Fuzzy group decision-making with
generalized probabilistic OWA operators

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Abstract. This paper presents a new fuzzy group decision-making approach by using probabilities and ordered weighted averages (OWA). Thus, we are able to assess the problem considering probabilistic information and the attitudinal character of the decision maker. For doing so, the fuzzy generalized probabilistic OWA (FGPOWA) operator is presented. It is an aggregation operator that unifies the probability and the OWA operator considering the degree of importance that each concept has in the aggregation. It also uses fuzzy numbers that deal with uncertain environments where the available information is imprecise. Moreover, it includes a wide range of particular cases including the fuzzy generalized probabilistic aggregation, the fuzzy POW A operator and the fuzzy GOWA operator. A further generalization to this approach is presented by using quasi-arithmetic means forming the fuzzy quasi-arithmetic POW A (Quasi-FPOWA) operator. An application in fuzzy group decision making is developed by using the multi-person – FGPOWA (MP-FGPOWA) operator.

Keywords: Decision making, fuzzy numbers, OWA operator, probabilities, fuzzy group decision-making

1. Introduction

Decision making is a very common process in our life. People are always making decisions [9]. When studying the decision making process we can use a wide range of mathematical and statistical techniques. A very useful tool for doing so is the aggregation operator. In the literature, there are a lot of aggregation operators [1, 2, 11, 32, 39, 40]. For example, we can mention the probabilistic aggregation based on the use of probabilities in the analysis. Another useful aggregation operator is the ordered weighted averaging (OWA) operator [34]. It provides a parameterized family of aggregation operators between the minimum and the maximum.

Usually, when using the OWA operator we assume that the information is clearly known. However, in real world situations this may not be the case because the information may be vague or imprecise. Therefore, it is necessary to use another approach such as the use of the fuzzy numbers (FNs). By using FNs we can consider the minimum and the maximum results and the possibility that the internal values of the fuzzy interval will occur. If we extend the OWA operator to a fuzzy environment, we obtain the fuzzy OWA (FOWA) operator. Since its introduction, it has been studied by a lot of authors. For example, Chen and Chen [6] developed an extension by using induced aggregation operators. Liu [13] studied a process by using interval-valued trapezoidal fuzzy numbers. Merigó and Casanovas [18, 19], Merigó and Gil-Lafuente [24] and Wang and Luo [26] developed several generalizations by using generalized and quasi-arithmetic means. Wei [27] and Xu [31] suggested the use of harmonic means in the aggregation. Zeng and Su [52–57] introduced several distance measures by using fuzzy numbers, linguistic variables and intuitionistic fuzzy sets. Wei et al. [28] and Zhao et al.
Another interesting extension of the OWA operator is the generalized OWA (GOWA) operator [37]. It uses generalized means in the analysis. By using fuzzy information, we form the fuzzy generalized OWA (FGOWA) operator [19]. It includes the FOWA operator as a special case and a lot of other cases such as the quadratic FOWA (FOWQA) operator and the fuzzy generalized average (FGA). The FGOWA operator can be further extended by using quasi-arithmetic means obtaining the fuzzy quasi-arithmetic OWA (Quasi-FOWA) operator.

A further interesting case is the probabilistic OWA (POWA) operator. It unifies the probability and the OWA operator in the same formulation considering the degree of importance that each concept has in the aggregation. Note that the use probabilities and OWA operators in the same formulation has already been considered by other authors. Especially, it is worth noting the concept of immediate probabilities [8, 14, 36, 38]. Its main advantage is that it can consider the maximum and the minimum can represent in a more complete way the information by a lot of authors [7, 12]. Its main advantage is that it can represent in a more complete way the information.

2. Preliminaries

In this Section we briefly review some basic preliminaries regarding the fuzzy numbers (FNs), the POWA operator, the FOWA operator and the FGOWA operator.

2.1 Fuzzy numbers

The FNs [3, 42–48] have been studied and applied by a lot of authors [7, 12]. Its main advantage is that it can represent in a more complete way the information because it can consider the maximum and the minimum and the possibility that the internal values may occur.

