Second-Order Consensus of Networked Thrust-Propelled Vehicles on Directed Graphs

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Abstract—This paper addresses the second-order consensus problem of multiple underactuated thrust-propelled vehicles (TPVs) on directed graphs containing a directed spanning tree. We propose an adaptive approach to resolve this problem, and the proposed control does not rely on the commonly used backstepping methodology and the exact knowledge of the mass properties of the TPVs. In addition, the resultant complexity is greatly reduced and the requirement on the interaction graph is relaxed. Using an input-output analysis, we show that the proposed control realizes second-order consensus of the TPVs, i.e., the position and velocity consensus errors between the TPVs converge to zero, and moreover the velocities of the TPVs converge to the weighted average of their initial values. The performance of the proposed control approach is shown by a numerical simulation.

Index Terms—Second-order consensus, under-actuation, directed graphs, networked thrust-propelled vehicles.

I. INTRODUCTION

Formation control of multiple thrust-propelled vehicles (TPVs) has been actively studied in recent years (see, e.g., [1], [2], [3], [4], [5], [6]). As is described in [7], [6], TPV is such an autonomous vehicle which is controlled by three independent torque inputs, yet by only one thrust force along certain body-fixed direction. The major challenge involved in the control of a single TPV lies in its under-actuation [8], [9], [10], [11], [7]. This under-actuation becomes much more difficult to handle in the consensus problem of multiple TPVs (consensus is generally recognized as one basic requirement for realizing the formation), and specifically, many second-order consensus (i.e., position and velocity consensus) algorithms for fully-actuated agents (e.g., [12], [13], [14], [15]) are no longer applicable to TPVs due to their under-actuation.

To address the under-actuation of TPVs, more complicated control schemes have been proposed for multiple TPVs (e.g., [1], [4], [5], [16], [2], [3], [6]), which can be grouped, in accordance with the interaction graphs among the TPVs, into two categories. The first category of schemes (e.g., [1], [4], [5]) achieves the consensus of multiple TPVs on undirected graphs, and the stability analysis of the network is performed by the potential-energy-based approach. The second category of schemes (e.g., [16], [2], [3], [17], [6]) achieves the consensus on directed graphs, and due to the well-recognized asymmetrical property of the Laplacian matrix associated with a directed graph, the potential-energy-based analysis approach is no longer applicable. Two common handling strategies for the under-actuation of TPVs are the thrust-force/desired-attitude extraction approach [5] and the dynamic control approach [7], [6] (i.e., the control law includes the output of a dynamic differential equation), additionally combined with the standard backstepping technique (as is typically done—see, e.g., [9], [10], [7]).

Although the backstepping approach is advantageous in many aspects (e.g., its flexibility for designing controllers for various systems and its facility for constructing a suitable Lyapunov function), however, it usually gives rise to relatively complex control laws due to the necessity of canceling the indefinite terms during backstepping [18]. This deficiency becomes especially prominent for the consensus problem of multiple TPVs [5], [6] and nonholonomic mobile robots [17].

In this paper, we propose a new approach to resolve the second-order consensus problem of multiple TPVs in SE(3) on directed graphs containing a directed spanning tree, without relying on the now commonly used backstepping methodology. This is achieved via considering the problem from an input-output perspective (inspired by [19] and [20]), with which the design for realizing the second-order consensus of the networked TPVs and the design for the desired thrust force and angular velocities are separated. The separation design approach here makes us able to derive an adaptive consensus control without the need of constructing a complicated Lyapunov function based on backstepping and canceling the indefinite terms and that of accurately knowing the physical parameters of the TPVs, in comparison to the one in [6]. In the resultant control law with the new design perspective, the thrust input is a dynamic control law that is an extension of the one for the tracking problem of a single TPV [7], yet benefiting from the input-output perspective, our approach gives a sufficient and necessary condition for realizing the second-order consensus and the graph is relaxed from balanced graphs considered in [6] to the more general directed graphs only with a directed spanning tree.

