Multiple solutions for discrete boundary value problems

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Abstract

A recent multiplicity theorem for the critical points of a functional defined on a finite–dimensional Hilbert space, established by Ricceri, is extended. An application to Dirichlet boundary value problems for difference equations involving the discrete p–Laplacian operator is presented.

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1 Introduction

In the present paper we will deal with the following boundary value problem, with homogeneous Dirichlet conditions, for a difference equation, depending on a real parameter λ:

$$(P_\lambda)\ \begin{cases} -\Delta_p x(k-1) = \lambda f(k, x(k)) & \text{for every } k \in [1,T] \\ x(0) = x(T+1) = 0 \end{cases},$$

where $T \geq 2$ is an integer, $[1,T]$ denotes the discrete interval $\{1, 2, \dots T\}$, $p > 1$ is a real number, Δ_p is the *discrete p–Laplacian operator* defined by

$$\Delta_p x(k-1) = \Delta \left[|\Delta x(k-1)|^{p-2} \Delta x(k-1) \right]$$

and f is a continuous function defined on $[1,T] \times \mathbb{R}$ (see Section 3 below for details).

Under convenient assumptions on the function f, we will prove the existence of a positive λ^* such that the problem (P_{λ^*}) admits at least three solutions (see Theorem 6 below).

Boundary value problems for difference equations have been extensively studied (see the monographs of Lakshmikantham and Trigiante [9] and of Agarwal [1]): the classical theory of difference equations employs numerical analysis and features from the linear and nonlinear operator theory, such as fixed point methods; we remark that, usually, the application of the fixed point methods yields existence results only.

Recently, although, many new results have been established by applying variational methods: we recall here the works of Agarwal, Perera and O'Regan [2], [3], Cai, Guo and Yu [4], Cai and Yu [5], Faraci and Iannizzotto [6], Guo and Ma [7], Jiang and Zhou [8], Mihăilescu, Rădulescu and Tersian [10]; the variational approach represents and important advance as it allows to prove multiplicity results as well.

In all the aforementioned papers, discrete boundary value problems involving a variety of operators and boundary conditions are studied in a variational framework: solutions are seen as critical points of a convenient energy functional, defined on a function space; in general, such function spaces have *finite dimension*, which makes things easier (in comparison with the variational methods for differential equations).

In the present paper, we study the problem (P_λ) following a variational approach, based on a recent result of Ricceri (see [12]): such result assures the existence of at least three critical points for a certain class of functionals defined on a *finite–dimensional* normed space.

Thus, Ricceri's result is suitable for applications in the field of difference equations: such application yields a multiplicity result for a discrete boundary value problem of the type (P_λ) involving the discrete Laplacian operator $(p = 2)$.

In the present paper, we extend Ricceri's abstract result (see Theorem 3 below) and its application (see Theorem 6 below) to the case of the p–Laplacian, for any $p > 1$, and provide some new information about the intrinsic properties of the function space involved: namely, we establish the precise embedding constants of the function space involved into the space $\mathbb{R}^T$ with the maximum norm (see Lemma 4 below), improving a previous result of Jiang and Zhou [8].

The paper is organized as follows: in Section 2 we state and prove our abstract result; in Section 3 we apply it to the problem (P_λ); in Section 4 we discuss some limit cases and give examples.

2 The abstract result

Before introducing our result, let us recall, for the convenience of the reader, a recent theorem of Ricceri (see [12], Theorem A or [11], Theorem 1) which will be employed in our proof.

