Reliability-Based System-Level Optimization of Bridge Maintenance and Replacement Decisions

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This paper addresses the problem of optimizing bridge maintenance and replacement (M&R) decisions for a heterogeneous system of facilities. The objective is to determine optimal M&R policies for each facility over a finite planning horizon based on the knowledge of the current facilities conditions and on the prediction of future conditions.

The system-level problem is based on the results of an M&R optimization problem for each facility. The results of the facility-level optimization are incorporated in a reliability-based, bottom-up, system-level formulation that provides recommendations for each individual facility. We derive sufficient conditions for optimality and prove the result for the continuous case. A parametric study shows that the results obtained in the discrete-case implementation of the solution are valid approximations of the continuous case results. The computational efficiency of the system-level solution makes the formulation suitable for systems of realistic sizes.

Key words: reliability; optimization; infrastructure management

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1. Introduction

This paper addresses the optimization of bridge maintenance and replacement (M&R) decisions for a heterogeneous system of facilities. The objective of bridge management optimization models is to determine optimal M&R decisions based on the knowledge of the current condition of the system and on the prediction of future conditions.

The determination of optimal policies for a system of facilities cannot be reduced to the determination of optimal facility-level policies, repeated independently for each facility. At the system level, the facility-level decisions are interdependent due to the presence of a budget constraint that introduces trade-offs between facilities.

A system-level optimization model of infrastructure M&R can be designed using two different approaches: top-down or bottom-up. The former approach allows for systematic optimization of maintenance activities but makes simplifying assumptions about bridge component-deterioration behavior. Moreover, the top-down approach is limited to homogeneous systems, i.e., systems consisting of similar facilities. Examples of the top-down approach include Golabi and Shepard (1997); Kuhn and Madanat (2005); and Madanat, Park, and Kuhn (2006).

At the other end, the bottom-up approach allows for comprehensive representation of bridge deterioration but has never been extended to problems of realistic size and scope, primarily because the computational complexity associated with solving at the system level for dissimilar facilities is daunting. Examples include the reliability-based models of Mori and Ellingwood (1994); Chung, Manuel, and Frank (2003); and Estes and Frangopol (2001) at the facility level and Kong and Frangopol (2003) at the system level.

As such, existing system-level optimization methods provide either a detailed optimization of maintenance activities using unrealistic assumptions about bridge performance or realistic models of the facilities with limited optimization of maintenance activities. Our approach, described in the present paper, solves the system-level problem for realistic problem sizes and can accommodate any model of bridge component deterioration, including physical reliability-based models favored by bridge engineers.

The system-level problem is formulated as follows: The M&R optimization is first carried out at the facility level independently for each facility; the results are then used to solve the system-level problem, which consists of minimizing the probability of failure subject to a budget constraint. The facility-level problem is only one step toward solving the system-level problem. The former problem is not budget constrained; its solution is an optimal maintenance and repair policy for each facility given a maximum allowable
probability of failure \( p \). The latter problem, the solution to which is the main contribution of this paper, is a resource-allocation problem. In other words, what are the optimal values of the \( p_i \)s that minimize overall system risk given a budget constraint?

2. Problem Definition and Formulation

2.1. Definitions and Assumptions

The system considered in the present article is a system of facilities managed by a single agency such as a state department of transportation. Although the condition of each bridge in the system changes over time, the system remains constant over the planning horizon: bridges are neither added to the system nor decommissioned. The system can be composed of heterogeneous bridges. Maintenance and replacement decisions are made every year for every facility.

The condition of a facility is measured by its instantaneous probability of failure, defined at time \( t \) as \( P(t \leq T_f < t + \delta | T_f \geq t) \), where \( T_f \) is the time of failure and \( \delta \) is a time interval equal to one year. Failure can be defined differently by different authorities. For example, according to the Colorado Department of Transportation, deck failure corresponds to the situation when active corrosion is underway in at least 50% of the deck (Estes and Frangopol 2000). The specific definition of failure that is used is not important as long as the same definition applies to all facilities in the system.