Definition 1. Let $R$ be $(-\infty, \infty)$, the set of all real numbers. A FN is a fuzzy subset [41] of $R$ with membership function $m: R \rightarrow [0, 1]$ satisfying the following conditions:

- Normality: There exists at least one number $a_0 \in R$ such that $m(a_0) = 1$.
- Convexity: $m(t)$ is nondecreasing on $(-\infty, a_0]$ and nonincreasing on $[a_0, \infty)$.

The FN can be considered as a generalization of the interval number. In the literature, we find a wide range of FNs such as triangular FNs (TFN), trapezoidal FNs (TpFN), interval-valued FNs, intuitionistic FNs, generalized FNs, type 2 FNs and more complex structures.

For example, a trapezoidal FN (TpFN) of a universe of discourse $R$ can be characterized by a trapezoidal membership function $A = (a, \bar{a})$ such that

$$g(a) = a_1 + a(a_2 - a_1),$$
$$h(a) = a_1 - a(a_2 - a_1).$$

(1)
where \( \alpha \in [0, 1] \) and parameterized by \((a_1, a_2, a_3, a_4)\) where \(a_1 \leq a_2 \leq a_3 \leq a_4\) are real values. Note that if \(a_1 = a_2 = a_3 = a_4\), then the FN is a crisp value and if \(a_2 = a_3\), the FN is represented by a triangular FN (TFN). Note that the TPFN can be parameterized by \((a_1, a_2, a_4)\).

The TPFN can also be represented in the following way:

\[
m(t) = \begin{cases} 
1 & \text{if } t = [a_2, a_3], \\
\frac{t - a_1}{a_3 - a_1} & \text{if } t \in [a_1, a_2], \\
\frac{a_4 - t}{a_4 - a_3} & \text{if } t \in [a_3, a_4], \\
0 & \text{otherwise},
\end{cases}
\]

where \(a_1, a_2, a_3, a_4 \in \mathbb{R}\) and \(a_1 \leq a_2 \leq a_3 \leq a_4\). Note that in this paper, especially when developing the illustrative example, we denote the TPFN as \((a_1, a_2, a_3, a_4)\). Furthermore, we will denote all the FNs in a general way as \(\tilde{a}\). Thus, by providing this abbreviation we will be able to represent all the FNs in the same formulation.

In the following, we are going to review the FNs arithmetic operations as follows. Let \(A\) and \(B\) be two TPFNs, where \(A = (a_1, a_2, a_3, a_4)\) and \(B = (b_1, b_2, b_3, b_4)\), such that:

1. \(A + B = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4)\),
2. \(A - B = (a_1 - b_1, a_2 - b_2, a_3 - b_3, a_4 - b_4)\),
3. \(A \times k = (ka_1, ka_2, ka_3, ka_4)\), for \(k > 0\).

Among the wide range of methods existing in the literature for ranking FNs, we recommend the use of the methods commented by Merigó [15] such as the literature for ranking FNs, we recommend the use of POWA.

**Definition 2.** A POWA operator of dimension \(n\) is a mapping \(\text{POWA}: \mathbb{R}^n \longrightarrow \mathbb{R}\) that has an associated weighting vector \(W\) of dimension \(n\) such that \(w_j \in [0, 1]\) and \(\sum_{j=1}^{n} w_j = 1\), according to the following formula:

\[
\text{POWA}(a_1, \ldots, a_n) = \sum_{j=1}^{n} \beta_j b_j
\]

where \(b_j\) is the \(j\)th largest of the \(a_i\), each argument \(a_i\) has an associated probability \(p_i\) with \(\sum_{j=1}^{n} p_j = 1\) and \(p_j \in [0, 1]\), \(b_j = \beta w_j + (1 - \beta) p_j\) with \(\beta \in [0, 1]\) and \(p_j\) is the probability \(p_j\) ordered according to the \(j\)th largest of the \(a_i\).