Some control schemes in the current literature that do not rely on the backstepping design approach are given in [16], [3], [21]. The input-transformation-based approach [16] is relatively complex and confined to the planar formation of identical kinematic mobile agents (i.e., the formation in SE(2) is considered and the control torque and force are not explicitly given). The control scheme in [3] achieves the formation in SE(2) where only the line tracking is realized. The scheme in [21], based on Jacobi shape coordinates, realizes the decomposition of the formation-shape/formation-center dynamics of the
agents, which, as compared with ours, also seems limited in that the control is centralized and applicable only to SE(2). In addition, the results in [16], [3], [21] do not take into account the possible parametric uncertainties of the system.

II. PRELIMINARIES

A. Graph Theory

Let us first introduce the theory of directed graphs [22], [23], [24] in the scenario that \( n \) TPVs are involved. As is now standard, we employ a directed graph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \) to describe the interaction topology among the \( n \) TPVs, where \( \mathcal{V} = \{1, 2, \ldots , n\} \) is the vertex set that denotes the collection of the \( n \) TPVs and \( \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V} \) is the edge set that denotes the information flow among the \( n \) TPVs. The neighbors of TPV \( i \) is denoted by a set \( N_i = \{j | (i, j) \in \mathcal{E} \} \). A directed graph is said to have a directed spanning tree if there exists a vertex \( k_0 \in \mathcal{V} \) such that any other vertex of the graph has a directed path to vertex \( k_0 \). The weighted adjacency matrix \( \mathcal{W} = [w_{ij}] \) associated with the graph \( \mathcal{G} \) is defined on the basis of the rule that \( w_{ij} > 0 \) if \( j \in N_i \), and \( w_{ij} = 0 \) otherwise. In addition, as is now typically done, we assume that \( w_{ii} = 0 \), \( \forall i = 1, 2, \ldots , n \). The Laplacian matrix \( \mathcal{L}_w = [\ell_{w,ij}] \) associated with the graph \( \mathcal{G} \) is defined as \( \ell_{w,ij} = \Sigma_{k=1}^n w_{ik} \) if \( i = j \), and \( \ell_{w,ij} = -w_{ij} \) otherwise. Several useful properties related to \( \mathcal{L}_w \) are given by the following lemma.

**Lemma 1** ([25], [26], [24]): Let the Laplacian matrix \( \mathcal{L}_w \) be associated with a directed graph containing a directed spanning tree. Then, \( \lambda_1 = 0 \) is a simple eigenvalue of \( \mathcal{L}_w \) and all other eigenvalues of \( \mathcal{L}_w \) (denoted by \( \lambda_i \), \( i = 2, 3, \ldots , n \)) are in the open right half plane (RHP). Furthermore, there exists a nonnegative vector \( \gamma = [\gamma_1, \gamma_2, \ldots , \gamma_n]^T \) such that \( \gamma^T \mathcal{L}_w \gamma = 0 \) and \( \Sigma_{i=1}^n \gamma_i = 1 \), and \( \mathcal{L}_w 1_n = 0 \) with \( 1_n = [1, 1, \ldots , 1]^T \).

B. Equations of Motion of Thrust-Propelled Vehicles

The equations of the translational and rotational motion of the \( i \)-th TPV can be written as [7], [10]

\[
\begin{align*}
& m_i \ddot{x}_i = -\sigma_i R_i e_3 + m_i g e_3 \\
& \dot{R}_i = R_i S(\omega_i)
\end{align*}
\]  

(1)

where \( \sigma_i \in \mathcal{R} \) is the thruster force, \( m_i \) is the mass, \( g \) denotes the gravitational acceleration, \( e_3 = [0, 0, 1]^T \) denotes the down direction while \( e_1, e_2 \in \mathcal{R}^3 \) are the two other orthogonal unit vectors that are defined such that the resultant frame is right-handed, \( x_j \in \mathcal{R}^3 \) is the position vector of the TPV with respect to the inertial frame, \( R_i \in \text{SO}(3) \) is the orientation matrix of the TPV with respect to the inertial frame, \( \omega_i \in \mathcal{R}^3 \) is the angular velocity of the TPV with respect to the inertial frame (expressed in the body-fixed frame), and the skew-symmetric form \( S(\cdot) \) is defined as

\[
S(b) = \begin{bmatrix}
0 & -b_3 & b_2 \\
-b_3 & 0 & -b_1 \\
b_2 & b_1 & 0
\end{bmatrix}
\]  

(2)

for a 3-dimensional vector \( b = [b_1, b_2, b_3]^T \).

