Unified BER and Optimum Threshold Analysis of Binary Modulations in Simple and Cascaded Rayleigh Fading Channels with Switched Combining

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SUMMARY

Following a unified analytical framework, the bit error rate (BER) of several coherent and non-coherent binary modulation schemes are derived for a switched diversity system. The two variants of switched combining (SWC) that have been investigated are switch and stay combining (SSC) and switch and examine combining (SEC). For channel modelling, at first a simple slow flat fading channel is assumed where the amplitude attenuation obeys the Rayleigh distribution. Later the BER calculations are repeated for cascaded Rayleigh fading channel case. Rayleigh fading is the most popular model for electromagnetic signal propagation in wireless media when both or either of the transmitter/receiver is fixed. On the other hand, when both the transmitter and the receiver are mobile a cascaded (or double) Rayleigh fading model is better suited. The applicability of these two models, namely simple and cascaded Rayleigh model, has been indicated by several theoretical studies and their suitability is established by various field measurements. In our paper, simple closed-form BER expressions as a function of switching threshold have been found and optimum switching thresholds have been computed for both these models as well as for both types of diversity combining described earlier. The results presented in this article can be very useful for communication system designers to analyze link quality of switched diversity assisted systems in various wireless environments.

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1. INTRODUCTION

Rayleigh distribution had been used for more than half a century [1] for modelling electromagnetic signal propagation through multipath wireless environments. Interestingly, far from being outdated, it remains the most acceptable model till now. There are mainly three reasons behind this: first, in urban and suburban land mobile communication hardly any direct line-of-sight (LoS) exists between the transmitter (Tx) and the receiver (Rx). The mobile antenna receives a large number of reflected, diffracted and scattered waves from buildings, foliage, and rough terrain. In such an environment Rayleigh distribution closely matches with the channel attenuation profile as available from several field measurements. Second, Rayleigh is a good fit for
not only the cellular communication channel but several other non-LoS (nLoS) propagations e.g., troposcateter and ionospheric propagation, maritime ship-to-ship communication, and underwater acoustic communication. Third, quite contrary to the complex physical basis of Rayleigh model, the PDF and associated formulas involved in the model are remarkably simple. Thus the end results of many performance assessment problems are mostly in closed-form for Rayleigh channels. In particular, compared to other fading models, the transmission error probability calculations are much simpler in this case. All these facts accelerated the rapid growth of research work based on Rayleigh models (Ref. [2] contains a detailed list of them).

Traditionally Rayleigh and other statistical distributions (Rice, Nakagami-m, Hoyt, Weibull, Gamma etc.) were introduced to model fixed-to-mobile (F2M) cellular radio channels, where the base-station is stationary, elevated, and relatively free from local scattering. However in mobile-to-mobile (M2M) communication, the conventional statistical models are not strictly applicable due to the mobility of transmitter and receiver as well as obstructions around their low elevation antennas. One simple example of such an M2M wireless scenario may be some emergency, where military/ rescue squad vehicles form a vehicular ad-hoc network (VANET) and each vehicle needs to constantly communicate with all other moving vehicles. M2M situations can also be observed in propagation via diffracting wedges (such as street corners) in urban micro cells, in ad-hoc mobile networks, dedicated short-range communication systems for intelligent highways, relay-based cellular networks and so on.

The received envelope in M2M wireless channels is normally modeled using a cascaded or double Rayleigh distribution under nLoS conditions [3-4]. Existence of this double Rayleigh scattering phenomenon has been well established through various channel measurement experiments [5-10] and validated by some simulation studies [11-12]. The baseband equivalent channel transfer function in a cascaded Rayleigh fading model is given by a product of two independent zero-mean complex Gaussian

![Figure 1. Physical realization of simple and cascaded Rayleigh fading in the absence of direct LoS path.](image)
random variables (RVs) [13]. As shown in Fig. 1, the cascaded fading model is based on the two-ring path geometry [14-15] where each ring consists of independent scatterers around the mobile terminals. All the scattered waves travel between these two rings through a narrow pipe like channel, popularly known as keyhole or pinhole. When a system employs multiple antennas to achieve diversity, keyhole effect induces uncorrelated spatial fading at both ends of the channel, and is likely to improve the system performance. However, the channel transfer matrix becomes rank-deficient (have a single or reduced degree of freedom) in this mode of propagation. This rank deficiency in keyhole channels reduces effective channel capacity and severely degrades the link quality in multiple input multiple output (MIMO) systems [5-6].

Recent studies reveal that the classical double Rayleigh model has been already extended to models containing the product of two or more Rician [5, 16-17], Nakagami-\(m\) [18-19], and Weibull [20] RVs. The references cited here, though by no means comprehensive, highlight the growing research interest about keyhole M2M communication. In spite of this, we find that literatures dealing with error performance of different modulation schemes in keyhole channels are still scarce. Exceptions include an approximate solution of probability of error by Salo et al. [21], where the authors addressed coherent modulation schemes (having Gaussian \(Q\) function in the error probability expression) and evaluated the error rates in a cascaded Rayleigh channel. In a separate work, Uysal [22] calculated \(M\)-ary phase shift keying (MPSK) error performance in cascaded Rayleigh channels and presented the results in the form of single finite range integral. Some more studies [23-25] on double Rayleigh fading channel have also been done recently with space-time codes. However, to the best of the authors’ knowledge, error performance of different diversity systems in cascaded fading channel has never been investigated.