2.3. The fuzzy OWA operator

The FOWA operator is an extension of the OWA operator that uses uncertain information represented in the form of FNs. The FOWA operator provides a parameterized family of aggregation operators that include the fuzzy maximum, the fuzzy minimum and the fuzzy average criteria, among others. It is defined as follows.

**Definition 3.** Let \(\Psi\) be the set of FNs. A FOWA operator of dimension \(n\) is a mapping \(\text{FOWA}: \Psi^n \longrightarrow \Psi\) that has an associated weighting vector \(W\) of dimension \(n\) with \(w_j \in [0, 1]\) and \(\sum_{j=1}^{n} w_j = 1\), such that:

\[
\text{FOWA}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \sum_{j=1}^{n} w_j b_j,
\]

where \(b_j\) is the \(j\)th largest of the \(\tilde{a}_i\), and the \(\tilde{a}_i\) are FNs.

The FOWA operator accomplishes similar properties than the OWA operator such as the distinction between the descending FOWA (DFOWA) operator and the ascending FOWA (AFOWA) operator and so on.

2.4. The fuzzy generalized OWA operator

The fuzzy generalized OWA (FGOWA) operator [19] is an extension of the GOWA operator that uses uncertain information in the aggregation represented in the form of FNs. The reason for using this operator is that sometimes, the uncertain factors that affect our decisions are not clearly known and in order to assess the problem we need to use FNs in order to consider the different uncertain results that could happen in the future. It can be defined as follows.

**Definition 4.** Let \(\Psi\) be the set of FNs. An FGOWA operator of dimension \(n\) is a mapping \(\text{FGOWA}: \Psi^n \longrightarrow \Psi\) that has an associated weighting vector \(W\) of dimension \(n\) such that the sum of the weights is 1 and \(w_j \in [0, 1]\), then:
The fuzzy generalized probabilistic OWA (FGOWA) operator is an aggregation operator that provides a unified formulation between the probability and the OWA operator in an uncertain environment that can be assessed with FNs. It also uses generalized means providing a general framework that includes a wide range of particular cases. Its main advantage is that it can deal with subjective or objective information and with the attitudinal character of the decision maker in a unified framework. Thus, it is possible to evaluate the probabilities according to our degree of optimism or pessimism being able to under or overestimate them. It can be defined as follows.

**Definition 5.** Let $\Psi$ be the set of FNs. A FGOWA operator of dimension $n$ is a mapping $FGOWA: \Psi \rightarrow \Psi$ that has an associated weighting vector $\hat{w}$ of dimension $n$ with $\hat{w}_i \in [0,1]$ and $\sum_{i=1}^{n} \hat{w}_i = 1$, and a probabilistic vector $P$ of dimension $n$ with $p_j \in [0,1]$ and $\sum_{i=1}^{n} p_j = 1$, such that:

$$FGOWA(\tilde{a}_1, \tilde{a}_2, \ldots \tilde{a}_n) = \hat{A} \left( \sum_{j=1}^{m} \hat{w}_j b_j \right)^{1/\lambda} + \left(1 - \hat{A} \right) \left( \sum_{j=1}^{m} \hat{w}_j b_j \right)^{1/\lambda}$$

where $b_j$ is the $j$th largest $\tilde{a}_j$, the $\tilde{a}_j$ are FNs, $\hat{A} \in [0,1]$, and $\lambda$ is a parameter such that $\lambda \in (-\infty, \infty)$.

Note that different types of FNs could be used in the aggregation including TFNs, TpFNs, UFNs, type-2 FNs and more complex structures. Often, when dealing with FNs in the FGOWA operator, it is not clear which FN is higher. Therefore, we need to establish criteria for ranking the FNs. For simplicity, we recommend the following criteria. First, we analyze if there is an order between the FNs. That is, if the lowest value of an interval is higher than the highest value of another interval: $A = (a_1, a_2, a_3)$ is strictly higher than $C = (c_1, c_2, c_3)$ if $a_1 > c_1$. Note that in this case we can guarantee 100% that $A > C$. If not, we select the FN with the highest value in its highest membership level, usually, when $\alpha = 1$. Note that if the membership level $\alpha = 1$ is an interval, then, we calculate the average of the interval.