The challenge in the control of a TPV governed by (1), as is frequently mentioned in the existing results (see, e.g., [7], [5], [6]), is that there is only one control input in the first equation of (1) while the number of the translational DOFs (degrees of freedom) is three. The case becomes much more complicated when it comes to the formation problem of multiple TPVs (see, e.g., [5], [6]). To facilitate the handling of this challenge, we differentiate the first equation of (1) with respect to time and obtain [using the second equation of (1)]

\[
m_i \ddot{x}_i = -\dot{\sigma}_i R_i e_3 - \sigma_i R_i S(\omega_i) e_3
\]  

(3)

Let \( \alpha \) be a positive design constant and combining the first equation of (1) and (3) with \( \alpha \) yields

\[
m_i (\dddot{x}_i + \alpha \dot{x}_i) = -(\dot{\sigma}_i + \alpha \sigma_i) R_i e_3 - \sigma_i R_i S(\omega_i) e_3 + \alpha m_i g e_3.
\]  

(4)

Denote the three components of \( \omega_i \) by \( \omega_i^{(1)}, \omega_i^{(2)}, \) and \( \omega_i^{(3)} \) then equation (4) can be rewritten as

\[
m_i (\dddot{x}_i + \alpha \dot{x}_i) = -R_i \begin{bmatrix}
\sigma_i \omega_i^{(2)} \\
-\sigma_i \omega_i^{(1)} \\
\alpha \sigma_i + \sigma_i \omega_i^{(3)}
\end{bmatrix} + \alpha m_i g e_3.
\]  

(5)

Consider the following input transformation

\[
u_i = R_i \begin{bmatrix}
\sigma_i \omega_i^{(2)} \\
-\sigma_i \omega_i^{(1)} \\
\alpha \sigma_i + \sigma_i \omega_i^{(3)}
\end{bmatrix}
\]  

(6)

and so long as \( \sigma_i \neq 0 \), any new control \( u_i \) can be realized by designing suitable \( \omega_i^{(2)}, \omega_i^{(1)}, \) and \( \sigma_i \). Under the new control \( u_i \), we have

\[
m_i (\dddot{x}_i + \alpha \dot{x}_i) - \alpha m_i g e_3 = -u_i
\]  

(7)

which can now be considered as a fully-actuated 3-DOF system. The transformation of (1) to (7) is quite similar to the standard dynamic feedback linearization of nonholonomic mobile robots (see, e.g., [27]). Let \( a_i = [m_i, m_i g]^T \) and then the left side of equation (7) can obviously be linearly parameterized with respect to the parameter vector \( a_i \), which gives

\[
m_i (\dddot{x}_i + \alpha \dot{x}_i) - \alpha m_i g e_3 = Y_i(\zeta) a_i
\]  

(8)

where \( \zeta \in \mathcal{R}^3 \) is a vector and \( Y_i(\zeta) = [\zeta, -\alpha e_3] \) is a regressor matrix.

III. SECOND-ORDER CONSENSUS OF MULTIPLE TPVs

In this section, we study the second-order consensus for the \( n \) TPVs on a directed graph having a directed spanning tree, and the control objective is to realize the simultaneous position and velocity consensus of the \( n \) TPVs, i.e., \( x_i - x_j \rightarrow 0 \) and \( \dot{x}_i - \dot{x}_j \rightarrow 0 \) as \( t \rightarrow \infty \), \( \forall i, j = 1, 2, \ldots , n \). The controller design will be completed in two steps. First, design \( u_i \) based on (7) to realize the objective of second-order consensus. Second, solve the required values for \( \omega_i^{(2)}, \omega_i^{(1)} \) and \( \sigma_i \) based on (6), which act as the actual control inputs.

