New results on the eccentric digraphs of the digraphs

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Abstract Let $G$ be a digraph. For two vertices $u$ and $v$ in $G$, the distance $d(u, v)$ from $u$ to $v$ in $G$ is the length of the shortest directed path from $u$ to $v$. The eccentricity $e(v)$ of $v$ is the maximum distance of $v$ to any other vertex of $G$. A vertex $u$ is an eccentric vertex of $v$ if the distance from $v$ to $u$ is equal to the eccentricity of $v$. The eccentric digraph $ED(G)$ of $G$ is the digraph that has the same vertex set as $G$ and the arc set defined by: there is an arc from $u$ to $v$ if and only if $v$ is an eccentric vertex of $u$. In this paper, we determine the eccentric digraphs of digraphs for various families of digraphs and we get some new results on the eccentric digraphs of the digraphs.

Keywords Eccentricity; Eccentric vertex; Distance; Directed graph

1. Introduction

Let $G$ be a digraph with vertex set $V(G)$ and arc set $A(G)$. For two vertices $u$ and $v$ in $G$, if there is a directed path from $u$ to $v$, then we say that $v$ is reachable from $u$ and the distance $d(u, v)$ from $u$ to $v$ is the length of the shortest directed path from $u$ to $v$. If there is no directed path from $u$ to $v$ in $G$, then we define $d(u, v) = \infty$. The eccentricity $e(v)$ of $v$ in $G$, is the distance from $v$ to a vertex farthest from $v$. A vertex $u$ in $G$ is an eccentric vertex of vertex $v$ if the distance from $v$ to $u$ is equal to $e(v)$. The eccentric digraph of
$G$, denoted $ED(G)$, is the digraph on the same vertex set as $G$, in which there is an arc from $v$ to $u$ if and only if $u$ is an eccentric vertex of $v$.

Given a positive integer $k \geq 1$, the $k$th iterated eccentric digraph of $G$ is defined as $ED^k(G) = ED(ED^{k-1}(G))$ where $ED^1(G) = ED(G)$ and $ED^0(G) = G$. Since the number of the digraphs on $n$ vertices is finite, there is a positive integer $p$ and a non-negative integer $k$ such that $ED^k(G) = ED^{p+k}(G)$. The smallest $p$ and $t$, which make the equality hold, are called the period and the tail of $G$ respectively. The period and tail of $G$ are denoted by $p(G)$ and $t(G)$ respectively. We say that a graph is periodic if $t(G) = 0$.

Besides, we define the following digraphs in this paper.

The directed path $P_n = v_1v_2\ldots v_n$ is a directed graph with vertex set $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and arc set $A(P_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\}$.

The directed cycle $C_n = v_1v_2\ldots v_nv_1$ is a directed graph with vertex set $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and arc set $A(C_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$.

The in-directed fan $F_n^i$ is the digraph with vertex set $V(F_n^i) = \{c, v_1, v_2, \ldots, v_n\}$ and arc set $A(F_n^i) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\} \cup \{v_1c, \ldots, v_nc\}$.

The out-directed fan $F_n^o$ is the digraph with vertex set $V(F_n^o) = \{c, v_1, v_2, \ldots, v_n\}$ and arc set $A(F_n^o) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\} \cup \{cv_1, \ldots, cv_n\}$.

Let $F_n^* \, = \, \{c, v_1, v_2, \ldots, v_n\}$ and arc set $A(F_n^*) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\} \cup \{cv_1, \ldots, cv_n\} \cup \{v_1c, \ldots, v_nc\}$.

The out-directed wheel $W_n^o$ is the digraph with vertex set $V(W_n^o) = \{c, v_1, v_2, \ldots, v_n\}$ and arc set $A(W_n^o) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\} \cup \{cv_1, \ldots, cv_n\}$.

The in-directed wheel $W_n^i$ is the digraph with vertex set $V(W_n^i) = \{c, v_1, v_2, \ldots, v_n\}$ and arc set $A(W_n^i) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\} \cup \{v_1c, \ldots, v_nv_1\}$.

For a graph $G$, $G^*$ is the digraph obtained from $G$ by replacing each edge of $G$ by a symmetric pair of arcs.

For two vertex disjoint digraphs $G_1$ and $G_2$, $G_1 \oplus G_2$ is the digraph obtained by joining each vertex of $G_1$ to each vertex of $G_2$.

The complement of a digraph $G$ with $n$ vertices is the digraph $(K_n)^* - A(G)$, denoted $\overline{G}$.

In [1], Bolland and Miller introduced the concept of the eccentric digraph of a digraph and obtained some useful results as follow.