Note also that it is possible to distinguish between the descending FGOWA (DFGOWA) and the ascending FGOWA (AFGOWA) operator by using $w_j = w_{n-j+1}$, where $w_j$ is the $j$th weight of the DFGOWA and $w_{n-j+1}$ the $j$th weight of the AFGOWA operator. Note that this reordering is in the OWA aggregation. However, it is possible to consider a more general reordering process by using $p_j = p_{n-j+1}$. In this case, we consider descending and ascending orders in the OWA and in the fuzzy probabilistic aggregation (FPA) at the same time.

Another interesting transformation is found [35] by using $w_1 w_n (1 + w_2) / (m - 1)$. Furthermore, we can also analyze situations with buoyancy measures [35]. In this case, we assume that $w_i \geq w_j$, for $i < j$. Note that it is also possible to consider a stronger case known as buoyancy measure extensive where $w_i > w_j$, for $i < j$. And we can also consider the contrary case, that is, $w_i \leq w_j$, for $i < j$, and the extensive measure $w_i < w_j$, for $i < j$. The FGOWA operator is monotonic, bounded and idempotent.

Additionally, if the weights of the probabilities and the OWAs are also fuzzy, then, we have to establish a method for dealing with these fuzzy weights. Note that in these situations it is very common that $W = \sum_{j=1}^{m} \hat{w}_j \neq 1$ and $P = \sum_{j=1}^{m} p_j \neq 1$. Thus, a very useful method for dealing with these situations is by using:

$$FGOWA(\tilde{a}_1, \tilde{a}_2, \ldots \tilde{a}_n) = \frac{\hat{A}}{W} \left( \sum_{j=1}^{m} \hat{w}_j b_j \right)^{1/\lambda} + \left(1 - \frac{\hat{A}}{W} \right) \left( \sum_{j=1}^{m} \hat{w}_j b_j \right)^{1/\lambda}$$

A further interesting issue is the measures for characterizing the weighting vector. Following a similar methodology as it has been developed for the OWA operator [19, 35] we could formulate the orness measure (attitudinal character) and the entropy of dispersion. The orness measure of the FGOWA operator is formulated as follows:
It is defined as follows.

\[
\alpha(P) = \hat{\beta} \left( \sum_{j=1}^{n} \frac{a_j}{\sum_{i=1}^{n} \frac{n}{n-1}} \right)^{1/\beta} + (1 - \hat{\beta}) \sum_{j=1}^{n} \beta_j \left( \frac{n - j}{n - 1} \right)^{1/\beta},
\]

where \( \beta_j \) represents the probabilistic weights reordered according to the values of the arguments \( b_j \). Note that if \( \hat{\beta} = 1 \), we get the orness measure of the FGOWA operator and if \( \hat{\beta} = 0 \), the orness measure of the fuzzy generalized probabilistic aggregation (FGPA).

However, it is worth noting that the orness measure seen from the perspective of the attitudinal character, can also be formulated assuming that the probabilistic aggregation is neutral. Thus, we can establish a fixed value for this part of the aggregation, for example, assuming it is 0.5.

\[
\alpha(P) = \hat{\beta} \left( \sum_{j=1}^{n} \frac{a_j}{\sum_{i=1}^{n} \frac{n}{n-1}} \right)^{1/\beta} + (1 - \hat{\beta}) \times 0.5,
\]

In this case, if \( \hat{\beta} = 0 \), the attitudinal character is neutral, that is, 0.5. Note that we could also use other values instead of 0.5 depending on the assumptions we make regarding neutrality since a lot of decision making theories may establish at a different level, especially when dealing with utility theory.

