

Teleparallel Gravitational Energy in the Gamma Metric[‡]

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Abstract. The Møller energy (due to matter and fields including gravity) distribution of the gamma metric is studied in tele-parallel gravity. The result is the same as those obtained in general relativity by Virbhadra in the Weinberg complex and Yang-Radincshi in the Møller definition. Our result is also independent of the three teleparallel dimensionless coupling constants, which means that it is valid not only in the teleparallel equivalent of general relativity, but also in any teleparallel model.

Keywords: Energy; gamma metric; teleparallel gravity.
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1. Introduction

The well-known gamma metric [1, 2], a static asymptotically flat exact solution of Einstein vacuum equations, is given by

$$ds^2 = \left(1 - \frac{2m}{r}\right)^\gamma dt^2 - \left(1 - \frac{2m}{r}\right)^{-\gamma} \left[\left(\frac{\Omega}{\Xi}\right)^{\gamma^2-1} dr^2 + \frac{\Omega^{\gamma^2}}{\Xi^{\gamma^2-1}} d\theta^2 + \Omega \sin^2 \theta d\phi^2 \right] \quad (1)$$

where

$$\Omega = r^2 - 2mr, \quad \Xi = r^2 - 2mr + m^2 \sin^2 \theta. \quad (2)$$

This metric has three special cases given below.

- $|\gamma| = 1$, the metric is spherically symmetric. The gamma metric reduces to the Schwarzschild space-time.
- $|\gamma| \neq 1$, this space-time is axially symmetric. The gamma metric gives the Schwarzschild space-time with negative-mass, as putting $m = -m'$ ($m' > 0$) and carrying out a coordinate transformation $r \rightarrow R = r + 2m'$, then one obtains the Schwarzschild space-time with positive mass.

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Recently, the energy distribution associated with the gamma metric is calculated by using the Weinberg and Møller energy-momentum complexes in general relativity[3, 4]. The energy in general relativity was found as

$$E = m\gamma. \quad (3)$$

It is really tempting to investigate the teleparallel gravitational energy distribution in this space-time model.

The problem of energy localization is one of the oldest and most controversial problem which remain unsolved since the advent of Einstein's theory of general relativity[5]. Recently, this problem argued in tele-parallel gravity; It has been worked out by many physicists[6, 7, 8]. After Einstein's original work[9] on energy-momentum formulations, various definitions for the energy-momentum densities were proposed: e.g. Tolman, Papapetrou, Landau-Lifshitz, Bergmann-Thomson, Møller, Weinberg, Qadir-Sharif and also tele-parallel gravity analogs of them. Except for the Møller formulation, these energy-momentum definitions are restricted to calculate energy-momentum distribution in quasi-Cartesian coordinates. Møller proposed an expression which could be applied to any coordinate system. So, the notion of energy-momentum complexes was severely criticized for a number of reasons. First, the nature of a symmetric and locally conserved object is non-tensorial one; thus its physical interpretation appeared obscure[10]. Second, different energy-momentum complexes could yield different energy-momentum distributions for the same gravitational backgrounds[11]. Finally, energy-momentum complexes were local objects while it was usually believed that the suitable energy-momentum of the gravitational field was only total, i.e. it cannot be localized[12]. For a long time, attempts to deal with this problem were made only by proposers of quasi-local approach[13, 14].

There have been several attempts to calculate energy-momentum densities by using these energy-momentum definitions associated with many different space-times[15, 16, 17]. Virbhadr[18] showed different energy-momentum formulations gave the same energy distributions as in the Penrose energy-momentum formulation by using the energy and momentum definitions of Einstein, Landau-Lifshitz, Papapetrou and Weinberg for a general non-static spherically symmetric metric of the Kerr-Schild class. Vargas[8] found, using the definitions of Einstein and Landau-Lifshitz in tele-parallel gravity, that the total energy was zero in Friedmann-Robertson-Walker space-times. This result agrees with the previous works[19] of Cooperstock-Israelit, Rosen, Johri *et al.*, Banerjee-Sen. Later on, Saltı and his collaborators considered different space-times for various definitions in tele-parallel gravity to obtain the energy-momentum distributions in a given model. They found the same results as obtained in general relativity, and showed that teleparallel gravity and general relativity agree with each other[20, 21, 22].

The basic purpose of this paper is to obtain the total energy in the gamma metric by using the energy-momentum expression of Møller in teleparallel gravity. We will proceed according to the following scheme. In section 2, we give the energy and momentum

definition of Møller in teleparallel gravity and calculate the total energy for given space-time. Finally, we summarize and discuss our results. Throughout this paper, the Latin indices (i, j, \dots) represent the vector number, and the Greek ones (μ, ν, \dots) represent the vector components; all indices run from 0 to 3. We use units where $G = 1, c = 1$.