2.2. Formulation of the System-Level Problem

The system-level problem is based on the results of a facility-level problem for each facility. The facility-level problem is solved independently for each facility. The specific facility-level formulation that is used is not important as long as its results provide a function \( f_i(p) \) for each facility \( i \), where \( f_i(p) \) is the present value of the optimal maintenance and replacement cost of facility \( i \) over the planning horizon if the probability of failure of that facility is kept below \( p \). Note that in the facility-level problem, \( p \) is defined by the user and will be referred to as threshold of probability of failure.

The set of decision variables for the system-level problem is the set \( (p_1, \ldots, p_n) \) of thresholds of probability of failure for each facility, where \( n \) is the number of facilities. The optimization consists of minimizing a quantity \( J(p_1, \ldots, p_n) \) expressing the performance in terms of probability of failure or risk, subject to a budget constraint. In the context of reliability, it is relevant to choose \( J(p_1, \ldots, p_n) \) to be the highest threshold of probability of failure among all facilities, denoted by

\[
J(p_1, \ldots, p_n) = \max\{p_1, \ldots, p_n\}. \tag{1}
\]

Another expression of performance in terms of risk could be the average threshold of probability of failure over all facilities. However, this expression is not suitable in the context of reliability-based optimization because the following unsafe situation would be possible: the majority of the facilities with low probabilities of failure and a few facilities with probabilities of failure close to one. The formulation of the system-level optimization is therefore

\[
\min_{(p_1, \ldots, p_n) \in [0, 1]^n} \max\{p_1, \ldots, p_n\}
\]

subject to

\[
\sum_{i=1}^{n} f_i(p_i) \leq B, \tag{2}
\]

where \( f_i(p_i) \) is the optimal cost of maintenance and replacement for facility \( i \in [1, \ldots, n] \) determined using a facility-level optimization method, and \( B \) is the present value of a multiyear available budget. Most authorities have yearly budget constraints; however, it is not uncommon that they are allowed to carry over funds that were not spent in a given year to subsequent years.

2.3. Additional Assumptions

It is easy to show that for each facility \( i \), the function \( f_i(p) \) is nonincreasing. Intuitively, if larger probabilities of failure are allowed for a facility (i.e., larger values of \( p_i \)), fewer or less expensive M&R actions are required, thus leading to a lower cost \( f_i(p_i) \). More formally, if \( p_i < p_i' \), then the set of feasible actions for the threshold of probability of failure \( p_i \) is included in the set of feasible actions for the threshold of probability of failure \( p_i' \). Therefore, decreasing the threshold \( p_i \) of the probability of failure for each facility increases the total cost and the objective of decreasing the \( p_i \)s is constrained by the budget.

It is assumed that \( f_i(p) \) is defined for an interval of \( p_i \in [p_i^{\min}, 1] \), where \( p_i^{\min} \) is the lowest probability of failure that can be achieved (most likely by performing the most severe action on the facility at every time period). The assumption that \( f_i(p) \) is defined for an interval cannot be achieved in practice because it would require solving an extremely large number of facility-level problems. However, because the functions \( f_i \) are nonincreasing, they can be estimated accurately by interpolation between a small number of known points.

Each function \( f_i \) is piecewise constant for the following three reasons: (i) it is defined over an interval \( [p_i^{\min}, 1] \), (ii) it is nonincreasing over this interval, and (iii) it can only take a finite number of different values (each value corresponds to a combination of M&R actions taken from a finite set of available actions). Therefore, each function \( f_i \) is piecewise continuous. The size of the discontinuities of the functions \( f_i \) decreases as the number of available actions and the length of the planning horizon increase.

Moreover, the system considered is heterogenous,
i.e., composed of different facilities. For two different facilities $i$ and $j$, the set of $p_i$, where $f_i$ has discontinuities, is different from the set of $p_j$, where $f_j$ has discontinuities. This contributes to making $\sum_{i=1}^{n} f_i$ “smooth” enough (i.e., the discontinuities are small relative to the values of the function) so that approximating it by a continuous function $F$ is unlikely to lead to large errors. In the rest of the paper, the functions $F$ and $\sum_{i=1}^{n} f_i$ will be used interchangeably.