The entropy of dispersion [34] measures the amount of information being used in the aggregation. For the FGPOWA operator, it is defined as follows.

\[
H(P) = - \hat{\beta} \sum_{j=1}^{n} a_j \ln(a_j) + (1 - \hat{\beta}) \sum_{j=1}^{n} \beta_j \ln(\beta_j).
\]

Note that \( \beta_j \) is the \( i \)-th weight of the FPA aggregation. As we can see, if \( \hat{\beta} = 1 \), we get the entropy of dispersion of the FGOWA operator and if \( \hat{\beta} = 0 \), we extend the classical Shannon entropy [25] by using fuzzy generalized probabilities.

### 3.2. Families of FGPOWA operators

Another interesting issue is the analysis of different families of FGPOWA operators by analyzing particular cases in the coefficient \( \beta \), in the parameter \( \lambda \) and \( \delta \) and in the weighting vector \( W \). If we analyze the coefficient \( \beta \), we get the following:

- If \( \beta = 1 \), we get the FGOWA operator.
- If \( \beta = 0 \), we get the fuzzy generalized probabilistic aggregation (FGPA).

The more \( \beta \) approaches to 1, the more importance we give to the FGOWA operator, and vice versa. If we analyze different values of the parameter \( \lambda \) and \( \delta \), we obtain another group of particular cases.

- The FPPOWA operator: When \( \lambda = 1 \).
- The geometric FPPOWA (GFPOWA) operator: When \( \lambda = 0 \).
- The harmonic FPPOWA (HFPOWA) operator: When \( \lambda = 0 \).
- The quadratic FPPOWA (FPQOWA) operator: When \( \lambda = 2 \).
- The fuzzy maximum: When \( \lambda \rightarrow \infty \).
- The fuzzy minimum: When \( \lambda \rightarrow -\infty \).

Moreover, it is possible to consider other families by using different values in the parameter \( \lambda \) and \( \delta \). For example, we can form the following aggregation operators:

- If \( \lambda = 2 \) and \( \delta = 1 \), we get the fuzzy probabilistic ordered weighted quadratic average (FPQOWA).

That is:

\[
FGPOWA (\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) = \hat{\beta} \left( \sum_{j=1}^{n} \frac{a_j}{\sum_{i=1}^{n} \frac{n}{n-1}} \right)^{1/2} + (1 - \hat{\beta}) \sum_{j=1}^{n} \beta_j a_j.
\]

- If \( \lambda = 1 \) and \( \delta = 2 \), then, we obtain the fuzzy probabilistic quadratic ordered weighted averaging (FPQOWA) operator:

\[
FGPOWA (\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) = \hat{\beta} \left( \sum_{j=1}^{n} \frac{a_j}{\sum_{i=1}^{n} \frac{n}{n-1}} \right)^{1/2} + (1 - \hat{\beta}) \sum_{j=1}^{n} \beta_j a_j.
\]

- If \( \lambda = 1 \) and \( \delta = 3 \), we obtain the fuzzy probabilistic cubic ordered weighted averaging (FPCOWA) operator:

\[
FGPOWA (\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) = \hat{\beta} \left( \sum_{j=1}^{n} \frac{a_j}{\sum_{i=1}^{n} \frac{n}{n-1}} \right)^{1/3} + (1 - \hat{\beta}) \sum_{j=1}^{n} \beta_j a_j.
\]
Note that other families of FGPOWA operators could be studied following [15, 16, 35, 40].

4. The Quasi-FPOWA operator

The FGPOWA operator can be further generalized by using quasi-arithmetic means [1, 10, 20–22]. Thus, we obtain the Quasi-FPOWA operator that can be defined as follows.