Diversity techniques in wireless communications, especially predetection diversity combining have long been studied for mitigating the detrimental effects of channel fading. The conventional combining methods such as maximal ratio combining (MRC) and equal gain combining (EGC) need some of the channel state information (CSI) from all the received signals and require dedicated radio frequency (RF) chain for each diversity branch. On the other hand, selection combining (SC) involves a continuous monitoring of all the diversity branches but dispenses with the need of multiple RF chains. In SC, the selector switch chooses the branch with highest signal to noise ratio (SNR) while discarding any information that may be available through other branches. Thus only one RF chain at a time needs to be activated. A low-complexity low-cost alternative of SC is SWC where the receiver scans the branches sequentially and selects the first one having SNR above a predetermined threshold. If the SNR on the selected branch falls below the threshold value, the combiner switches to another branch. Basically SWC is a suboptimum version of SC; in SWC no switching would take place as long as the current branch SNR is higher than the threshold whereas with SC the switch moves immediately if it finds a better branch. There are a number of ways to decide the switching logic for SWC, out of which the two most popular variants are switch and stay combining (SSC) [26-27] and switch and examine combining (SEC) [28]. In SSC, the receiver switches to, and stays with (for the next transmission slot) the next branch, no matter what the received signal power of the next branch is. On the other hand, an SEC receiver examines the signal strength of the next branch and switches back if it is not acceptable. The process is repeated until either an acceptable branch is found or no branch is left to be examined.

The contribution of the present text is twofold. In a previous paper, using the compact form given by Wojner [29], the authors presented unified bit error rate (BER) expressions for MRC, EGC, SC and SWC for coherent and non-coherent binary signalling schemes in Rayleigh channels [30]. However, as far as the analysis for switched diversity is concerned, the study was limited to a fixed threshold value for the dual branch SSC case. In this paper, we present BER analysis for general \(L\) branch SSC and SEC diversity systems over independent and identically distributed (IID) Rayleigh fading channels. In addition, optimal switching threshold values are also tabulated. Next, we address the switched diversity case in cascaded fading channel and derived average BER expressions as well as optimal switching thresholds. The resultant
expressions are original, presented in a unified manner, attractive for simplicity and can be extended to cover other modulation formats.

The rest of the paper is organized as follows: section 2 outlines the system and channel model used throughout the analytical derivations. Performance of binary modulation over Rayleigh fading channel with SSC and SEC is discussed in section 3. The following section, section 4, presents the performance with SSC and SEC over cascaded fading channel. Selected numerical and simulation results are shown in graphical form in section 5. Finally, a summary of the present paper and some concluding remarks are offered in section 6.

2. SYSTEM MODEL

The communication link model with \( L \) branch predetection switched diversity combiner operating over IID fading channels is described in Fig. 2. There are \( L \) available antennas receiving signals with statistically independent random amplitudes and random phases. Let the transmitted signal \( s(t) \) be a bandpass signal whose frequency domain representation \( S(f) \) is non-zero for frequencies in a small neighborhood of some high carrier frequency \( f_c \), i.e. \( S(f) \approx 0 \) for \( |f - f_c| > W \) where \( W << f_c \). Under slow flat fading assumption, the received equivalent baseband signal at the \( n \) th diversity branch can be written as,

\[
\tilde{s}_n(t) = \alpha_n \exp(i\theta_n) s(t) + n_n(t) \quad ; n=1,2,...,L
\]

where \( \tilde{s}(t) \) is baseband equivalent of the transmitted signal \( s(t) \). For constant amplitude binary modulations like phase shift keying (PSK) and frequency shift keying (FSK), average power of \( \tilde{s}(t) \) is fixed at \( \tilde{E}_b^2 = \frac{2}{T_b} \), with \( \tilde{E}_b \) and \( T_b \) being the bit energy and bit duration. The parameters \( \alpha_n \) and \( \theta_n \) denote simple (or cascaded) Rayleigh distributed attenuation factor and uniformly distributed phase shift of the \( n \) th channel respectively. To each signal there is an additive zero mean white Gaussian noise component \( n_n(t) \) with two sided power spectral density (PSD) \( \frac{N_0}{2} \) that is assumed to be independent of the signal and uncorrelated with the noise in any other branch.

The diversity combiner operates in discrete time fashion, i.e. the branch switching occurs at time \( t = nT_s \), where \( n \) is any integer. The slow fading assumption guarantees that \( T_s \leq T_c \), \( T_c \) being the channel coherence time. For the general \( L \) branch SSC scheme, if the \( j \) th antenna is in use at time \( t = (n-1)T_s \) and in the next instant \( t = nT_s \) SNR of the \( j \) th branch drops below some given threshold, output selector of the combiner switches to \( k \) th \( (k = \lfloor j + 1 \rfloor \mod L) \) antenna regardless of the SNR available in the \( k \) th branch. It has been shown by Yang and Alouini [28] that the error performance of \( L \)-branch SSC is identical with the dual-branch case. This explains why only \( L = 2 \) case (for SSC) has been considered in the present paper.