Finally, the function $F$ is defined over the interval $[p_{n}^{\text{min}}, 1]$, where $p_{n}^{\text{min}} = \max[p_{1}^{\text{min}}, \ldots, p_{n}^{\text{min}}]$.

### 3. Continuous Case Solution

The solution is first determined in the case where the facility-level optimal cost $f_i(p_i)$ is defined for an interval of $p_i$ for each facility $i \in \{1, \ldots, n\}$. This assumption will be relaxed in §4 of the paper. Moreover, the function $p \mapsto \sum_{i=1}^{n} f_i(p)$ is assumed to be continuous. The validity of this assumption was discussed in §2.3.

#### 3.1. Optimal Solution

**3.1.1. Notation.** For any integer $n$, the notation $\{1, \ldots, n\}$ represents the set of integers between 1 and $n$, and $n$ being included. The set $\mathbb{R}^+$ is the set of nonnegative real numbers. For any set $E \subset \mathbb{R}$, $E^+$ indicates the set of $n$-tuples of elements of $E$. For any $(p_1, \ldots, p_n) \in \mathbb{R}^n$, the notation $\max[p_1, \ldots, p_n]$ represents the largest element of the $n$-tuple $(p_1, \ldots, p_n)$. For any real numbers $a$ and $b$ such that $a \leq b$, the set of real numbers $x$ such that $a \leq x \leq b$ is denoted by $[a, b]$.

**3.1.2. Optimality Conditions.** Let us define $B_{n}^{\text{min}}$ as the minimum present value of a multiyear budget available so that the threshold of probability of failure of all facilities is strictly less than one. Let us define $B_{n}^{\text{max}}$ as $\sum_{i=1}^{n} f_i(p_{n}^{\text{min}})$, with $p_{n}^{\text{min}}$ as defined at the end of §2.3.

For any $B \in [B_{n}^{\text{min}}, B_{n}^{\text{max}}]$, if the following conditions are met:

$$\forall i \in \{1, \ldots, n\}, \quad p_i = \min \left\{ p \in [p_{n}^{\text{min}}, 1] : \sum_{i=1}^{n} f_i(p) \leq B \right\},$$

(3)

then the tuple $(p_1, \ldots, p_n)$ of thresholds of probability of failure of $n$ heterogeneous facilities is an optimal solution of Equation (2).

In practice, $B_{n}^{\text{min}}$ and $B_{n}^{\text{max}}$ are extreme values and the present value $B$ of a multiyear budget, which is defined by the manager or by some other authority, will always be between $B_{n}^{\text{min}}$ and $B_{n}^{\text{max}}$.

#### 3.2. Intuition for the Proof of Optimality

The second condition in Equation (3), which indicates that the entire budget is used at optimality, is intuitive. To obtain an intuitive explanation for the first condition, which indicates that all $p_i$s are equal at optimality, let us consider a system of two facilities and the following situation: the threshold of probability of failure of facility 1, $p_1$, is strictly lower than that of facility 2, $p_2$, as shown in Figure 1, and the combination of decision variables $(p_1, p_2)$ is feasible (i.e., $f_1(p_1) + f_2(p_2) \leq B$).

It is easy to show that this combination cannot be optimal. Because $f_1(p_1)$ decreases as $p_1$ is increased and $f_2(p_2)$ increases as $p_2$ is decreased, it is possible to decrease $p_2$ by a small amount $\delta_2$ and increase $p_1$ by $\delta_1$ so as to keep $f_1(p_1) + f_2(p_2) = f_1(p_1 + \delta_1) + f_2(p_2 - \delta_2)$. Thus, the new combination $(p_1 + \delta_1, p_2 - \delta_2)$ is feasible and the objective function is improved. This gives an intuitive explanation as to why $p_1$ and $p_2$ must be equal at optimality.