Definition 6. Let $\Psi$ be the set of FNs. A Quasi-FPOWA operator of dimension $n$ is a mapping $QFPOWA: \Psi^n \rightarrow \Psi$ that has an associated weighting vector $W$ of dimension $n$ with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, and a probabilistic vector $P$ of dimension $n$ with $p_i \in [0, 1]$ and $\sum_{i=1}^{n} p_i = 1$, such that:

$$QFPOWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \tilde{b} = (1 - \tilde{b})^{-1} (\sum_{i=1}^{n} \tilde{a}_i b_i) + \tilde{b} (\sum_{i=1}^{n} \tilde{a}_i b_i),$$

(34)

where $b_i$ is the $i$th largest $a_i$, the $\tilde{a}_i$ are FNs, $\beta \in [0, 1]$, and $\tilde{b}$ is a strictly continuous monotonic function.

Note that if $\tilde{b}(b) = b^\beta$ and $\delta(b) = b^\delta$, then, the Quasi-FPOWA operator becomes the FGPOWA operator. The Quasi-FPOWA operator includes a wide range of families of aggregation operators in a similar way as the FGPOWA operator. Moreover, it also includes a lot of other extensions could be developed by using Choquet integrals [21, 23, 24], continuous aggregations [60, 61, 64, 65], logarithmic aggregation operators [59, 63] and power averages [29, 62].

5. Group decision-making with the FGPOWA

The FGPOWA and the Quasi-FPOWA operators can be applied in a lot of fields including statistics, economics and engineering. The main reason is that they include the classical probabilistic aggregation (expected value) and the OWA operator as particular cases. Therefore, all the previous studies that use the probability or the OWA operator can be revised an extended with this new approach.

In this Section, we analyze the use of the FGPOWA operator in fuzzy multi-person problems. Observe that the FGPOWA operator is able to create a unified decision making process between risk and uncertain environments by integrating probabilities and OWA operators in the same formulation. We analyze a fuzzy multi-person decision-making problem in the selection of strategies. We focus on the selection of general strategies in an enterprise. As these decisions are very complex and relevant, it requires the opinion of several experts that assess the problem. Thus, by using a multi-person analysis we can assess the information in the most efficient way.

The procedure to select general strategies with the FGPOWA operator in fuzzy multi-person decision-making is described as follows.

Step 1: Let $A = A_1, A_2, \ldots, A_m$ be a set of finite alternatives, $S = S_1, S_2, \ldots, S_m$, a set of finite states of nature (or attributes), forming the payoff matrix $(\tilde{a}_{ij})_{m \times n}$. Let $E = e_1, e_2, \ldots, e_p$ be a finite set of decision-makers. Let $U = (u_1, u_2, \ldots, u_m)$ be the weighting vector of the decision-makers such that $\sum_{i=1}^{m} u_i = 1$ and $u_i \in (0, 1)$. Each decision-maker provides his own payoff matrix $(\tilde{a}_{ij}^{(k)})_{m \times n}$.

Step 2: Calculate the weighting vector $\tilde{P} = \tilde{b} \times \tilde{W} = (1 - \tilde{b}) \times \tilde{P}$ to be used in the FGPOWA aggregation. Note that $W = (w_1, w_2, \ldots, w_n)$ such that $\sum_{i=1}^{n} w_i = 1$ and $w_i \in [0, 1]$ and $P = (p_1, p_2, \ldots, p_m)$ such that $\sum_{i=1}^{m} p_i = 1$ and $p_i \in [0, 1]$.

Step 3: Aggregate the information of the decision-makers $E$ using the weighting vector $U$. In this example, we use the fuzzy weighted average (FWA). The result is the collective payoff matrix $(\tilde{a}_{ij}^{(U)})_{m \times n}$. Thus, $\tilde{a}_{ij}^{(U)} = \sum_{k=1}^{p_i} \tilde{a}_{ij}^{(k)}$. Note that we can use different types of aggregation operators instead of the FWA to aggregate this information such as different types of FPOWA operators.

Step 4: Calculate the aggregated results using the FGPOWA operator explained in Eq. (6). Consider different families of FGPOWA operators as described in Section 3.
Step 5: Adopt decisions according to the results found in the previous steps. Select the alternative(s) that provides the best result(s) and establish a ranking of the alternatives.