2. Møller's Energy in the Teleparallel Gravity

The teleparallel theory of gravity (tetrad theory of gravitation) is an alternative approach to gravitation and corresponds to a gauge theory for the translation group based on Weitzenböck geometry[23]. In the theory of teleparallel gravity, gravitation is attributed to torsion[24], which plays the role of a force[25], and the curvature tensor vanishes identically. The essential field is acted by a nontrivial tetrad field, which gives rise to the metric as a by-product. The translational gauge potentials appear as the nontrivial item of the tetrad field, so induces on space-time a teleparallel structure which is directly related to the presence of the gravitational field. The interesting place of teleparallel theory is that, due to its gauge structure, it can reveal a more appropriate approach to consider some specific problem. This is the situation, for example, in the energy and momentum problem, which becomes more transparent.

Møller modified general relativity by constructing a new field theory in teleparallel space. The aim of this theory was to overcome the problem of the energy-momentum complex that appears in Riemannian Space[26]. The field equations in this new theory were derived from a Lagrangian which is not invariant under local tetrad rotation. Saez[27] generalized Møller theory into a scalar tetrad theory of gravitation. Meyer[28] showed that Møller theory is a special case of Poincare gauge theory[29, 30].

In teleparallel gravity, the super-potential of Møller is given by Mikhail *et al.*[?] as

$$U_{\mu}^{\nu\beta} = \frac{(-g)^{1/2}}{2\kappa} P_{\chi\rho\sigma}^{\tau\nu\beta} [\Phi^{\rho} g^{\sigma\chi} g_{\mu\tau} - \lambda g_{\tau\mu} \xi^{\chi\rho\sigma} - (1 - 2\lambda) g_{\tau\mu} \xi^{\sigma\rho\chi}] \quad (4)$$

where $\xi_{\alpha\beta\mu} = h_{i\alpha} h^i{}_{\beta;\mu}$ is the con-torsion tensor and $h_i{}^{\mu}$ is the tetrad field and defined uniquely by $g^{\alpha\beta} = h_i^{\alpha} h_j^{\beta} \eta^{ij}$ (here η^{ij} is the Minkowski space-time). κ is the Einstein constant and λ is free-dimensionless coupling parameter of teleparallel gravity. For the teleparallel equivalent of general relativity, there is a specific choice of this constant.

Φ_{ρ} is the basic vector field given by

$$\Phi_{\mu} \doteq \xi^{\rho}{}_{\mu\rho} \quad (5)$$

and $P_{\chi\rho\sigma}^{\tau\nu\beta}$ can be found by

$$P_{\chi\rho\sigma}^{\tau\nu\beta} = \delta_{\chi}^{\tau} g_{\rho\sigma}^{\nu\beta} + \delta_{\rho}^{\tau} g_{\sigma\chi}^{\nu\beta} - \delta_{\sigma}^{\tau} g_{\chi\rho}^{\nu\beta} \quad (6)$$

with $g_{\rho\sigma}^{\nu\beta}$ being a tensor defined by

$$g_{\rho\sigma}^{\nu\beta} = \delta_{\rho}^{\nu} \delta_{\sigma}^{\beta} - \delta_{\sigma}^{\nu} \delta_{\rho}^{\beta}. \quad (7)$$

The energy-momentum density is defined by

$$\Xi_{\alpha}^{\beta} = U_{\alpha,\lambda}^{\beta\lambda} \quad (8)$$

where comma denotes ordinary differentiation. The energy is expressed by the surface integral;

$$E = \lim_{r \rightarrow \infty} \int_{r=\text{constant}} U_0^{0\zeta} \eta_\zeta dS \quad (9)$$

where η_ζ (with $\zeta = 1, 2, 3$) is the unit three-vector normal to surface element dS .

The general form of the tetrad, h_i^μ , having spherical symmetry was given by Robertson[31]. In the Cartesian form it can be written as

$$\begin{aligned} h_0^0 &= iA, & h_a^0 &= Cx^a, & h_0^\alpha &= iDx^\alpha, \\ h_a^\alpha &= B\delta_a^\alpha + Ex^\alpha x^\alpha + \epsilon_{\alpha\beta} Fx^\beta \end{aligned} \quad (10)$$

where A, B, C, D, E , and F are functions of t and $r = \sqrt{x^\alpha x^\alpha}$, and the zeroth vector h_0^μ has the factor $i^2 = -1$ to preserve Lorentz signature§.