#### 3.3. Proof of Optimality Results

This section provides the proof that a tuple $(p_1, \ldots, p_n)$ satisfying conditions (3) exists, as well as the proof that conditions (3) are sufficient conditions of optimality for problem (2). The assumptions are presented in §§2.1 and 2.3.

**3.3.1. Notation for the Proofs.** Let $F$ be the function $p \mapsto \sum_{i=1}^{n} f_i(p)$ and $I_B$ be the set $\{p \in [p_{n}^{\text{min}}, 1] : F(p) \leq B \}$.

**3.3.2. Proof of Existence of a Tuple Satisfying Conditions (3).** The set $I_B$ is the preimage of the compact set $[0, B]$ by the function $F$, which is assumed to be continuous. Therefore, the set $I_B$ is compact. Moreover, $F(1) = B_{n}^{\text{max}}$ is assumed to be less than $B$, which means that one is in the set $I_B$ and the set $I_B$ is not empty.

Therefore, the set $I_B$ has a minimum and there exists a tuple $(p_1, \ldots, p_n)$ satisfying conditions (3). Q.E.D.

**3.3.3. Proof of Sufficiency Conditions.** We will consider a tuple $(p_1, \ldots, p_n)$ satisfying the sufficiency conditions (3), i.e., such that $p_i = p^*$ for all $i \in \{1, \ldots, n\}$, where $p^* = \min I_B$.

Let us show that the tuple $(p_1, \ldots, p_n)$ is an optimal solution, i.e., if a tuple $(q_1, \ldots, q_n) \in [p_{n}^{\text{min}}, 1]^n$ is such that

$$\max[q_1, \ldots, q_n] < p^* = \max[p_1, \ldots, p_n],$$

then it is not a feasible solution of problem (2).
Let \( q = \max\{q_1, \ldots, q_n\} \) and suppose \( q < p^* \).

Because \( q_i \leq q \) and \( f_i \) is nondecreasing for all \( i \in \{1, \ldots, n\} \), then \( f_i(q_i) \geq f_i(q) \), and we have

\[
\sum_{i=1}^{n} f_i(q_i) \geq \sum_{i=1}^{n} f_i(q) = F(q).
\]  

(5)

Because \( q < p^* \), by definition of \( p^* = \min I_B \), \( q \) is not in the set \( I_B \) and \( F(q) > B \). From Equation (5), we have

\[
\sum_{i=1}^{n} f_i(q_i) > B,
\]

(6)

which shows that the tuple \( (q_1, \ldots, q_n) \) is not a feasible solution of problem (2). Q.E.D.

4. Discrete Case Implementation

4.1. Facility-Level Optimization

The system-level optimization presented in this section is based on the results of the optimization of maintenance and replacement decisions at the facility level as described in Robelin and Madanat (2007). We consider bridge decks because the deck is the bridge component that deteriorates fastest.

The facility-level problem uses a Markovian deterioration model that accounts for aspects of the history of deterioration and maintenance. These aspects include the amount of time since the performance of the latest maintenance action and the type of action that was performed. As required for the system-level optimization, the facility-level optimization provides the optimal cost of maintaining a facility as a function of the threshold of probability of failure for each facility.

4.2. Empirical Evidence of the Validity of the Discrete Case Results

In practice, the values \( f_i(p_i) \) cannot be determined for an interval of \( p_i \) because this would require solving the facility-level optimization problem a very large number of times for each facility. The values \( f_i(p_i) \) can therefore be determined only for a finite set of values of \( p_i \) for each facility. Here, we aim to show empirically that the results of the discrete case implementation represent a good approximation of the continuous case solution.

The numerical application considers a heterogeneous system of 742 bridge decks. The deterioration of each bridge deck is modeled according to Frangopol, Kong, and Gharaiibe (2001), and maintenance and replacement cost information is adapted from Kong and Frangopol (2003). A heterogeneous system of bridge decks is created by changing the parameters provided in these papers within realistic ranges. The facility-level optimization problem is solved for each facility for various values of the threshold of probability of failure. The condition of each facility is measured by its reliability index \( \beta = -\Phi^{-1}(p) \), where \( \Phi \) is the cumulative standard normal distribution and \( p \) is the probability of failure of the facility. The condition of the facility is discretized and various discretization stepsizes are tried.