Let us first define a sliding vector (similar to [19])

\[
s_i = \ddot{x}_i + \theta_3 \Sigma_{j \in N_i} w_{ij} (\hat{x}_i - \hat{x}_j) + k \Sigma_{j \in N_i} w_{ij} (x_i - x_j)
\]  

(9)

where \( b, k > 0 \) are the damping and stiffness gains.
Next, define a reference acceleration
\[
\ddot{x}_{r,i} = -b\sum_{j\in N_i} w_{ij} (\dot{x}_i - \dot{x}_j) - k\sum_{j\in N_i} w_{ij} (x_i - x_j) - \Lambda \int_0^t s_i(r) dr.
\]
where \(\Lambda\) is a symmetric positive definite matrix.

Let us now define another sliding vector
\[
\xi_i = \ddot{x}_i - \ddot{x}_{r,i} = s_i + \Lambda \int_0^t s_i(r) dr.
\]
The inclusion of the integration of the sliding vector \(s_i\) above is similar to [19], and it will be shown later that this is important for ensuring the boundedness of the TPVs' velocities and further the stability of the whole network.

Now the control input \(u_i\) is designed as
\[
u_i = -Y_i (\dddot{x}_{r,i} + \alpha \ddot{x}_{r,i}) \hat{a}_i,
\]
where \(\hat{a}_i\) is the estimate of the parameter \(a_i\), which is updated by the adaptation law
\[
\hat{a}_i = -\Gamma_i Y_i^T (\dddot{x}_{r,i} + \alpha \ddot{x}_{r,i}) \Delta a_i
\]
where \(\Delta a_i = a_i - \hat{a}_i\) is the parameter estimation error.

The closed-loop translational behavior of the \(i\)-th TPV can then be described by
\[
\begin{align*}
\dot{\xi}_i &= \dddot{x}_{r,i} + \alpha \ddot{x}_{r,i} + \Delta a_i \\

\dot{s}_i &= \dddot{x}_i - \dddot{x}_{r,i} + \Delta a_i
\end{align*}
\]
Substituting the control law (12) into (7) yields
\[
m_2(\xi_i + \alpha \xi_i) = Y_i (\ddot{x}_{r,i} + \alpha \ddot{x}_{r,i}) \Delta a_i
\]
where \(m_2(\xi_i + \alpha \xi_i)\) is the parameter estimation error.

Stacking up all the equations like the first one in (15) yields
\[
\ddot{x} + b (L_w \otimes I_3) \dot{x} + k (L_w \otimes I_3) x = s\]
where \(x = [x_1^T, x_2^T, \ldots, x_n^T]^T\), \(s = [s_1^T, s_2^T, \ldots, s_n^T]^T\) is the Kronecker product [29], and \(I_3\) is the 3×3 identity matrix.

In the current literature, numerous analysis approaches have been presented to examine the stability of the system (16) in the case that \(s = 0\) (see, e.g., [12], [13], [14], [15]). Yet, due to the under-actuation of the TPV's translational dynamics, \(s\) is generally nonzero. One attempt is made in [6], which relies on the backstepping-based Lyapunov analysis approach to derive the updating law for \(s\) for ensuring the second-order consensus of the networked TPVs. The possible limitations of [6] lie in three aspects: 1) the use of the Lyapunov approach can only give rise to the sufficient condition (which means that the result in [6] is conservative), and adopting the backstepping methodology involves the cross-term canceling, making the derived control law in [6] rather complex (see [6, equation (23)]); 2) the determination of the damping and stiffness gains relies on the knowledge of the mass properties of the TPVs; 3) the interaction graph among the agents is required to be balanced and strongly connected. The control scheme here, benefiting from the input-output perspective and the adaptive algorithm, rules out the above mentioned three limitations.

At the first step, we clarify what kind of \(s\) will result in second-order consensus of the TPVs with bounded velocities, mainly based on the result in [19]. To this end, applying the Laplace transformation to both sides of (16) yields
\[
pV(p) - \dot{x}(0) = -b (L_w \otimes I_3) V(p) - k (L_w \otimes I_3) V(p) + x(0) + S(p)
\]
where \(p\) denotes the Laplace variable, and \(V(p)\) and \(S(p)\) denote the Laplace transforms of \(\dot{x}\) and \(s\), respectively.