If the combiner is following SEC scheme instead, the selector, just like SSC, switches from the current branch to the next branch at \( t = nT_s \) if SNR in current branch falls below some threshold value. However the selector examines the target branch too. If the next or target branch also has a SNR lower than threshold value then the selector continues searching for a better branch until all branches are exhausted. If no other branch can provide a SNR higher than the threshold, the selector may stay with the last examined branch or switch back to the branch used in previous time instant at \( t = (n-1)T_s \). Both the strategies lead to identical performance. This examination process offers additional diversity gain over SSC when \( L > 2 \) and render SEC a better scheme than SSC, of course at the cost of some added complexity. For \( L = 2 \), the performance is same as dual-diversity SSC or \( L \) branch SSC [28], as pointed out earlier.
Fig. 2 depicts some additional blocks other than the basic selector switch that are required for successful operation of a switched combiner. A channel estimator estimates the current SNR in different branches and feeds a comparator with these values. The comparator compares the SNR values against the predetermined threshold SNR value (generally found from a table that stores the optimum thresholds for different SNR) and triggers the switching logic block. Depending on the switching strategy (SSC/SEC) the switching logic block controls the branch selector accordingly.

3. UNIFIED BER ANALYSIS FOR SIMPLE RAYLEIGH FADING CHANNEL

In this paper we consider an $L$ branch switched diversity system employing binary modulation and operating in a slow non-selective IID fading environment corrupted by additive white Gaussian noise (AWGN). With the assumption of statistical independence between fading and noise, the overall average BER ($P_e$) with diversity can be calculated by averaging the non-fading error probability $P_e(\gamma)$ over the underlying fading random variable $\gamma \Delta \alpha^2 (E_s/N_0)$ as

$$P_e = \int P_e(\gamma)f_{\gamma,\text{awgn}}(\gamma)d\gamma$$

(2)

where $P_e(\gamma)$ is simply the conditional error probability in AWGN channel, $\gamma$ is instantaneous SNR per bit at the combiner output, and $f_{\gamma,\text{awgn}}(\gamma)$ denotes the probability density function (PDF) of $\gamma$ in a specified
fading environment with switched diversity. After Wojnar [29], a compact form of conditional BER for different binary modulations is

\[ P(\gamma) = \frac{1}{2} \frac{\Gamma(b, a\gamma)}{\Gamma(b)} \quad : a, b \in \left\{ \frac{1}{2}, 1 \right\} \]  

(3)

where \( a \) depends on type of modulation (1/2 for orthogonal FSK, 1 for antipodal PSK), \( b \) depends on type of detection (1/2 for coherent, 1 for non-coherent), and \( \Gamma(\cdot, \cdot) \) are gamma function \([31, (6.5.3)]\) and complementary incomplete gamma function \([31, (6.5.3)]\) respectively.

3.1 BER for Switch and Stay Combining

For Rayleigh channels with SSC, the PDF \( f_{\gamma, \text{SSC}}(\gamma) \), regardless of the number of diversity branches is \([32]\)

\[ f_{\gamma, \text{SSC}}(\gamma) = \begin{cases} \frac{1}{\gamma} \left( \frac{1}{\gamma} \right)^{\frac{1}{\gamma} - 1} \left( 1 - \frac{1}{\gamma} \right)^{\frac{1}{\gamma} - 1} & : \gamma < \gamma_s \\ \frac{1}{\gamma} \left( \frac{1}{\gamma} \right)^{\frac{1}{\gamma} - 1} \left( 2 - \frac{1}{\gamma} \right)^{\frac{1}{\gamma} - 1} & : \gamma \geq \gamma_s \end{cases} \]  

(4)

where \( \gamma \) is the common average SNR per bit per branch, i.e. \( \gamma = \gamma_i \forall i, i \in \{1, 2, \cdots, L\} \) and \( \gamma_s \) is the common switching threshold. Inserting (3) and (4) in (2) and after considerable algebraic manipulation, the unified BER for binary modulations with SSC as a function of switching threshold SNR \( \gamma_s \) was found by the authors \([30, (18)]\) as

\[ P_{\text{SSC}} = \frac{1}{2} \left[ 1 - \exp\left( -\hat{\gamma} \right) \left( 1 - \frac{\Gamma(b, a\gamma_s)}{\Gamma(b)} \right) \right] \]  

(5)

where \( \hat{\gamma} = \gamma_s / \gamma \) denotes the normalized threshold.

