a version of the classical variation of constants formula it was shown that
\[
[\mathcal{A}_t(X_0)]^z = \left[\frac{\lambda}{2} e^{-\lambda t}, \left(1 - \frac{\lambda}{2}\right) e^{-\lambda t}\right].
\]
Now, for each \( t \in [0, T] \), applying Zadeh’s extension principle to the deterministic solution \( L_t(x_0) = x_0 e^{-2t} \), and taking into account that \( L_t \) is non-decreasing with respect to \( x_0 \), we have

\[
[\tilde{L}_t(X_0)]^z = \left[ \frac{z}{2} e^{-2t}, \left(1 - \frac{z}{2}\right) e^{-2t} \right].
\]

Consequently, \( \tilde{L}_t(X_0) = \mathcal{A}_t(X_0) \).

**Example 2.** Consider the differential equation \( x'(t) = \lambda x^2(t) \), where \( \lambda < 0 \). Then, the solution, for the initial condition \( x(0) = x_0 \), is given by \( x(t, x_0) = \frac{x_0}{1-e^{-2\lambda t}} \). Therefore, \( L_t(x_0) \) is continuous with respect to \( x_0 \) on the open interval \( I = (0, +\infty) \) for each \( t \in [0, +\infty) \). Consequently, there exists \( \tilde{L}_t(X_0) \) for all \( X_0 \in \mathcal{F}(I) \), and we have that \( \tilde{L}_t(X_0) = \mathcal{A}_t(X_0) \) is the attainable set.

Now, if we consider \( \lambda > 0 \), then \( L_t(x_0) \) is continuous with respect to \( x_0 \) on the open interval \( J = (0, z) \) for each \( t \in [0, 1/z] \). Therefore, the attainable set \( \tilde{L}_t(X_0) \) exists for each \( t \in [0, \sup_{x \in [X_0]} z] \).

In [14], Kaleva studied the previous problem for \( \lambda = 1 \), i.e., \( x'(t) = x^2(t) \) where the initial condition \( X_0 \) is a triangular fuzzy number

\[
X_0(y) = \begin{cases} 
3 - y & \text{if } 2 \leq y \leq 3, \\
y - 1 & \text{if } 1 \leq y \leq 2, \\
0 & \text{elsewhere}.
\end{cases}
\]

In this case there exists an attainable set \( \tilde{L}_t(X_0) \) for \( t \in [0, 1/3] \). Also, for each \( t \in [0, 1/3] \) the function \( L_t(x_0) \) is non-decreasing with respect to \( x_0 \). Then, for each \( z \in [0, 1] \) we have

\[
[\tilde{L}_t(X_0)]^z = L_t([X_0]^z) = L_t([1 + \alpha, 3 - \alpha]) = \left[ L_t(1 + \alpha), L_t(3 - \alpha) \right] = \left[ 1 + \frac{\alpha}{1 - t - tz}, \frac{3 - \alpha}{1 - 3t - tz} \right].
\]

In this Example we see that if \( \lambda < 0 \) the attainable set \( \tilde{L}_t(X_0) \) exists for \( X_0 \in \mathcal{F}(I, +\infty) \), with \( t \in [0, +\infty) \). However, if \( \lambda > 0 \) there exists \( \tilde{L}_t(X_0) \) for \( X_0 \in \mathcal{F}(0, z) \), with \( t \in [0, 1/\lambda z] \).

**Remark 2.** In [14] Kaleva studied the previous problem for \( \lambda = 1 \), considering \( x'(t) \) the Hukuhara derivative and the initial condition \( X_0 \) as a triangular fuzzy number. In that case he obtained the same solution which was obtained here via differential inclusion. Therefore, from Theorem 2, the three processes studied lead to the same solution for Example 2.

### 3.3. Fuzzy coefficient and initial condition

In this section we will discuss the fuzzy differential equations obtained from a deterministic differential equation introducing an uncertainty coefficient and fuzzy initial condition.

For this, we will consider that parameter \( w \) and initial condition \( x_0 \) are uncorrelated [6]. Moreover, we assume that \( W \) and \( X_0 \) are non-interactive, that is, \([X_0 \times W]^z = [X_0]^z \times [W]^z \) for all \( z \in [0, 1] \).