The previous fuzzy multi-person decision process can be summarized using the following aggregation operator that we call the multi-person – FGPOWA (MP-FGPOWA) operator.

Definition 7. A MP-FGPOWA operator is a mapping $MP-FGPOWA: \Psi^3 \times \Psi^3 \rightarrow \Psi$ that has a weighting vector $U$ of dimension $p$ with $\sum_{i=1}^{p} \tilde{a}_i = 1$ and $\tilde{b}_i \in [0, 1]$ and a weighting vector $W$ of dimension $n$ with $\sum_{j=1}^{n} \tilde{w}_j = 1$ and $\tilde{v}_j \in [0, 1]$, such that:

$$MP - FGPOWA((\tilde{a}_1, \ldots, \tilde{a}_n), (\tilde{b}_1, \ldots, \tilde{b}_m)) = \sum_{i=1}^{n} \tilde{p}_i \tilde{b}_i,$$

where $\tilde{p}_i$ is the $i$th largest of the $\tilde{a}_i$, each argument $\tilde{a}_i$ has an associated probability $\tilde{b}_i$ with $\sum_{i=1}^{m} \tilde{b}_i = 1$ and $\tilde{b}_i \in [0, 1]$, $\tilde{p}_i = \tilde{b}_i \tilde{p}_i + (1 - \tilde{b}_i)\tilde{p}_{i-1}$ with $\tilde{p}_0 \in [0, 1]$ and $\tilde{p}_m$ is the probability $\tilde{b}_m$ ordered according to $\tilde{b}_i$, that is, according to the $i$th largest of the $\tilde{a}_i$, $\tilde{a}_i = \sum_{k=1}^{n} \tilde{d}_i$, $\tilde{d}_i$ is the argument variable provided by each person (or expert).

Note that the MP-FGPOWA operator has similar properties to those explained in Section 3, such as the distinction between descending and ascending orders, and so on.

The MP-FGPOWA operator includes a wide range of particular cases following the methodology explained in Section 3. Thus, it includes:

- The multi-person – fuzzy probabilistic aggregation (MP-FPA) operator.
- The multi-person – FOWA operator.
- The multi-person – fuzzy arithmetic mean (MP-FAM) operator.
- The multi-person – fuzzy generalized probabilistic aggregation (MP-FGPA).
- The multi-person – FGOWA (MP-FGOWA).
- The multi-person – fuzzy generalized mean.

Note that it is possible to consider other types of aggregation operators to aggregate the experts opinions, although in Definition 6 we assume that the opinions are aggregated by using FWA operators.

6. Illustrative example

In the following, we present a numerical example of the new approach in a fuzzy multi-person decision-making problem regarding the selection of strategies. We focus on an economic problem regarding the general strategy of an enterprise for the next year. Note that there are many other types of decision making problems in the literature [4, 5, 17, 49–51, 66].

Step 1: Assume an enterprise that it is planning its general strategy for the next year and considers 5 possible strategies.

- $A_1$ = Expand to the Asian market.
- $A_2$ = Do not make any expansion.

In order to evaluate these strategies, the company is assessed by three independent experts. These experts consider that the key factor is the economic situation of the world economy for the next period. They consider 5 possible states of nature that could happen in the future:

- $S_1$ = Very bad economic situation.
- $S_2$ = Poor economic situation.
- $S_3$ = Regular economic situation.
- $S_4$ = Good economic situation.
- $S_5$ = Very good economic situation.

Each expert gives different opinions. The results of the available strategies, depending on the state of nature $S_i$ and the alternative $A_k$ that the decision-maker chooses, are shown in Tables 1, 2 and 3.

Step 2: In this problem, we assume the following weighting vector for the three experts: $U=(0.4, 0.3, 0.3)$. The experts assume the following weighting vector for the FOWA: $W=(0.1, 0.1, 0.2, 0.3, 0.3)$. The three groups assume the following fuzzy probabilistic information for each state of nature: $P=(0.1, 0.2, 0.3, 0.3, 0.1)$. Note that for simplicity we assume that the weights are not fuzzy so it is easier to deal with them.