Noting that \( \Gamma(z, 0) = \Gamma(z) \) \([33, (8.350.3)]\) one can identify that the SSC performance with \( \gamma_s = 0 \) becomes identical with the no diversity case

\[ P = \frac{1}{2} \left[ 1 - \left( \frac{a\gamma}{1 + a\gamma} \right)^{\gamma} \right] \]  

(6)

for no switching occurring at all. As a double check, it can be easily verified that by averaging (3) over the Rayleigh PDF, \( f_{\gamma}(\gamma) = (1/\gamma)\exp(-\gamma/\gamma) \) while making use of a definite integration formula \([31, (6.5.36)]\)

\[ \int_{0}^{\infty} \exp(-\alpha x) \Gamma(\beta, \nu x) dx = \frac{\Gamma(\beta)}{\alpha} \left[ 1 - \left( \frac{\nu}{\alpha + \nu} \right)^{\beta} \right] : \Re(\alpha + \nu) > 0, \Re(\beta) > -1 \]  

(7)

the BER expression given in (6) may be obtained.
Transmitting data at a fixed SNR i.e. keeping the value of average SNR $\bar{\gamma}$ constant, as the switching threshold increases from 0, the probability of switching

$$\Pr(\gamma < \gamma_t) = \int_0^{\gamma} f_\gamma(\gamma)d\gamma$$

also increases causing the BER to decrease until an optimum threshold SNR value $\gamma^*_t$ is reached. Naturally, the optimum threshold $\gamma^*_t$ ensures minimum average error probability and can be obtained from the following identity [34, (7)]

$$\frac{\partial P_t}{\partial \gamma_t} \bigg|_{\gamma_t = \gamma^*_t} = 0$$

In the range $\gamma_t > \gamma^*_t$ the BER starts increasing once again. Finally, for $\gamma_t \to \infty$ there will be constant switching because no branch can satisfy threshold requirement. In this case the performance will be equivalent to the performance of diversity branch selected at random, which resembles the behaviour of single branch diversity. Mathematically, (5) boils down to (6) once again as $\Gamma(l, \infty) = 0$ [33, (8.350.4)]. A quick look at (4) reveals that in both the cases i.e., for $\gamma_t = 0$ and $\gamma_t = \infty$, (4) reduces to simple Rayleigh PDF.

Despite the fact that closed-form expressions for $\gamma^*_t$ with specific signalling constellation can be easily obtained, a unified expression, including all the modulation schemes under consideration, cannot be extracted. Differentiating (5) with respect to $\gamma_t$, equating the differential to zero, and noting that $\Gamma(l, z) = \exp(-z)$ [33, (8.352.7)], $\Gamma(l/2, \sqrt{z}) = \sqrt{\pi} \operatorname{erfc}(z)$ [31, (6.5.17)] we get a set of two different equations. For non-coherent detection $\gamma^*_t$ can be written directly as an analytic function of $\bar{\gamma}$,

$$\gamma^*_t = \frac{1}{a} \ln(1 + a\bar{\gamma})$$

while for coherent detection $\gamma^*_c$ is

$$\gamma^*_c = \frac{1}{a} \left[ \operatorname{erfcinv}\left(1 - \frac{a\bar{\gamma}}{\sqrt{1 + a\bar{\gamma}}} \right) \right]$$

where $\operatorname{erfcinv}(\cdot)$ denotes inverse complementary error function.

### 3.2 BER for Switch and Examine Combining

The PDF $f_{\gamma,\text{sec}}(\bar{\gamma})$ for $L$ branch SEC under Rayleigh fading is given as [28]
\[ f_{\gamma, \text{SEC}}(\gamma) = \begin{cases} 
\frac{1}{\gamma} \exp\left(-\frac{\gamma}{\gamma} \left[1 - \exp\left(-\frac{\gamma}{\gamma} \right)\right]\right)^{\gamma-1} & ; \gamma < \gamma_t \\
\frac{1}{\gamma} \exp\left(-\frac{\gamma}{\gamma} \sum_{j=0}^{\gamma} \left[1 - \exp\left(-\frac{\gamma}{\gamma} \right)\right]\right) & ; \gamma \geq \gamma_t 
\end{cases} \quad (11) \]

which is same as (4) for \( L = 2 \). This observation mathematically proves that \( L \) branch SSC is just a special case of SEC for \( L = 2 \) which is quite in harmony with the theoretical discussion presented in section 2. Inserting \( P_s(\gamma) \) given by (3) and \( f_{\gamma, \text{SEC}}(\gamma) \) from (11) in (2) gives

\[ P_s = \frac{1}{2\gamma} \left[1 - \exp(-\hat{\gamma})\right] + \frac{1}{2\gamma} \left[1 - \exp(-\hat{\gamma})\right]^2 \quad (12) \]

where \( I_1 = \int \psi(\gamma) d\gamma, \hat{\gamma} = \gamma_t/\gamma, \text{ and } \psi(\gamma) = \Gamma(\gamma, \alpha a) \exp(-\gamma/\gamma). \)

Using (7), the first integral can be readily evaluated as

\[ I_1 = \Gamma(\hat{\gamma}, \alpha a) \left[1 - \left(\frac{\alpha \gamma}{1 + \alpha \gamma}\right)^{\hat{\gamma}}(1 + \alpha \gamma)\right] \quad (13) \]

However, calculation of the second integral is not straightforward like the first as the lower limit of the integration \( \gamma_t \) is having a finite non-zero value. Solving the same through the method of integration by parts, the result can be expressed with yet another complementary incomplete gamma function

\[ I_2 = \Gamma(\hat{\gamma}, \alpha a) \exp(-\hat{\gamma}) - \Gamma(\hat{\gamma}, \gamma_l) \quad (14) \]

where \( \hat{\gamma} = \gamma_t/\gamma. \)

Finally, accumulating the results from (11)-(13), the overall unified BER expression for binary modulations with SEC can be written as

\[ P_{s, \text{SEC}} = \frac{1}{2} \left[1 - \exp(-\hat{\gamma})\right] + \frac{1}{2\Gamma(\gamma)} \left[1 - \left(\frac{\alpha \gamma}{1 + \alpha \gamma}\right)^{\hat{\gamma}}(1 + \alpha \gamma)\right] \quad (15) \]

involving only elementary functions. Note that, for \( L = 2 \), (15) reduces to (5) as expected.