Theorem 1 *Let (X, τ) be a Hausdorff space and $\Phi, J : X \to \mathbb{R}$ be functionals; moreover, let M be the (possibly empty) set of all the global minimizers of J and define*

$$\alpha = \inf_{x \in X} \Phi(x),$$

$$\beta = \begin{cases} \inf\limits_{x \in M} \Phi(x) & \text{if } M \neq \emptyset \\ \sup\limits_{x \in X} \Phi(x) & \text{if } M = \emptyset \end{cases}.$$

Assume that the following conditions are satisfied:

(1.1) *for every $\mu > 0$ and every $\rho \in \mathbb{R}$ the set $\{x \in X \ : \ \Phi(x) + \mu J(x) \leq \rho\}$ is sequentially compact (if not empty);*

(1.2) *$\alpha < \beta$.*

Then, at least one of the following conditions holds:

(1.3) *there exists a continuous mapping $h : (\alpha, \beta) \to X$ with the following property: for every $t \in (\alpha, \beta)$, one has*

$$\Phi(h(t)) = t$$

and for every $x \in \Phi^{-1}(t)$, $x \neq h(t)$

$$J(x) > J(h(t));$$

(1.4) *there exists $\mu^* > 0$ such that the functional $\Phi + \mu^* J$ admits at least two global minimizers in X.*

We will also use the following consequence of the finite–dimensional version of the Mountain Pass Theorem (see Struwe [13], p. 74). Let $C^1(X, \mathbb{R})$ denote the set of functionals that are differentiable and whose derivatives are continuous on X.

Theorem 2 *Let $(X, \|\cdot\|)$ be a Banach space, $\dim(X) < \infty$, and $E \in C^1(X, \mathbb{R})$ be a coercive functional having at least two strict local minimizers $x_0, x_1 \in X$. Then, E has a critical point $x_2 \in X \setminus \{x_0, x_1\}$.*

Now we can introduce our abstract result, which is a simple extension of the main result of Ricceri [12]: here, an arbitrary real number $p > 1$ replaces 2 (we include the proof for the sake of completeness).

Theorem 3 *Let $(X, \|\cdot\|)$ be a Banach space, $\dim(X) < \infty$, $p > 1$ a real number such that the functional $x \mapsto \|x\|^p$ is continuously Gâteaux differentiable in X, $J \in C^1(X, \mathbb{R})$, $\bar{x} \in X$ and $r, s \in \mathbb{R}$ with $0 < r < s$. Assume that the following conditions are satisfied:*

(3.1) $\displaystyle \liminf_{\|x\| \to +\infty} \frac{J(x)}{\|x\|^p} \geq 0$;

(3.2) $\displaystyle \inf_{x \in X} J(x) < \inf_{\|x - \bar{x}\| \leq s} J(x)$;

(3.3) $\displaystyle J(\bar{x}) \leq \inf_{r \leq \|x - \bar{x}\| \leq s} J(x)$.

Then, there exists $\lambda^* > 0$ such that the functional

$$x \mapsto \frac{\|x - \bar{x}\|^p}{p} + \lambda^* J(x)$$

admits at least three critical points in X.

Proof. We are going to apply Theorem 1: so, we denote by τ the norm topology on X and define a continuous functional Φ by putting for every $x \in X$

$$\Phi(x) = \begin{cases} \|x - \bar{x}\|^p & \text{if } \|x - \bar{x}\| < r \\ r^p & \text{if } r \leq \|x - \bar{x}\| \leq s \\ \|x - \bar{x}\|^p - s^p + r^p & \text{if } s < \|x - \bar{x}\| \end{cases} .$$

First, we prove the inequality

(1) $\beta > r^p$,

distinguishing two cases:

- if $M \neq \emptyset$, since M is closed and Φ is coercive, there is some $\bar{y} \in M$ such that $\Phi(\bar{y}) = \beta$, which, by (3.2), implies that $\|\bar{y} - \bar{x}\| > s$, in particular

$$\beta = \|\bar{y} - \bar{x}\|^p - s^p + r^p > r^p;$$

- if $M = \emptyset$, clearly $\beta = +\infty$.