Note that the FOWA operator has an importance of 30% while the FPA has 70% in this particular example.

Step 3: First, we aggregate the information of the three experts into one collective matrix that represents the information of all the experts of the problem. The results are shown in Table 4.

Step 4: With this information, we can aggregate the expected results for each state of nature in order to make a decision. For this, we use Eq. (6) to calculate the FGPOWA aggregation. In Table 5, we present...
Table 1
Decision matrix provided by expert 1

<table>
<thead>
<tr>
<th></th>
<th>S₁</th>
<th>S₂</th>
<th>S₃</th>
<th>S₄</th>
<th>S₅</th>
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<td>(70,80,90)</td>
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<tr>
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Table 2
Decision matrix provided by expert 2

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<th>S₃</th>
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<th>S₅</th>
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Table 3
Decision matrix provided by expert 3

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<th>S₅</th>
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Table 4
Collective results

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<th>S₅</th>
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<tr>
<td>A₁</td>
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<td>(50,60,70)</td>
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Table 5
Aggregated results by different fuzzy aggregation operators

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<tr>
<th></th>
<th>FAM</th>
<th>FPA</th>
<th>FOWA</th>
<th>FPOWA</th>
<th>A-FPA</th>
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<tr>
<td>A₁</td>
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<td>(64,74,84)</td>
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<td>(60.70,79.79,89.79)</td>
<td>(83.63,93.63,103.63)</td>
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<tr>
<td>A₂</td>
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<td>(46.76,56.66)</td>
<td>(36.34,46.34,56.34)</td>
<td>(44.18,54.18,64.18)</td>
<td>(46.66,56.66,66.66)</td>
</tr>
<tr>
<td>A₃</td>
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<td>(66.2,76.2,86.2)</td>
<td>(58.48,68.48,78.48)</td>
<td>(63.86,73.86,83.86)</td>
<td>(65.1,75.1,85.1)</td>
</tr>
<tr>
<td>A₄</td>
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<td>(57.5,67.5,77.5)</td>
<td>(54.1,64.1,74.1)</td>
<td>(56.46,66.46,76.46)</td>
<td>(57.4,67.4,77.4)</td>
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</table>

7. Conclusions

The FGPOWA operator has been presented. It is an aggregation operator that unifies the FGPA and the FGOWA operator in the same formulation considering the degree of importance that each concept has in the aggregation. It also uses generalized means providing a general framework that includes a wide range of particular cases including the FPOWA operator and the FPWAQA operator. By using fuzzy information it is able to deal with uncertain environments where the information is vague or imprecise. We have further generalized this approach by using quasi-arithmetic means forming the Quasi-FPOWA operator.

We have also studied the applicability of this new approach in a fuzzy group decision-making problem regarding the selection of general strategies in a company. We have used the MP-FGPOWA operator in order to assess the opinion of several experts in the analysis. We have seen that each particular case of FGPOWA operator may lead to different results and decisions. The main advantage of this approach in decision making is that it integrates the attitudinal character of the decision maker and the probabilistic information in the same formulation and considering the degree of importance that each concept has. Thus, it is possible to underestimate or overestimate the neutral probabilistic information according to our attitude. Thus, the FGPOWA permits to be more risk averse or risk taker than the normal expected value. This is useful because in risky and uncertain environments we do not know what is going to happen in the future. Therefore, it is important to consider the normal situations that should occur but also any possibility from the minimum to the maximum.

Note that the applicability of the FGPOWA operator is very broad because all the previous studies that
use the probability or the OWA operator can be revised and extended with this new approach. For example, it can be implemented in decision theory, economics, business administration, engineering, soft computing, physics and biology.

In future research, we expect to develop further developments by adding more concepts in the analysis such as the use of distance measures, induced aggregation operators, moving averages, weighted averages and more complex structures.

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References