For SEC the optimal threshold \( \gamma_t^* \) is an increasing function of both \( \gamma_t \) and \( L \). Indicative values of the optimum switching threshold \( \gamma_t^* \) for minimum BER are computed for binary phase shift keying (BPSK) and differential phase shift keying (DPSK) through numerical minimization using \( \text{fminunc}() \) function available in the well-known mathematical software package MATLAB and are listed in Table I.
Table I. Optimum switching threshold $\gamma^*_T$ for BPSK and DPSK operating over simple Rayleigh fading channel.

<table>
<thead>
<tr>
<th>$L$ (dB)</th>
<th>$BPSK$ ($a=1, b=1/2$)</th>
<th>$DPSK$ ($a=1, b=1$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L=8$</td>
<td>$L=4$ SSC</td>
<td>$L=8$ SSC</td>
</tr>
<tr>
<td>0</td>
<td>0.84</td>
<td>-1.71</td>
</tr>
<tr>
<td>3</td>
<td>3.08</td>
<td>1.63</td>
</tr>
<tr>
<td>6</td>
<td>5.05</td>
<td>5.11</td>
</tr>
<tr>
<td>9</td>
<td>6.77</td>
<td>7.01</td>
</tr>
<tr>
<td>12</td>
<td>8.24</td>
<td>8.51</td>
</tr>
<tr>
<td>15</td>
<td>9.58</td>
<td>11.19</td>
</tr>
<tr>
<td>18</td>
<td>10.73</td>
<td>11.50</td>
</tr>
<tr>
<td>21</td>
<td>11.73</td>
<td>12.63</td>
</tr>
<tr>
<td>24</td>
<td>12.61</td>
<td>13.99</td>
</tr>
<tr>
<td>27</td>
<td>13.39</td>
<td>15.10</td>
</tr>
</tbody>
</table>

It may be noted that part of some derivations presented in this section are somewhat similar to those obtained in earlier work [2]. Specifically, for the BPSK case ($a = 1, b = 1/2$), (5) and (15) reduce to eq. (9.306) and (9.343) of ref. [2] and (10.b) becomes eq. (9.313) of ref. [2] for $a = 1$. In this paper the results are, however, obtained in a unified form without considering each and every modulation scheme separately and thereby avoiding repeated calculations. For SEC, the optimum threshold values have been evaluated as given in Table 2 although nothing about optimum threshold with SEC is discussed in ref. [2].

4. UNIFIED BER ANALYSIS FOR CASCADED RAYLEIGH FADING CHANNEL

For BER calculation over a cascaded Rayleigh fading channel we start our derivation with a formal definition of the channel SNR statistics. Let us consider two independent Rayleigh distributed signal envelopes $\alpha_1$ and $\alpha_2$ with mean power $E[\alpha^2] = \Omega_1$ and $E[\alpha^2] = \Omega_2$ respectively. Following the arguments in Kovacs’ thesis [35, (A.2.10)], we can express the PDF of the cascaded signal envelope $\alpha = \alpha_1\alpha_2$ i.e. $f_\alpha(\alpha)$ as

$$f_\alpha(\alpha) = \frac{4\alpha \Omega_1}{\Omega_2} K_0\left(\frac{2\alpha \sqrt{\Omega_1 \Omega_2}}{\sqrt{\Omega_2}}\right) ; \alpha \geq 0$$

where $E[\alpha^2] = \Omega_1\Omega_2 = \Omega$ and $K_0(.)$ denotes zero order modified Bessel function of second kind.

Next, with a simple transformation of variable $f_\gamma(\gamma) = f_\gamma(\sqrt{\Omega_1/\gamma}/2\sqrt{\Omega_2/\Omega_1})$ [2, (2.3)], the PDF of instantaneous SNR per bit $\gamma$ may be obtained as

$$f_\gamma(\gamma) = \frac{2}{\gamma} K_0\left(\frac{\sqrt{\gamma}}{\sqrt{\gamma}}\right) ; \gamma \geq 0$$

which was first mentioned by Chizhik et al. [36, (5)]. Further, integrating (17) using Luke’s indefinite integration formula [37, (5.2.31)]
\[
\int_0^\infty x^\nu K_{\nu-1}(x)dx = 2^{-\nu} \Gamma(\nu) - z^\nu K_{\nu}(z) \quad ; \Re(\nu) > 0
\]  

(18)

one arrives at the corresponding cumulative distribution function (CDF) of \( \gamma \)

\[
F_\gamma(\gamma) = 1 - 2 \sqrt{\frac{\gamma}{\gamma}} K_{\frac{\gamma}{\gamma}} \left( 2 \sqrt{\frac{\gamma}{\gamma}} \right) \quad ; \gamma \geq 0
\]  

(19)

4.1. BER for Switch and Examine Combining

It is quite evident from section 2 and 3 that the performance of SSC diversity receivers can be derived as a special case \((L = 2)\) of diversity systems employing SEC. Thus for cascaded Rayleigh fading channels we deduce the unified BER for SEC systems first and then specialize the end expression to obtain BER for SSC systems in the next subsection.