Now we prove that all the assumptions of Theorem 1 hold in the present case, starting with (1.1): by (3.1) we get for every $\mu > 0$

$$\lim_{\|x\| \to +\infty} [\Phi(x) + \mu J(x)] = +\infty,$$

which implies that for every $\rho \in \mathbb{R}$ the corresponding sublevel set of $\Phi + \mu J$ is bounded and closed; hence, such set is (sequentially) compact, if not empty.

In order to prove that (1.2) is satisfied, we observe that

$$\inf_{x \in X} \Phi(x) = 0$$

and we invoke (1).

By Theorem 1, either (1.3) or (1.4) holds: actually, we will prove that (1.4) is true, arguing by contradiction.

Assume that (1.4) is false: then, (1.3) must be satisfied, so let the continuous mapping $h : (0, \beta) \to X$ be defined as above; by using (1), it is easily seen that

$$\|h(t) - \bar{x}\| < r \text{ iff } t < r^p,$$

$$r \leq \|h(r^p) - \bar{x}\| \leq s,$$

$$\|h(t) - \bar{x}\| > s \text{ iff } t > r^p,$$

which contradicts the continuity of h.

By (1.4), there exists $\mu^* > 0$ such that the functional $\Phi + \mu^* J$ has at least two global minimizers $x_0, x_1 \in X$ ($x_0 \neq x_1$): we prove that

$$(2) \qquad \|x_i - \bar{x}\| < r \text{ or } \|x_i - \bar{x}\| > s \ (i = 0, 1),$$

arguing again by contradiction; indeed, if $r \leq \|x_i - \bar{x}\| \leq s$, by (3.3) we obtain

$$\Phi(x_i) + \mu^* J(x_i) = r^p + \mu^* J(x_i) > \mu^* J(x_i) \geq \mu^* J(\bar{x}) = \Phi(\bar{x}) + \mu^* J(\bar{x}),$$

against the fact that x_i is a global minimizer for $\Phi + \mu^* J$.

Set

$$\lambda^* = \frac{\mu^*}{p}.$$

From (2) and the definition of Φ it follows that both x_0 and x_1 are *local* minimizers of the functional $E \in C^1(X, \mathbb{R})$ defined for all $x \in X$ by putting

$$E(x) = \frac{\|x - \bar{x}\|^p}{p} + \lambda^* J(x).$$

We prove that E has at least one critical point $x_2 \in X \setminus \{x_0, x_1\}$, considering two cases:

- if both x_0, x_1 are *strict* local minimizers of E, an application of Theorem 2 gives the desired result;
- if either x_0 or x_1 is not a strict local minimizer, E obviously admits infinitely many local minimizers (in particular, critical points) at the same level.

Thus, the proof is complete. □

Some comments are now in order: in the proof, we have used the fact that X has finite dimension (in proving (1.1)); Ricceri has shown that, if the dimension of X is infinite, the conclusion of Theorem 3 does not hold for $p = 2$ (see [12], Example 1 and Remark 1 for a further discussion about possible extensions to the infinite–dimensional case).

In particular, our abstract result has not a direct application to variational problems involving infinite–dimensional Banach spaces (such as boundary value problems for differential equations).

Finally, we remark that the hypothesis that the functional $x \mapsto \|x\|^p$ is continuously Gâteaux differentiable is here essential: such hypothesis does not hold in general (for instance, consider the case $X = \mathbb{R}^2$ with the maximum norm and an arbitrary $p > 1$), but it holds in most applications (see Lemma 5 below).

3 An application

In the present Section we are going to apply Theorem 3 to the problem (P_λ) introduced in Section 1: namely, we will prove that, under convenient assumptions on the function f, there exists $\lambda^* > 0$ such that (P_{λ^*}) admits at least three solutions.