**Proposition 1.1** For the complete digraph $(K_n)^*$, $ED((K_n)^*) = (K_n)^*$.

**Proposition 1.2** For the complete multipartite digraph $G$, $ED^2(G) = G$.

**Proposition 1.3** For a directed cycle $C_n$, $ED(C_n) = C_n$.

Note that the direction of any arc in $ED(C_n)$ is opposite to that in $C_n$.

**Proposition 1.4** A non-trivial eccentric digraph has no vertex of out-degree zero.

**Proposition 1.5**

1. $p = 1, t = 0$ if and only if $G = K_n$.
2. $p = 1, t = 1$ if and only if $G = K_n^r$.
3. $p = 2, t = 0$ when $G = K_{n_1, n_2, \ldots, n_k}$ or $G = K_{n_1} \cup K_{n_2} \cup \ldots \cup K_{n_k}$. 

In [1], Bolland and Miller introduced the concept of the eccentric digraph of a digraph and obtained some useful results as follow.
4. $p = 2, t = 1$ when $G = H_{n_1, n_2, \ldots, n_k}$ or $G = H_{n_1} \cup H_{n_2} \cup \ldots \cup H_{n_k}$
where $H_{n_1, n_2, \ldots, n_k}$ is a strongly connected subdigraph of $K_{n_1, n_2, \ldots, n_k}$ of order $n_1 + n_2 + \ldots n_k$.

$H_{n_1} \cup H_{n_2} \cup \ldots \cup H_{n_k}$ is a strongly connected subdigraph of $K_{n_1} \cup K_{n_2} \cup \ldots \cup K_{n_k}$ of order $n_1 + n_2 + \ldots n_k$.

**Proposition 1.6** Let $G$ be a digraph with $|V(G)| = n$ and no vertex of out-degree 0. Then $G$ has a vertex of out-degree $n - 1$ if and only if $ED(G)$ has a vertex of out-degree $n - 1$.

In this paper, we have obtained some results on the eccentric digraphs of the digraphs.

2. New results

**Lemma 2.1** The eccentric digraph of a directed path $P_n$ is a directed graph $G$, where $V(G) = V(P_n)$ and $A(G) = \{v_i,v_j : i > j, i,j = 1,2,\ldots,n\}$.

**Lemma 2.2** $ED((K_m \cup K_n)^*) = (K_m,n)^*$ and $ED((K_{m,n})^*) = (K_m \cup K_n)^*$. So $t((K_m \cup K_n)^*) = p((K_m \cup K_n)^*) = 1$.

**Lemma 2.3** The digraph $G_1$ in Figure 1 satisfies that $ED(G_1) = K_1 \oplus (K_n)^*$ and $ED^2(G_1) = G_1$. So $t(G_1) = 0$, $p(G_1) = 2$, i.e. $G_1$ is periodic.

**Proof:** Suppose $V(G_1) = \{v_1,v_2,\ldots,v_n,c\}$ and $c$ is the vertex of in-degree $n$ and out-degree $n$. Since $e(c) = 1$ then the other vertex $v_i$ is the eccentric vertex of $c$ for any $i = 1,2,\ldots,n$. Since $\text{indegree}(c) = \text{outdegree}(c) = n$ then $e(v_i) = 2$ for any $i = 1,2,\ldots,n$. Thus, $v_i$ is the eccentric vertex of $v_j$ if $i \neq j$ and $i,j = 1,2,\ldots,n - 1$. So $ED(G_1) = (K_1 \oplus (K_n)^*)$. Furthermore, since $ED(K_1 \oplus (K_n)^*) = G_1$ then $ED^2(G_1) = G_1$. □

**Lemma 2.4** Let $G = K_1 \oplus (K_n)^*$, then $ED(G) = G_1$ and $ED^2(G) = G$. So $t(G) = 0$, $p(G) = 2$, i.e. $G$ is periodic, where $G_1$ is the digraph in the Figure 1.

**Lemma 2.5** Let $G = rK_1 \oplus (K_n)^*$, then $ED^2(G) = G$. So $t(G) = 0$, $p(G) = 2$, i.e. $G$ is periodic, where $r$ is a positive integer.

**Lemma 2.6** The eccentric digraph of $F_n^1$ is the digraph in Figure 2. Furthermore, $ED^2(F_n^1) = G_1$ and $ED^2(F_n^1) = ED^1(F_n^1)$. So $t(F_n^1) = p(F_n^1) = 2$.

**Lemma 2.7** The eccentric digraph of $F_n^2$ is the digraph in Figure 3. Furthermore, $ED(F_n^2) = ED^3(F_n^2)$. So $t(F_n^2) = 1$, $p(F_n^2) = 2$.
Lemma 2.8 The eccentric digraph of $F_n^*$ is the digraph in Figure 4. Furthermore, $ED(F_n^*) = ED^3(F_n^*)$. So $t(F_n^*) = 1$, $p(F_n^*) = 2$.