Let \( f : \Omega \subset \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n \) be a continuous function and \((x_0, w) \in \Omega \). Additionally, consider the initial value problem

\[
\begin{align*}
\begin{cases} 
\dot{x}(t) = f(t, x(t), w), \\
x(t_0) = x_0, 
\end{cases}
\end{align*}
\]

where \( x(t) \) is defined on \([t_0, T]\). Supposing that the initial condition \( x_0 \) and parameter \( w \) are uncertain and modelled by fuzzy sets. Then we have problem (2),

\[
\begin{align*}
\begin{cases} 
\dot{x}(t) = \hat{f}(t, x(t), W), \\
x(t_0) = X_0. 
\end{cases}
\end{align*}
\]

We will obtain a solution to (15) by using the extension principle. In order to do that, let \( U \subset \Omega \) be an open set in \( \mathbb{R}^n \times \mathbb{R}^k \) such that there is a unique solution \( x(\cdot, x_0, w) \) of (14) in the interval \([t_0, T]\), with \((x_0, w) \in U \) and for all \( t \in [t_0, T] \). Then \( x(t, \cdot, \cdot) \) is continuous on \( U \) and the operator:

\[
L_t : U \to \mathbb{R}^n
\]
given by $L_r(x_0, w) = x(t, x_0, w)$, is the unique solution of (14) and continuous with respect to $(x_0, w)$.

So, applying the extension principle to the solution $L_r$, we obtain

$$
\hat{L}_r : \mathcal{F}(U) \rightarrow \mathcal{F}(\mathbb{R}^n),
$$

which is the solution to problem (15), with initial condition $X_0$ and parameter $W$, via extension principle, for all $t \in [t_0, T]$.

Next, we show that the fuzzy solution of (15) obtained by the extension principle coincides with the one obtained by differential inclusion.

By making the variables change $y = (x, w)$, the initial value problem (14) may be rewritten as

$$
\begin{cases}
  y' = F(t, y) = (f(t, x, w), 0), \\
  y(t_0) = y_0,
\end{cases}
$$

where $y(t) = (x(t), 0)$, $y(t_0) = (x(t_0), w) \in \mathbb{R}^n \times \mathbb{R}^k$ and $y$ should be defined on $[t_0, T]$.

Denoting the natural projection $\mathbb{R}^n \times \mathbb{R}^k \rightarrow \mathbb{R}^n$ by $pr_1$, the solutions to Eqs. (14) and (16) are related to each other in the following way: If $x$ is a solution of problem (14), then $y := (x, w)$ is a solution to (16). Conversely, a solution $y$ to (16) yields the solution $x := pr_1 \circ y$ to (14).

Taking up Hüllermeier’s proposal to formulate fuzzy differential equations in terms of systems of differential inclusions, the initial value problem (15), with initial condition $X_0$ and parameters $W$, leads to

$$
\begin{cases}
  x' \in f(t, x, [W]^x), \\
  x(t_0) \in X_0^x,
\end{cases}
$$

where $x \in [0, 1]$ and from Theorem 1 $f(t, x, [W]^x) = [f(t, x, W)]^x$.

Analogously, the initial value problem (16) extends to

$$
\begin{cases}
  y' = F(t, y), \\
  y(t_0) \in [Y_0]^x = [X_0]^x \times [W]^x.
\end{cases}
$$

We denote by $\mathcal{A}_r^F(Y_0)$ the attainable set of (18). Also, we denote by $\hat{L}_r^F(Y_0)$ the Zadeh extension principle of $L_r^F(y_0) = (x(t), y_0)$, solution of problem (16).

Consequently, from problems Eqs. (17) and (18) we obtain

$$
pr_1(\mathcal{A}_r^F([Y_0]^x)) = \mathcal{A}_r([X_0]^x)
$$

for each $x \in [0, 1]$. Also, we have

$$
[\hat{L}_r(X_0, W)]^x = pr_1([\hat{L}_r^F(Y_0)]^x).
$$

**Corollary 1.** Let $U$ be an open set in $\mathbb{R}^n \times \mathbb{R}^k$. Suppose that $f$ is continuous, that for each $(x_0, w) \in U$ there exists a unique solution $x(\cdot, x_0, w)$ of problem (14) and that $x(t, \cdot, \cdot)$ is continuous on $U$. Then, there exists $\hat{L}_r(X_0, W)$ for each $(X_0, W) \in \mathcal{F}(U)$ and

$$
\hat{L}_r(X_0, W) = \mathcal{A}_r(X_0)
$$

for all $t_0 \leq t \leq T$.

**Proof 2.** From Theorem 2 we have $\hat{L}_r^F(Y_0) = \mathcal{A}_r^F(Y_0)$. Then, from Eqs. (19) and (20), the result follows.