By generalizing (11) for arbitrary fading channels, the PDF \( f_{\gamma, \text{sec}}(\gamma) \) for \( L \) branch SEC would be

\[
f_{\gamma, \text{sec}}(\gamma) = \begin{cases} 
  f_\gamma(\gamma) F_\gamma(\gamma) \sum_{j=0}^{L-1} F_\gamma(\gamma) & ; \gamma < \gamma_s \\
  f_\gamma(\gamma) \sum_{j=0}^{L-2} F_\gamma(\gamma) & ; \gamma \geq \gamma_s
\end{cases}
\]

(20)

where \( f_\gamma(\gamma), F_\gamma(\gamma) \) are the PDF and CDF of the corresponding fading channel SNR. For cascaded Rayleigh fading channel the PDF and CDF are given by (17) and (19) respectively.

Inserting (20) and (3) in (2), and after some rearrangement of terms, the average BER with SEC thus can be found as

\[
P_{\text{sec}} = \sum_{j=0}^{L-1} F_\gamma(\gamma) \int_0^\infty P_\gamma(\gamma) f_\gamma(\gamma) k \gamma \frac{\partial}{\partial k} F_\gamma(\gamma) f_\gamma(\gamma) k \gamma \pi
\]

(21)

Utilizing the relation \( \frac{\partial \Gamma(\beta, x)}{\partial \alpha} = -x^{\beta-1} \exp(-x) \) [31, (6.5.25)] and (A.3), the first integral is evaluated easily through integration by parts

\[
\int_0^\infty P_\gamma(\gamma) f_\gamma(\gamma) k \gamma \pi = \frac{1}{2} \left[ 1 - \frac{\Gamma(b+1)}{\alpha^b} U \left( b + 1, 2, -\frac{1}{\alpha} \right) \right]
\]

(22)

where \( U(\cdot, \cdot) \) stands for Tricomi’s confluent hypergeometric function [31, (13.1.3)]. It may be noted that (22) basically gives the BER of different binary modulations in cascaded Rayleigh fading channel (no diversity, \( L = 1 \)) in closed-form and thus avoids computation of any infinite series, common in previous literatures [21].

Unfortunately a similar closed-form expression can not be found for the second integral which has a finite integration range. That’s why for the second integral, first, we represent the conditional error probability \( P_\gamma(\gamma) \) in series form [33, (8.354.2)]
and then restate the second integral in (21) as

\[
\sum_{j=0}^{\infty} \left[ F_j(\gamma_j) \right] \cdot P_r(\gamma_j) f_j(\gamma) d\gamma = \frac{1}{2} \sum_{j=0}^{\infty} \left[ F_j(\gamma_j) \right] \cdot \frac{a^k}{21!} \sum_{j=0}^{\infty} \left[ F_j(\gamma_j) \right] \sum_{k=0}^{\infty} \left( -\frac{a}{2} \right)^{k} \gamma^{k} F_j(\gamma) d\gamma
\]

Lastly, using (1.12.1.6) [38], the integral in (24) may be evaluated as follows

\[
\int_0^{\gamma} \gamma^{\alpha} f_j(\gamma) d\gamma = \frac{2\gamma^{\alpha-1}}{\Gamma(b+k+1)} \left\{ K_\alpha \left( 2\sqrt{\gamma} \right) F_1(1;b+k+1,b+k+2;\hat{\gamma}) + \sqrt{\gamma} K_\alpha \left( 2\sqrt{\gamma} \right) F_1(1;b+k+2,b+k+2;\hat{\gamma}) \right\}
\]

where \( F_1(a,b,c;z) \) refers to the generalized hypergeometric function [33, (9.14.1)].

From (21), (22), (24), and (25), putting everything together, unified BER expression for binary modulations with SEC in a cascaded Rayleigh fading channel is obtained as

\[
P_{sec} = \frac{1}{2} \left[ 1 - \sum_{j=0}^{\infty} \left[ F_j(\gamma_j) \right] \cdot \frac{\Gamma(b+1)}{\Gamma(b+k+1)} \right]
\]

\[
+ \frac{\hat{\gamma}}{\Gamma(b)} \sum_{j=0}^{\infty} \left[ F_j(\gamma_j) \right] \sum_{k=0}^{\infty} \left( -\frac{a}{2} \right)^{k} \gamma^{k} \left\{ K_\alpha \left( 2\sqrt{\gamma} \right) F_1(1;b+k+1,b+k+2;\hat{\gamma}) + \sqrt{\gamma} K_\alpha \left( 2\sqrt{\gamma} \right) F_1(1;b+k+2,b+k+2;\hat{\gamma}) \right\}
\]

where \( \hat{\gamma} = \gamma_i / \eta \) denotes normalized threshold as usual. The final expression involves simple hypergeometric functions which can be evaluated easily and efficiently. Also the series term converges very fast, an accuracy of \( 10^{-14} \) can be achieved within first 50 terms.