Equation (17) can be further written as
\[
V(p) = \left[ \begin{array}{c} pI_n + \frac{k}{p} L_w + \frac{b}{p} L_w \end{array} \right]^{-1} I_3 \left[ \begin{array}{c} \dot{x}(0) - \frac{k}{p} (L_w \otimes I_3) x(0) + S(p) \end{array} \right]
\]
where \(I_n\) is the \(n \times n\) identity matrix.

The transfer function \(G(p)\) in (18) has the following property [19].

**Lemma 2:** If the Laplacian matrix \(L_w\) is associated with a graph having a directed spanning tree, then, all the poles excluding the simple pole at the origin of \(G(p)\) are in the open left half plane (LHP) if and only if the damping and stiffness gains satisfy
\[
\frac{b^2}{k} > \max_{\lambda_i \geq 0} \frac{(\text{Re}\lambda_i)^2}{|\text{Im}\lambda_i|^2}
\]
where \(|\cdot|\) denotes the modulus of a complex number, and \(\text{Re}\lambda_i\) and \(\text{Im}\lambda_i\) are the real and imaginary parts of \(\lambda_i\), respectively.

We are presently ready to present the following lemma.

**Lemma 3:** If the interaction graph among the TPVs has a directed spanning tree and the gains \(b, k\) are chosen to satisfy (19), the output \(\dot{x}\) of the system (16) is bounded if and only if the integral of \(|s|\) with respect to time is bounded (i.e., \(\int_0^t s(r) dr \in \mathbb{L}_\infty\) for \(\forall t \geq 0\), i.e., the system (16) is integral-bounded-input bounded-output (iBIBO) stable in the sense of [19]. Furthermore, if \(\int_0^t s(r) dr \to 0\) as \(t \to \infty\), \(\dot{x}_i \to \sum_{k=1}^n \gamma_k \ddot{x}_k(0)\) and \(x_i - x_j \to 0\), \(\forall i,j = 1, 2, \ldots, n\).

**Proof:** According to Lemma 2, all the poles of the transfer function \(G(p)\) excluding the simple zero pole are in the open LHP. Then, the first assertion can be directly proved by following the result in [19]. In the case that \(\int_0^t s(r) dr \to 0\) as \(t \to \infty\), applying the final value theorem gives
\[
\lim_{p \to 0} S(p) = \lim_{t \to \infty} \int_0^t s(r) dr = 0.
\]
Then, following a procedure similar to [19], we can prove the second assertion of Lemma 3.
Next, we derive the control input signals $\sigma_i$, $\omega_i^{(1)}$, and $\omega_i^{(2)}$ based on the control input $u_i$ given by (12) and the transformation (6). Assuming that $\sigma_i \neq 0$ and letting $\bar{u}_i = R_i^T u_i$, we obtain the control input signals for the i-th TPV as

$$\dot{\sigma}_i + \sigma_i u_i = \bar{u}_i^{(3)}$$  \hspace{1cm} (20)

$$\omega_i^{(1)} = - \frac{\bar{u}_i^{(2)}}{\sigma_i}$$  \hspace{1cm} (21)

$$\omega_i^{(2)} = \frac{\bar{u}_i^{(1)}}{\sigma_i}$$  \hspace{1cm} (22)

where $\bar{u}_i^{(1)}$, $\bar{u}_i^{(2)}$, and $\bar{u}_i^{(3)}$ are the three components of $\bar{u}_i$.

**Remark 1:** The assumption $\sigma_i \neq 0$ is equivalent to requiring $\dot{x} \neq g e_3$ (i.e., there is no free fall, similar to [9], [7], [6]). Furthermore, since $\omega_i^{(3)}$ is not explicitly used to realize the second-order consensus, following the standard practice, we simply set $\omega_i^{(3)} = 0$, i.e., the yaw motion is not controlled.

We are presently ready to give the following theorem.