**Example 3.** Let us consider the normalized deterministic Verhulst population model

$$
\begin{cases}
  x' = ax(1 - x), \\
  x(0) = x_0,
\end{cases}
$$

where $f(x, a) = ax(1 - x)$ and $a$ is the intrinsic rate of growth.
The deterministic solution of (21) is

\[ L_t(x_0, a) = \frac{x_0}{x_0 - (x_0 - 1)e^{-at}}. \]

Let \( U \) be the region limited by \( 0 < x < 1 \) and \( a > 0 \). Then \( U \) is an open set in \( \mathbb{R} \times \mathbb{R} \) and \( L_t \) is continuous on \( U \) for each \( t > 0 \).

Since \( L_t(x_0, a) \) is continuous with respect to \( (x_0, a) \) on \( U \), there exists \( \tilde{L}_t(X_0, A) \) for \( (X_0, A) \) with support contained in \( U \) and their \( \alpha \)-level sets are given by

\[
[\tilde{L}_t(X_0, A)]^a = L_t([X_0]^a \times [A]^a) = \{L_t(x_0, a)/x_0 \in [X_0]^a \text{ and } a \in [A]^a\}
\]

for all \( t \geq 0 \) and \( \alpha \in [0, 1] \).

Consequently, from Corollary 1, the attainable set of problem (21), considering \( x_0 \) and \( a \) as fuzzy sets, is given by

\[
[\tilde{L}_t(X_0, A)]^a = \left\{ \frac{x_0}{x_0 - (x_0 - 1)e^{-at}} \mid x_0 \in [X_0]^a \text{ and } a \in [A]^a \right\} = \mathcal{A}_t([X_0]^a).
\]

A discrete version of Verhulst population model (21) was studied in [4], where the authors obtained a solution, studied the equilibrium, cycles and stability of attractors.

**Remark 3.** For solving the fuzzy differential equation (15), the idea is not to consider particular values of \( x_0 \) and \( w \) before getting the fuzzy for \( L_t(x_0, w) \). Actually, the idea is, precisely, to solve the deterministic equation (14) symbolically for arbitrary values of \( x_0 \) and \( w \), and then apply the extension principle to \( L_t(x_0, w) \) obtaining, \( \tilde{L}_t(X_0, W) \), which is the attainable set of fuzzy differential equation (15).

For example, let us consider the following problem

\[
\begin{cases}
x' = w - x \\
x(0) = x_0,
\end{cases}
\]

where \( w, x_0 \in \mathbb{R} \). Then the deterministic solution is

\[
x(t) = x_0 e^{-t} + w(1 - e^{-t})
\]

and the function \( L_t \) is defined by

\[
L_t(x_0, w) = x_0 e^{-t} + w(1 - e^{-t}),
\]

which is linear and continuous in \( \mathbb{R}^2 \). Then there exists \( \tilde{L}_t(X_0, W) \), for any \((X_0, W) \in \mathcal{F}(\mathbb{R}^2)\), and by linearity we have

\[
\tilde{L}_t(X_0, W) = X_0 e^{-t} + W(1 - e^{-t}).
\]

From Corollary 1, \( \tilde{L}_t(X_0, W) = \mathcal{A}_t(X_0) \) is the attainable set of problem (22).

Note that \( L_t \) depends on \( x_0 \) and \( w \), and it is the solution of deterministic Eq. (22). Since we are considering that \( X_0 \) and \( W \) are non-interactive [6], \( L_t \) must be extended both in \( x_0 \) and \( w \) even if \( x_0 = w \) in (22).

4. Conclusion

We have reviewed three methods for study problem (2), and focused on two of them: (a) by using a family of differential inclusions, expressed level setwise on the fuzzy velocities; (b) by finding the crisp solution and then fuzzifying this solution via the Zadeh extension principle. The choice of which approach to use depends on the problem at hand. If, for a particular problem, the state variables are uncertain, then the suitable approaches are, usually, Hüllermeier’s or the one exploiting the Hukuhara derivative. On the other hand, if the IVP under consideration comes from a deterministic case where only a parameter and/or the initial
condition are uncertain, then the use of Zadeh’s extension principle might be preferable, since it is simpler than the other alternatives. It requires neither the concept of derivative of a fuzzy function nor the use of selection theory in order to obtain a solution to the fuzzy IVP. In this case, we only need Zadeh’s extension principle, which is a rather basic concept in fuzzy sets.

Under certain conditions, such as the monotonicity of the vector field, Kaleva [12] proved that both methods via Hüllermeier and Hukuhara yield the same solution. Therefore, we may conclude that, if the fuzzy IVP comes from a deterministic problem whose vector field is given by an increasing function, then the three methods produce the same solution.

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References