We need to introduce some notation: first of all, for every $a, b \in \mathbb{Z}$, $a \leq b$, we define the discrete interval

$$[a, b] = \{a, a+1, \dots b\}.$$

Let $T \in \mathbb{N}$, $T \geq 2$ and $p \in \mathbb{R}$, $p > 1$: we will deal with functions $x : [0, T+1] \to \mathbb{R}$, for which we introduce the *forward difference operator* Δ by putting for every $k \in [1, T+1]$

$$\Delta x(k-1) = x(k) - x(k-1);$$

moreover, we introduce for every real $\gamma > 1$ the mapping $\varphi_\gamma : \mathbb{R} \to \mathbb{R}$ by putting for every $t \in \mathbb{R}$

$$\varphi_\gamma(t) = |t|^{\gamma-2}t$$

and, for any $p > 1$, the *discrete p–Laplacian operator* Δ_p defined by

$$\Delta_p x(k-1) = \Delta\varphi_p(\Delta x(k-1)).$$

Finally, let $f : [1, T] \times \mathbb{R} \to \mathbb{R}$ be such that $f(k, \cdot)$ is continuous for every $k \in [1, T]$.

The solutions of the boundary value problem (P_λ) (for an arbitrary $\lambda > 0$) can be found as elements of a convenient function space: we define the real vector space

$$X = \{x : [0, T+1] \to \mathbb{R} \ : \ x(0) = x(T+1) = 0\}$$

and for every $x \in X$ we denote

$$\|x\| = \left[\sum_{k=1}^{T+1} |\Delta x(k-1)|^p\right]^{\frac{1}{p}},$$

so $(X, \|\cdot\|)$ is a Banach space and $\dim(X) = T$; we also put for every $x \in X$

$$\|x\|_\infty = \max_{k \in [1,T]} |x(k)|.$$

Note that the function $N(x) = \|x\|$ is continuously differentiable because the function $\varphi_p, p > 1$ is continuously differentiable and partial derivatives $\frac{\partial N}{\partial x(k)}$, $k \in [1, T]$ are continuous functions. By classical results, the norms $\|\cdot\|$ and $\|\cdot\|_\infty$ are equivalent on X: the following Lemma yields the precise constants determining the relation between the two.

Denote

$$c_1 = \begin{cases} \left[\left(\dfrac{2}{T}\right)^{p-1} + \left(\dfrac{2}{T+2}\right)^{p-1}\right]^{\frac{1}{p}} & \text{if } T \text{ is even} \\ \dfrac{2}{(T+1)^{\frac{p-1}{p}}} & \text{if } T \text{ is odd} \end{cases}$$

and

$$c_2 = [2 + 2^p(T-1)]^{\frac{1}{p}}.$$

Lemma 4 *Let*

$$S = \{x \in X \; : \; \|x\|_\infty = 1\}.$$

Then, the following conditions hold:

(4.1) $\min_{x \in S} \|x\| = c_1$;

(4.2) $\max_{x \in S} \|x\| = c_2$.

Proof. First, we observe that the set S is compact.

We prove (4.1): by compactness, there exists $x \in S$ which minimizes $\|\cdot\|$ over S; there is, also, $\tau \in [1, T]$ such that $|x(\tau)| = 1$ and $|x(k)| < 1$ for every $k \in [0, \tau - 1]$; without any loss of generality, we may assume that $x(\tau) = 1$.

Next, we will deduce from the minimality property of x some information about the geometry of such function.

We prove that

(3) $x(k-1) \leq x(k)$ for every $k \in [1, \tau]$,

arguing by contradiction: indeed, let $h \in [1, \tau]$ be such that $x(h-1) > x(h)$; then, clearly $h \leq \tau - 1$ and there is some $j \in [h, \tau - 1]$ fulfilling

$$x(j) \leq x(h-1) \leq x(j+1);$$

hence, we define $y \in S$ by putting

$$y(k) = \begin{cases} x(k) & \text{if } k \in [0, h-1] \\ x(h-1) & \text{if } k \in [h, j] \\ x(k) & \text{if } k \in [j+1, T+1] \end{cases}$$

and we get

$$\begin{aligned} \|x\|^p - \|y\|^p &= \sum_{k=1}^{T+1} \left[|\Delta x(k-1)|^p - |\Delta y(k-1)|^p \right] \\ &= \sum_{k=h}^{j} |\Delta x(k-1)|^p + |x(j+1) - x(j)|^p - |x(j+1) - x(h-1)|^p \\ &\geq \sum_{k=h}^{j} |\Delta x(k-1)|^p > 0, \end{aligned}$$

which implies $\|y\| < \|x\|$, a contradiction.