4.2 BER for Switch and Stay Combining

The BER for SSC case may be obtained by putting \( L = 2 \) in (26) as

\[
P_{ssc} = \frac{1}{2} \left[ 1 - \left( 1 + F_i(\gamma_i) \right) \cdot \frac{\Gamma(b+1)}{\Gamma(b+k+1)} \right]
\]

\[
+ \frac{\hat{\gamma}}{\Gamma(b)} \sum_{i=0}^{\infty} \left( -\frac{a}{2} \right)^{i} \gamma^{i} \left\{ K_\alpha \left( 2\sqrt{\gamma} \right) F_1(1;b+k+1,b+k+2;\hat{\gamma}) + \sqrt{\gamma} K_\alpha \left( 2\sqrt{\gamma} \right) F_1(1;b+k+2,b+k+2;\hat{\gamma}) \right\}
\]
Further, by inserting values of the parameter set \( \{a, b\} \) in (27) one may obtain error rates for different modulation methods. For example, in case of BPSK \((a = 1, b = 1/2)\) the BER would be,

\[
P_{e, \text{SSC BPSK}} = \frac{1}{2} \left[ 1 - \left(1 + F_s(y_r)\right) \frac{\exp\left(\frac{1}{27}\right)}{27} \left( K\left(\frac{1}{27}\right) - K\left(\frac{1}{27}\right) \right) \right] + \frac{\gamma}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k (\gamma_{ SS C})^{k+1}}{k! (k+1/2) (k+3/2)} \left( \gamma \left( \frac{2\sqrt{\gamma}}{k+3/2} \right) F_s\left(\frac{1}{k+3/2, k+5/2, \gamma}\right) \right) \]

\[
+ \sqrt{\gamma} K\left(\frac{2\sqrt{\gamma}}{k+3/2} \right) F_s\left(\frac{1}{k+3/2, k+5/2, \gamma}\right)
\]

where \( \Gamma(1/2) = \sqrt{\pi} \) \[31, (6.1.8)\], \( \Gamma(3/2) = \sqrt{\pi}/2 \) \[31, (6.1.9)\], and

\[
U\left(\frac{3}{2}, 2, x\right) = \frac{1}{\sqrt{\pi}} \exp\left(\frac{x}{2}\right) \left[ K\left(\frac{x}{2}\right) - K\left(\frac{x}{2}\right) \right]
\]

have been used to derive (28).

### Table II. Optimum switching threshold \( \gamma^*_T \) for BPSK and DPSK operating over cascaded Rayleigh fading channel.

<table>
<thead>
<tr>
<th>L</th>
<th>BPSK ((a=1, b=1/2))</th>
<th>DPSK ((a=1, b=1))</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>9</td>
<td>0.12</td>
<td>0.12</td>
</tr>
<tr>
<td>10</td>
<td>0.24</td>
<td>0.24</td>
</tr>
<tr>
<td>11</td>
<td>0.35</td>
<td>0.35</td>
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<tr>
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<td>0.59</td>
</tr>
<tr>
<td>14</td>
<td>0.70</td>
<td>0.70</td>
</tr>
<tr>
<td>15</td>
<td>0.82</td>
<td>0.82</td>
</tr>
<tr>
<td>16</td>
<td>0.94</td>
<td>0.94</td>
</tr>
</tbody>
</table>

For finding the optimal threshold under cascading fading, we opted for an indirect numerical method as differentiation of (26) or (27) would generate infinite number of terms. The method finds minimum error probability which ensures optimality of the derived threshold value. The threshold values are made accurate to two decimal places. Detailed results for BPSK and DPSK modulation that were found through this procedure are depicted in Table II.

From Table II one finds that the optimal values follow a trend similar to single Rayleigh fading, i.e. the values are increasing with both \( \gamma \) and \( L \). Nevertheless, for a given average SNR and diversity order, \( \gamma^*_T \) values are lesser for the cascaded channel no matter whatever modulation scheme is in use.
5. RESULTS AND DISCUSSIONS

In this section we provide selected numerical and simulation results to show the generality of the newly derived results.

The BER performances of BPSK and DPSK with SSC and SEC (operating with common optimum switching threshold) in single Rayleigh fading channel are presented in Fig. 3. The BER values are shown for different orders of diversity ($L = 2, 4, \text{ and } 8$). When $L$ is greater than two ($L > 2$), error performance of a system with SEC diversity improves while for the systems using SSC diversity, the performance remains same. For comparison purpose, BER of BPSK for the no diversity case ($L = 1$), with dual diversity SC, and with SSC (or SEC with $L = 2$) when the threshold value is fixed ($\gamma_T = 3\text{dB}$) have also been shown. It is easily recognized that for SSC/SEC, systems with optimum threshold outperform those with fixed threshold. However, SC error rates are still well below any kind of SWC.