**Theorem 1:** The adaptive controller given by (12), (13), (20), (21), and (22) gives rise to the second-order consensus of the n TPVs on directed graphs a directed spanning tree if and only if the gains $b_k$ satisfy (19), i.e., $\dot{x}_i \rightarrow \Sigma_{k=1}^{n} \gamma_k \dot{x}_k(0)$ and $x_i - x_j \rightarrow 0$ as $t \rightarrow \infty$, $\forall i,j = 1,2,\ldots,n$.

**Proof:** Similar to [30], [28], [31], for the third and fourth subsystems in (15), we take into consideration the Lyapunov-like function candidate below

$$V_i = \frac{m_i}{2} \xi_i^T \xi_i + \frac{1}{2} \Delta \alpha_i^T \Gamma_i^{-1} \Delta \alpha_i$$  \hspace{1cm} (23)

and differentiating $V_i$ with respect to time along the trajectories of these two subsystems yields

$$\dot{V}_i = -\alpha m_i \xi_i^T \xi_i \leq 0$$  \hspace{1cm} (24)

which then implies that $\xi_1 \in L_2 \cap L_\infty$ and $\dot{a}_i \in L_2$, $\forall i$.

Consider the second subsystem in (15) with $\xi_i$ as the input and $\int_0^t s_i(r) dr$ as the output, and from the input-output properties of strictly proper and exponentially stable linear systems [32, p. 59], we have that $\int_0^t s_i(r) dr \in L_2 \cap L_\infty$, $s_i \in L_2 \cap L_\infty$, and $\int_0^t s_i(r) dr \rightarrow 0$ as $t \rightarrow \infty$, $\forall i$. In the case that $b_k$ satisfy the inequality (19), the poles of $G(p)$ ruling out the simple zero pole are all in the open LHP (by Lemma 2). From Lemma 3, we obtain that $\dot{x}_i \in L_\infty$, and that $\dot{x}_i \rightarrow \Sigma_{k=1}^{n} \gamma_k \dot{x}_k(0)$ and $x_i - x_j \rightarrow 0$ as $t \rightarrow \infty$, $\forall i,j$.

Integrating (16) with respect to time gives

$$b (L_w \otimes I_3) x + k (L_w \otimes I_3) \int_0^t x(r) dr$$

$$= \int_0^t s(r) dr - [\dot{x} - \dot{x}(0)] + b (L_w \otimes I_3) x(0)$$  \hspace{1cm} (25)

which implies that $(L_w \otimes I_3) \int_0^t x(r) dr \in L_\infty$ and $(L_w \otimes I_3) x \in L_\infty$ according to the standard input-output properties of linear systems. From (16), we then obtain that $\dot{x} \in L_\infty$. From (10), we have that $\dot{x}_i \in L_\infty$, and additionally it can be shown that $\dot{x}_i \in L_\infty$, $\forall i$. From (14), we have that $\dot{\xi}_i \in L_\infty$ and further $\dot{x}_i \in L_\infty$, $\forall i$.

**TABLE I**

**INITIAL POSITIONS AND VELOCITIES OF THE SEVEN TPVs**

<table>
<thead>
<tr>
<th>i-th TPV initial positions (m)</th>
<th>i-th TPV initial velocities (m · s⁻¹)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>[0.5, 0.5, 0.5]ᵀ</td>
</tr>
<tr>
<td>2</td>
<td>[0.0, -1.5, 1.0]ᵀ</td>
</tr>
<tr>
<td>3</td>
<td>[1.5, -2.0, -0.5]ᵀ</td>
</tr>
<tr>
<td>4</td>
<td>[3.0, 1.0, 1.5]ᵀ</td>
</tr>
<tr>
<td>5</td>
<td>[0.8, -1.5, -1.0]ᵀ</td>
</tr>
<tr>
<td>6</td>
<td>[0.0, 2.5, -2.0]ᵀ</td>
</tr>
<tr>
<td>7</td>
<td>[1.8, 0.0, -2.0]ᵀ</td>
</tr>
</tbody>
</table>

Then, from the standard linear system theory, the output $\dot{x}$ of the system (16) will be oscillatory or divergent, and therefore the second-order consensus can no longer be achieved.