Lemma 2.9 The eccentric digraph of $W_n^0$ is also the digraph $G_1$ in Figure 1. Furthermore, $ED(W_n^0) = ED^3(W_n^0)$. So $t(W_n^0) = 1$, $p(W_n^0) = 2$.

Lemma 2.10 The eccentric digraph of $W_n^1$ is the out-directed wheel $W_n^1$, while the direction of the rim of $W_n^1 = ED(W_n^1)$ is opposite to that in $W_n^1$ and it satisfies that $ED^1(W_n^1) = W_n^0$ and $ED^2(W_n^1) = ED^4(W_n^1)$. So $t(W_n^1) = p(W_n^1) = 2$.

Lemma 2.11 $ED(C_n) = C_n$.
Note that the direction of any arc in $ED(C_n)$ is the same to that in the given cycle $C_n$. By proposition 1.3, $ED^2(C_n) = C_n^*$, where the direction of any arc of $C_n$ is opposite to that in the directed cycle $C_n$.

Lemma 2.12 The eccentric digraph of the complement of $P_n$ satisfies that $ED(P_n) = F_{n-1}^*$ and $ED^2(P_n) = ED^4(P_n) = ED(F_{n-1}^*)$, where $v_n$ is the center of $F_{n-1}^*$. So $t(P_n) = p(P_n) = 2$.

Lemma 2.13 Let $rP_2$ be a digraph in the following Figure 5, then $ED(rP_2) = (K_{2r})^* - E(rP_2)$.

Lemma 2.14 Let the digraph $K_{m,n} - E(rP_2) = (mK_1 \oplus nK_1) - E(rP_2)$, then $ED(K_{m,n} - E(rP_2)) = ED^3(K_{m,n} - E(rP_2))$. So $t(K_{m,n} - E(rP_2)) = 1$, $p(K_{m,n} - E(rP_2)) = 2$, where $1 \leq r \leq \min\{m, n\}$.

Lemma 2.15 Let $S_{m,n}^i$ ($i = 1, 2$) be a directed double-star in the Figure 6 and Figure 7, then

(1) $ED^k(S_{m,n}^1) = \begin{cases} G_{1,1} & \text{if } k \text{ odd,} \\ G_{1,2} & \text{if } k \text{ even.} \end{cases}$

Note that $G_{1,1}$ is isomorphic to $G_{1,2}$ (See Figure 7 and Figure 8).

(2) $ED^3(S_{m,n}^2) = K_1 \oplus (K_{m+n+1})^*$ and $ED^2(S_{m,n}^2) = ED^4(S_{m,n}^2)$.

Theorem 2.1 Let $G$ be a digraph with $|V(G)| = n$. If there is one vertex of in-degree $n-1$ and out-degree 0, then the vertex has out-degree $n-1$ in the eccentric digraph $ED(G)$.

Proof: Let $V(G) = \{v_1, v_2, ..., v_{n-1}, c\}$ and $c$ be the vertex of in-degree $n-1$ and out-degree 0 in $G$. Then $e(c) = \infty$. Hence, every other vertex $v_i$ ($i = 1, 2, ..., n-1$) is the eccentric vertex of $c$. Thus, $c$ is a vertex of out-degree $n-1$ in $ED(G)$. □

Theorem 2.2 Let $G$ be a digraph with $|V(G)| = n + 1$. If there is one vertex of out-degree $n$ and in-degree 0 and others are reachable each other, then
$ED(G) = G_1$ and $ED(G) = ED^3(G)$.

**Proof:** Suppose that $V(G) = \{v_1, v_2, ..., v_n, c\}$ and $c$ is the vertex of out-degree $n$ and in-degree 0. Since $e(c) = 1$ then the other vertex $v_i$ is the eccentric vertex of $c$ for any $i = 1, 2, ..., n$. Since $\text{indegree}(c) = 0$, then $e(v_i) = \infty$ for $i = 1, 2, ..., n$. Since $v_i$ and $v_j$ are reachable for $i \neq j$, then $c$ is the only eccentric vertex of $v_i$ for any $i = 1, 2, ..., n$. From the above, we get that $ED(G) = G_1$. Furthermore, by lemma 2.3 we know that $ED^3(G) = ED(G)$. \qed

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{$G_1$}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{$ED(F^3_n)$}
\end{figure}
Figure 3 : \( ED(F_n) \)

Figure 4 : \( ED(F_n^*) \)

Figure 5 : \( rP_2 \)
Figure 6: $S^1_{m,n}$

Figure 7: $S^2_{m,n}$

Figure 8: $G_{1,1}$
Figure 9: $G_{1,2}$

References