Figure 3. Average BER of BPSK and DPSK with SSC/SEC versus average SNR per bit per branch ($\bar{\gamma}$) for various values of the diversity order ($L=2, 4, \text{ and } 8$) over simple Rayleigh fading channel.
Next, in Fig. 4, we present the average BER of SSC and SEC diversity receivers operating over a cascaded Rayleigh fading channel. Error rate curves for different diversity orders ($L = 2, 4, \text{ and } 8$) are shown. For calculating BER at each SNR point ($\gamma$), optimal threshold values given in Table 2 are used. Monte Carlo simulations were performed to validate the analytical results presented in section 3 and 4. The stochastic simulation process is used to estimate the BER by counting the erroneous bits at the receiver and then dividing the count by the total number of bits passed through the system. Usually the number of bits examined at a SNR point is at least 10 times higher than the inverse of the expected error rate, i.e. to test a BER of $10^{-4}$, $10^5$ bits were examined. Further, a mean of 10 individual runs was taken for plotting to average out any variation about the mean. The simulated points are superimposed (shown by asterisk mark) on the analytical values obtained in the paper and presented in Fig. 5. The simulation and analytical values almost coincide with each other and thereby testify the correctness of the derived results.

BER for any modulation scheme is a monotonic and rapidly decreasing function of SNR realized at receiver front end. This is why we see that all the error curves in Fig. 3-5 fall exponentially. Implementation of more than one antenna, i.e. increasing the diversity order ($L$) also helps in achieving a lower error rate, especially when the SNR cannot be increased. For wireless links operating over a power-limited channel this is often the case. Switched combining provides a means of achieving this diversity gain with minimal complexity. However the amount of diversity gain depends on the type of fading the channel experiences, i.e. whether the fading can be characterized with a simple or with a cascaded distribution.

Figure 4. Average BER of BPSK and DPSK with SSC/SEC versus average SNR per bit per branch ($\gamma$) for various values of the diversity order ($L=2, 4, \text{ and } 8$) over cascaded Rayleigh fading channel.
We compare the performance of switched combining in cascaded Rayleigh channel with respect to the simple Rayleigh channel in Fig. 5. The negative effect of keyhole channels is clear. Keeping the available SNR at receiver constant, for any particular modulation, if the channel characteristics change from simple Rayleigh to cascaded Rayleigh the error rates may go up at least an order of magnitude. This may result in irreducible error floor in some quality of service (QoS) limited systems and calls for additional fading mitigation endeavours (increasing diversity branches, channel coding etc.) on system engineers’ part.

6. CONCLUSIONS

In this paper, we have derived in a unified manner, the exact BER and optimum threshold expressions of binary signalling for generalized multi-branch switched diversity systems. Two different fading channel models, namely simple and cascaded Rayleigh fading, have been studied to incorporate both fixed and mobile wireless transmission scenarios. All the BER expressions presented here are in terms of well-known functions and can be evaluated numerically with ease. Moreover, closed-form BER of binary modulations operating over cascaded Rayleigh channel without diversity is also proposed. Simulated results have been presented to verify the mathematical analysis. Further extension of our derivations seems straightforward for other modulation formats as well as for fading channels obeying different statistical distributions.
APPENDIX A: INTEGRATION INVOLVING $z^{-\alpha}$, $\exp(-\beta z)$, AND $K_{\nu}(\sqrt{z})$

The integrals of this form can be evaluated as [38, (2.16.8.4)],

$$\int_0^\infty x^{-\alpha} \exp(-\beta x) K_{\nu}(\sqrt{z}) \, dx = \frac{\beta^{\frac{\alpha}{2}}}{2} \Gamma\left(\frac{\alpha + \nu}{2}\right) \Gamma\left(\frac{\alpha - \nu}{2}\right) \exp\left(-\frac{\beta^2}{4}\right) W_{\frac{\alpha - \nu}{2}, \frac{\alpha + \nu}{2}} \left(\frac{t^2}{4\beta}\right)$$  \hspace{1cm} (A.1)

where $\Re(\alpha) > \Re(\nu)$, $\Re(\beta) > 0$. In (A.1) $W_{\nu}(\cdot)$ is Whittaker’s function, bearing a close relation [31, (13.1.33)] with Tricomi’s confluent hypergeometric function,

$$W_{\nu}(z) = \exp\left(-\frac{z}{2}\right) z^{\frac{1-n}{2}} U\left(\frac{1}{2} + \mu - \kappa, 1 + 2\mu; z\right)$$  \hspace{1cm} (A.2)

Now, combining (A.1)-(A.2) with a substitution $x = \sqrt{z}$ and writing $\alpha$ instead of $\alpha/2$, we finally have

$$\int_0^\infty z^{-\alpha} \exp(-\beta z) K_{\nu}(\sqrt{z}) \, dz = \frac{1}{2\beta^\alpha} \left(\frac{t^2}{4\beta}\right)^{\frac{\nu}{2}} \Gamma\left(\frac{\alpha + \frac{\nu}{2}}{2}\right) \Gamma\left(\frac{\alpha - \frac{\nu}{2}}{2}\right) \exp\left(-\frac{\beta^2}{4}\right)$$  \hspace{1cm} (A.3)

REFERENCES

AUTHORS’ BIOGRAPHIES

Aniruddha Chandra received his B. E. (Hons.) degree in Electronics and Communication Engineering and M. E. degree in Communication Engineering from Jadavpur University (JU), Kolkata, India in 2003 and 2005 respectively and is currently pursuing a Ph. D. there.

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