An analogous argument leads to the following relation:

(4) $x(k-1) \geq x(k)$ for every $k \in [\tau + 1, T + 1]$.

We can obtain more precise information:

$$(5) \quad x(k) = \begin{cases} \dfrac{k}{\tau} & \text{if } k \in [0, \tau] \\ \dfrac{T+1-k}{T+1-\tau} & \text{if } k \in [\tau + 1, T + 1] \end{cases}.$$

Indeed, we already know that $x(0) = x(T+1) = 0$ and $x(\tau) = 1$; moreover, by relations (3) and (4) we are reduced to solving two constrained minimization problems:

- first, we put $z_k = \Delta x(k-1)$ for every $k \in [1, \tau]$ and consider the problem $\min_{z \in Q} \psi(z)$, where $z = (z_1, \dots z_\tau)$ and

$$Q = \left\{ z \in \mathbb{R}^\tau \ : \ 0 \leq z_k \leq 1, \ \sum_{k=1}^{\tau} z_k = 1 \right\}, \ \psi(z) = \sum_{k=1}^{\tau} z_k^p;$$

by the elementary inequality

$$\frac{1}{\tau} \sum_{k=1}^{\tau} z_k \leq \left(\frac{1}{\tau} \sum_{k=1}^{\tau} z_k^p \right)^{\frac{1}{p}},$$

where the equality holds for

$$z_1 = z_2 = \ldots = z_\tau = \frac{1}{\tau},$$

it follows that

$$\min_{z \in Q} \psi(z) = \psi \left(\frac{1}{\tau}, \dots \frac{1}{\tau} \right) = \frac{1}{\tau^{p-1}};$$

- analogously, we get for every $k \in [\tau + 1, T + 1]$

$$\Delta x(k-1) = -\frac{1}{T + 1 - \tau}.$$

The above equalities imply (5).
Thus, we obtain

$$\|x\|^p = \sum_{k=1}^{\tau} \frac{1}{\tau^p} + \sum_{k=\tau+1}^{T+1} \frac{1}{(T+1-\tau)^p} = \frac{1}{\tau^{p-1}} + \frac{1}{(T+1-\tau)^{p-1}}.$$

We still need to find τ: with this aim in mind, we observe that the function $\xi_{T,p} : (0, T+1) \to \mathbb{R}$ defined by

$$\xi_{T,p}(t) = \frac{1}{t^{p-1}} + \frac{1}{(T+1-t)^{p-1}}$$

attains its minimum at $t = \dfrac{T+1}{2}$, while the same function is decreasing in $\left(0, \dfrac{T+1}{2}\right)$ and increasing in $\left(\dfrac{T+1}{2}, T+1\right)$, see Figure 1.
Now we distinguish two cases:

- if T is even, we choose $\tau = \dfrac{T}{2}$ or, equivalently, $\tau = \dfrac{T+2}{2}$ and get

$$\|x\| = \left[\left(\frac{2}{T} \right)^{p-1} + \left(\frac{2}{T+2} \right)^{p-1} \right]^{\frac{1}{p}};$$