**Remark 2:** The control calculation scheme (20), (21), and (22) is structurally the same as the one in [6], yet the vector $u_i$ [which is given by (12) with $\dot{a}_i$ being updated by (13)] is different from that in [6]. This difference allows the interaction graph to be the one which only has a directed spanning tree and makes our scheme no longer require the exact knowledge of the TPVs' dynamics (certainly at the expense of measuring the translational accelerations of the TPVs). The integral action employed in (10) and (11) ensures that all TPVs’ velocities converge to the weighted average of their initial values, as opposed to [6].

**IV. SIMULATION RESULTS**

In this section, we examine the performance of the proposed controller by conducting a simulation using seven TPVs on a directed graph shown in Fig. 1 (which obviously contains a directed spanning tree). The masses of the seven TPVs (with the unit being kg) are set as $m_1 = 2$, $m_2 = 1$, $m_3 = 1.5$, $m_4 = 1.8$, $m_5 = 0.9$, $m_6 = 1.6$, and $m_7 = 2.5$. The gravitational acceleration is set as $g = 9.8$ m · s⁻². The sampling period employed in the simulation is 5 ms.

The entries of the weighted adjacency matrix $W$ are set as $w_{ij} = 1.0$ if $j \in N_i$, and $w_{ij} = 0$ otherwise. The initial positions and velocities of the seven TPVs are listed in Table I. The controller parameters $\alpha$, $\Lambda$, and $\Gamma_i$ are determined as $\alpha = 10$, $\Lambda = 10 I_3$, and $\Gamma_i = \text{diag}[0.001, 0.1]$, respectively, $i = 1,2,\ldots,7$. The damping and stiffness gains are chosen as $\alpha = 1.0$ and $\beta = 0.5$ which satisfy the condition

$$b > \sqrt{k \max_{i \geq 2} \frac{(\text{Im} \lambda_i)^2}{(\text{Re} \lambda_i)^2}}$$

$$= 0.2236.$$  \hspace{1cm} (25)

The initial parameter estimates are chosen as $\dot{a}_i = [0,0,0]^T$ and the initial thrust forces of the TPVs are set as $\sigma_i(0) = 5.0$, $i = 1,2,\ldots,7$. The left eigenvector of the Laplacian matrix $L_w$ associated with its zero eigenvalue is $\gamma = [0,0,0,0,0,0,0,0]^T$ and thus the weighted average of the initial velocities of the TPVs is $\gamma_2 \dot{x}_2(0) + \gamma_3 \dot{x}_5(0) + \gamma_6 \dot{x}_6(0) + \gamma_7 \dot{x}_7(0) = [-1.2, 0.4, 0.1]^T$. The simulation results are shown in Fig. 2 to Fig. 5. Fig. 2 shows that the norms of the position consensus errors between the seven TPVs, which, obviously, asymptotically converge to
Fig. 1. Interaction graph among the TPVs

Fig. 2. Norms of the position consensus errors of the TPVs ($\| \cdot \|$ denotes the 2-norm of a vector)

zero. The velocities of the seven TPVs are plotted in Fig. 3 to Fig. 5, and as is expected, they indeed converge to the weighted average of the TPVs’ initial velocities.

V. CONCLUSION

In this paper, we have examined the second-order consensus problem for multiple TPVs on directed graphs containing a directed spanning tree. Different from the usually adopted backstepping methodology to address the under-actuation of the TPVs, we present a new approach that exploits the input-output properties of a second-order linear interconnection system. The consequence of this new perspective is the simplicity of the control scheme, the aggressive design of the controller parameters, and the extension from balanced graphs to the more general directed graphs only containing a directed spanning tree. Through an input-output analysis based on the
iBIBO stability, we demonstrate that the proposed controller gives rise to the second-order consensus of the networked TPVs, and thanks to the employment of an integral action in the control input design, the velocities of the TPVs are ensured to converge to the weighted average of their initial values. A simulation using a network of seven TPVs is conducted to illustrate the performance of the proposed controller.

REFERENCES