Using the general coordinate transformation

$$h_{a\mu} = \frac{\partial \mathbf{X}^\nu}{\partial \mathbf{X}^\mu} h_{a\nu} \quad (11)$$

where \mathbf{X}^ν and \mathbf{X}^μ are, respectively, the isotropic and Schwarzschild coordinates (t, r, θ, ϕ) , we obtain the tetrad components of h_a^μ as:

$$\left(1 - \frac{2m}{r}\right)^{\frac{\gamma}{2}} \begin{pmatrix} i(1 - \frac{2m}{r})^{-\gamma} & 0 & 0 & 0 \\ 0 & (\frac{\Omega}{\Xi})^{\frac{1}{2} - \frac{\gamma^2}{2}} s\theta c\phi & \frac{\Omega^{-\frac{\gamma^2}{2}}}{\Xi^{\frac{1-\gamma^2}{2}}} c\theta c\phi & -\frac{s\phi}{\sqrt{\Omega}s\theta} \\ 0 & (\frac{\Omega}{\Xi})^{\frac{1}{2} - \frac{\gamma^2}{2}} s\theta s\phi & \frac{\Omega^{-\frac{\gamma^2}{2}}}{\Xi^{\frac{1-\gamma^2}{2}}} c\theta s\phi & \frac{c\phi}{\sqrt{\Omega}s\theta} \\ 0 & (\frac{\Omega}{\Xi})^{\frac{1}{2} - \frac{\gamma^2}{2}} c\theta & -\frac{\Omega^{-\frac{\gamma^2}{2}}}{\Xi^{\frac{1-\gamma^2}{2}}} s\theta & 0 \end{pmatrix} \quad (12)$$

where we have introduced the following notation: $s\theta = \sin \theta$, $c\theta = \cos \theta$, $s\phi = \sin \phi$ and $c\phi = \cos \phi$. For the gamma metric, $g_{\mu\nu}$ is defined by

$$\begin{pmatrix} (1 - \frac{2m}{r})^\gamma & 0 & 0 & 0 \\ 0 & -(1 - \frac{2m}{r})^{-\gamma} (\frac{\Omega}{\Xi})^{\gamma^2 - 1} & 0 & 0 \\ 0 & 0 & -(1 - \frac{2m}{r})^{-\gamma} \frac{\Omega^{\gamma^2}}{\Xi^{\gamma^2 - 1}} & 0 \\ 0 & 0 & 0 & -(1 - \frac{2m}{r})^{-\gamma} \Omega \sin^2 \theta \end{pmatrix} \quad (13)$$

and its inverse $g^{\mu\nu}$

$$\begin{pmatrix} (1 - \frac{2m}{r})^{-\gamma} & 0 & 0 & 0 \\ 0 & -(1 - \frac{2m}{r})^\gamma (\frac{\Omega}{\Xi})^{1 - \gamma^2} & 0 & 0 \\ 0 & 0 & -(1 - \frac{2m}{r})^\gamma \frac{\Omega^{-\gamma^2}}{\Xi^{1 - \gamma^2}} & 0 \\ 0 & 0 & 0 & -(1 - \frac{2m}{r})^\gamma \frac{1}{\Omega \sin^2 \theta} \end{pmatrix} \quad (14)$$

Next, in the case, the required non-vanishing component of $U_\mu^{\nu\beta}$ is[32, 33]

$$U_0^{01} = \frac{2m\gamma}{\kappa} \sin \theta \quad (15)$$

§ The tetrad of Minkowski space-time is $h_a^\mu = \text{diag}(i, \delta_a^\alpha)$ where (a=1,2,3)

From this point of view, using equation (9) with this result the energy associated with the gamma metric is found as:

$$E = m\gamma \tag{16}$$

It is the same energy as obtained in general relativity by Virbhadra and Yang-Radinschi.

3. Discussion

We evaluated the energy distribution associated with the gamma metric using the teleparallel gravity version of Møller's energy-momentum definition. The energy was found that it depends on the mass m . Therefore, we got the same result as Virbhadra[3] has obtained using the energy-momentum complex of Weinberg and Yang-Radinschi[4] have obtained using the general relativity version of the Møller energy-momentum complex. The energy distribution is also independent of the teleparallel dimensionless coupling constant, which means that it is valid not only in the teleparallel equivalent of general relativity, but also in any teleparallel model.

Further, this paper sustains (a) the importance of the energy-momentum definitions in the evaluation of the energy distribution of a given space-time, (b) the opinion that different energy-momentum expressions definitions could give the same result in a given space-time and (c) the Møller energy-momentum definition allows to make calculations in any coordinate system.

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