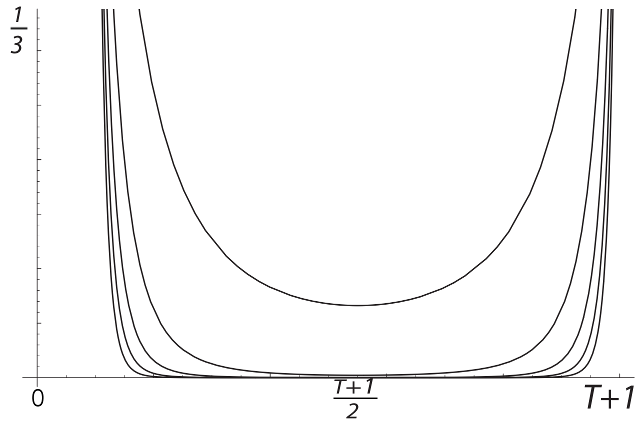

Figure 1: Graph of the function $\xi_{T,p}$ for $T = 10$, $1 \le t \le 10$ and $p = 2, 4, \dots, 10$.

- if T is odd, we choose $\tau = \dfrac{T+1}{2}$ and get

$$\|x\| = \frac{2}{(T+1)^{\frac{p-1}{p}}}.$$

This proves (4.1).
Now we prove (4.2): given $x \in S$, we observe that

$$\|x\|^p = |x(1)|^p + \sum_{k=2}^{T} |\Delta x(k-1)|^p + |x(T)|^p \le 2 + 2^p(T-1),$$

so

$$\max_{x \in S} \|x\| \le c_2;$$

on the other hand, we may define $x \in S$ by putting $x(k) = (-1)^k$ for every $k \in [1, T]$ and get

$$\|x\|^p = 2 + 2^p(T-1),$$

which implies (4.2) and concludes the proof. □
Lemma 4 above represents a refined version of Lemma 2.2 of Jiang and Zhou [8].
A variational framework for problem (P_λ) is provided as follows: for every $k \in [1, T]$ and every $t \in \mathbb{R}$ we put

$$F(k,t) = \int_0^t f(k,\tau)d\tau,$$

for every $x \in X$

$$J(x) = -\sum_{k=1}^{T} F(k, x(k))$$

and for every $\lambda > 0$ and every $x \in X$

$$E_\lambda(x) = \frac{\|x\|^p}{p} + \lambda J(x).$$

Lemma 5 *For every $\lambda > 0$, E_λ is continuously Gâteaux differentiable, and for every $x, y \in X$*

(5.1) $\langle E'_\lambda(x), y\rangle = -\sum_{k=1}^{T} [\Delta_p x(k-1) + \lambda f(k, x(k))]\, y(k).$

Proof. Clearly $E_\lambda \in C^1(X, \mathbb{R})$; in what follows we prove (5.1): choose $x, y \in X$.
Let $\varphi : \mathbb{R} \to \mathbb{R}$ be an arbitrary mapping: we recall the *summation by parts formula*

(6) $$\sum_{k=1}^{T} [\varphi(\Delta x(k-1))\, \Delta y(k-1) + \Delta\varphi(\Delta x(k-1))\, y(k)] = \varphi(\Delta x(T))\, y(T).$$

Using (6) with $\varphi = \varphi_p$, we get

$$\begin{aligned}\lim_{\delta\to 0^+} \frac{\|x+\delta y\|^p - \|x\|^p}{p\delta} &= \sum_{k=1}^{T+1} \varphi_p(\Delta x(k-1))\, \Delta y(k-1) \\ &= -\sum_{k=1}^{T} \Delta\varphi_p(\Delta x(k-1))\, y(k).\end{aligned}$$

Besides, we have

$$\lim_{\delta\to 0^+} \frac{J(x+\delta y) - J(x)}{\delta} = -\sum_{k=1}^{T} f(k, x(k)) y(k).$$

The equalities above imply (5.1). □

Now we can introduce our multiplicity result for the solutions of the problem (P_λ).