The results for the system of 742 bridges are shown in Figure 2. For each of the three discretization stepsizes, the optimization problem is solved for different values of the threshold \( p \) of probability of failure (two different values with stepsize 2, four different values with stepsize 1, and seven different values with stepsize 0.5). For a given discretization stepsize, it can be noted that the sum of the facility-level optimal costs decreases as the threshold \( p \) of probability of failure increases, which is intuitive and was one of the assumptions made in the continuous case. Moreover, the variation of the optimal cost with respect to \( p \) is relatively smooth; therefore, it is appropriate to interpolate between the results for different values of \( p \), and the continuity assumption made in the continuous case is likely to be verified.

The graph also shows that for a given value of \( p \), the sum of the facility-level optimal costs decreases as the discretization step size decreases. This is intuitive because the model becomes finer as the stepsize decreases, thus allowing for improvements in the optimization. It can also be noted that for any given value of \( p \), the difference in cost between stepsize 1 and stepsize 0.5 is much smaller than the difference between stepsize 2 and stepsize 1, which suggests that the results “converge” as the stepsize decreases to zero.

These results, based on a system of realistic size, provide empirical evidence that the results found in the discrete case implementation can confidently be considered valid approximations of the results in the continuous case.
4.3. Implementation and Derivation of Facility-Level Policies

The facility-level optimization problem is first solved for each facility for different values of the threshold $p$ of probability of failure (or, equivalently, for different values of the reliability index). This provides the optimal cost $f_i(p)$ for a discrete set of values of $p$ for each facility. After summing these values over all facilities and interpolating, the function $F$ defined as follows is known:

$$\forall p \in [p_{\min}, 1], \quad F(p) = \sum_{i=1}^{n} f_i(p). \quad (7)$$

Given a user-defined value $B$ of the budget, the optimal value $p^*$ of the threshold of probability of failure can be expressed as

$$p^* = \min\{p: F(p) \leq B\}. \quad (8)$$

For each facility, solving the facility-level optimization with the threshold of probability of failure taken as $p^*$ provides a set of maintenance and replacement policies. This set of policies is optimal at the system level. The implementation procedure is described in Figure 3.

4.4. Computational Complexity

The solution is such that the computational complexity of the system-level problem is low. Namely, the fact that the threshold of probability of failure is the same for all facilities at the system-level optimum reduces the optimization problem for $n$ facilities to $n$ independent facility-level problems. Therefore, the complexity is proportional to the number of facilities in the system, i.e., $O(n)$. For each facility, the facility-level optimization problem is solved a small number of times as seen earlier. Because the time to solve one facility-level problem is of the order of a few seconds on a personal computer, typical values of computation times for the system-level problem are five hours per thousand facilities. These computation times indicate that the present system-level optimization method can be applied to systems the size of that managed by a state department of transportation. For example, if applied to the system managed by Caltrans, composed of 24,000 bridges (California Department of Transportation 2006), the optimization would require a computation time of approximately five days, which is very short compared with the time scale of maintenance and replacement decisions.

5. Contribution

This paper addresses the determination of the optimal maintenance and replacement policies for a system of bridges. The results of the facility-level optimization are incorporated in a reliability-based, bottom-up, system-level formulation with the following characteristics:

- the system can be composed of dissimilar facilities; as opposed to top-down approaches, recommendations are provided for each individual facility;
- any facility-level optimization model can be accommodated as long as it provides the optimal cost of maintaining a facility as a function of a threshold of probability of failure of the facility;
- the computational efficiency of the system-level solution makes the formulation suitable for systems of realistic sizes; and
- the solution is proven to be optimal in the continuous case and a parametric study shows that the results obtained in the discrete case implementation of the solution are valid approximations of the continuous case results.

The reliability-based formulation for a system of facilities presented in this article is likely to be applicable to other systems: fleets of vehicles, other civil infrastructure systems, assembly lines, etc.

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