Theorem 6 *Let T, p, f, F be as above and $r, s \in \mathbb{R}$ satisfy $0 < r < s$. Moreover, assume that the following conditions hold:*

(6.1) $\limsup_{|t|\to+\infty} \frac{F(k,t)}{|t|^p} \le 0$ *for every $k \in [1, T]$;*

(6.2) $\sum_{k=1}^{T} \sup_{|t|\le \frac{s}{c_1}} F(k,t) < \sum_{k=1}^{T} \sup_{t\in\mathbb{R}} F(k,t);$

(6.3) $\sup_{\frac{r}{c_2}\le|t|\le\frac{s}{c_1}} F(k,t) \le -\sum_{h\ne k} \sup_{|t|\le\frac{s}{c_1}} F(h,t)$ *for every $k \in [1, T]$.*

Then, there exists $\lambda^* > 0$ such that (P_{λ^*}) admits at least three solutions.

Proof. We are going to apply Theorem 3 with X, J, p defined above and $\bar{x} = 0$: hence, we need to check that all hypotheses of that result are satisfied.

We prove that (3.1) holds: since X has finite dimension, there exists $c > 0$ such that for every $x \in X$

$$\|x\| \ge c \left[\sum_{k=1}^{T} |x(k)|^p\right]^{\frac{1}{p}}.$$

Choose $\varepsilon > 0$: by (6.1), there exists $K > 0$ such that for every $k \in [1, T]$ and $t \in \mathbb{R}$ with $|t| > K$

$$\frac{F(k,t)}{|t|^p} < \frac{c^p \varepsilon}{T};$$

let

$$M = \max_{k\in[1,T],\ |t|\le K} |F(k,t)|,$$

then for every $x \in X$ with

$$\|x\| > \left(\frac{MT}{\varepsilon}\right)^{\frac{1}{p}}$$

we get

$$\frac{J(x)}{\|x\|^p} \ge -\sum_{|x(k)|\le K} \frac{\varepsilon}{T} - \sum_{|x(k)|>K} \frac{|F(k,x(k))|}{c^p |x(k)|^p} \ge -\varepsilon,$$

which proves (3.1).
Obviously, if $\inf_{x\in X} J(x) = -\infty$, the inequality (3.2) is fulfilled.
Assume now that $\inf_{x\in X} J(x) > -\infty$
First, we prove that for every $\sigma > 0$ the following identity holds:

(7) $$\inf_{\|x\|_\infty \le \sigma} J(x) = -\sum_{k=1}^{T} \sup_{|t|\le\sigma} F(k,t).$$

To see this, notice that for every $x \in X$, $\|x\|_\infty \le \sigma$ we obviously have

$$J(x) = -\sum_{k=1}^{T} F(k, x(k)) \ge -\sum_{k=1}^{T} \sup_{|t|\le\sigma} F(k,t);$$

on the other hand, for every $\varepsilon > 0$ and every $k \in [1, T]$ there is some $t_k \in \mathbb{R}$, $|t_k| \le \sigma$ such that

$$F(k, t_k) > \sup_{|t|\le\sigma} F(k,t) - \frac{\varepsilon}{T},$$

so, defined $\tilde{x} \in X$ by putting $\tilde{x}(k) = t_k$ for every $k \in [1, T]$, we get $\|\tilde{x}\|_\infty \le \sigma$ and

$$J(\tilde{x}) < -\sum_{k=1}^{T} \sup_{|t|\le\sigma} F(k,t) + \varepsilon,$$

which proves (7).
In a similar way, we deduce that

(8) $$\inf_{x\in X} J(x) = -\sum_{k=1}^{T} \sup_{t\in\mathbb{R}} F(k,t).$$

Then, from (4.1), (6.2), (7) and (8) we deduce that

$$\inf_{x\in X} J(x) = -\sum_{k=1}^{T} \sup_{t\in\mathbb{R}} F(k,t) < -\sum_{k=1}^{T} \sup_{|t|\le \frac{s}{c_1}} F(k,t) = \inf_{\|x\|_\infty \le \frac{s}{c_1}} J(x) \le \inf_{\|x\|\le s} J(x).$$

We prove that (3.3) holds: clearly $J(0) = 0$, while for every $x \in X$ satisfying $r \le \|x\| \le s$ we have by (4.1) and (4.2)

$$\frac{r}{c_2} \le \|x\|_\infty \le \frac{s}{c_1};$$

there exists $k \in [1,T]$ such that $\|x\|_\infty = |x(k)|$, so by (6.3) we get

$$J(x) = -F(k,x(k)) - \sum_{h\neq k} F(h,x(h)) \ge -\sup_{\frac{r}{c_2}\le|t|\le\frac{s}{c_1}} F(k,t) - \sum_{h\neq k} \sup_{|t|\le\frac{s}{c_1}} F(h,t) \ge 0.$$

Thus, by Theorem 3 there exists $\lambda^* > 0$ such that E_{λ^*} admits at least three critical points in X: let us denote them x_0, x_1, x_2.

Finally, we prove that x_i $(i = 0,1,2)$ is a solution of (P_{λ^*}): indeed, recalling (5.1), we get for every $y \in X$

$$-\sum_{k=1}^{T} \left[\Delta_p x_i(k-1) + \lambda^* f(k,x_i(k))\right] y(k) = 0,$$

which obviously implies that x_i solves (P_{λ^*}). □

4 Remarks and examples

In this final Section, we are going to discuss the main features of Theorem 6, presenting some examples in connection.

We start with a simple example of a system complying with all hypotheses of Theorem 6: notice that $p \neq 2$, so the case under examination cannot be solved applying the results of [12].

Example 7 Consider the system

$$(9)\quad \begin{cases} -\varphi_5(\Delta x(1)) + \varphi_5(\Delta x(0)) = 2\lambda\left(x(1)^3 - x(1)\right) \\ -\varphi_5(\Delta x(2)) + \varphi_5(\Delta x(1)) = -4\lambda\left(x(2) - \dfrac{1}{10}\right)^3 \\ x(0) = x(3) = 0 \end{cases},$$

which is of the type (P_λ) with $T = 2$, $p = 5$ and

$$f(1,t) = 2(t^3 - t),\ f(2,t) = -4\left(t - \frac{1}{10}\right)^3,$$

that is,

$$F(1,t) = \frac{t^4}{2} - t^2,\ F(2,t) = \frac{1}{10^4} - \left(t - \frac{1}{10}\right)^4$$

(see Figure 2).

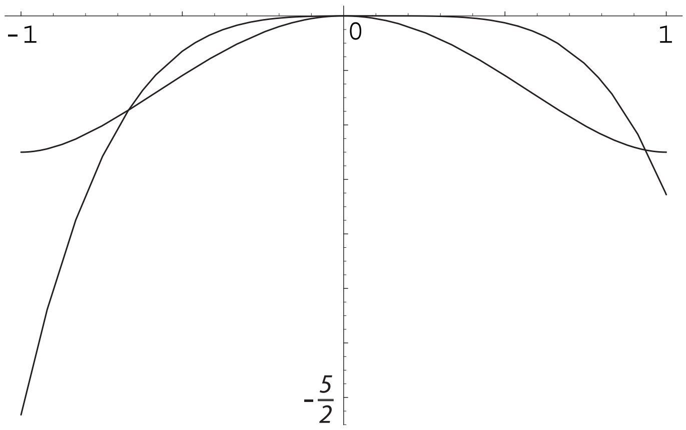

Figure 2: Graphs of the functions $F(1,\cdot)$ and $F(2,\cdot)$ for $-1 \leq t \leq 1$.

Note that in this case we have $c_1 = \left(\frac{17}{16}\right)^{\frac{1}{5}}$ and $c_2 = 34^{\frac{1}{5}}$.

By a straightforward computation, we see that the condition (6.1) is fulfilled; moreover, we put $r = \frac{c_2}{5}$ and $s = c_1$ and obtain

$$\sup_{0.2 \leq |t| \leq 1} F(1,t) = -0.0392, \ \sup_{|t| \leq 1} F(1,t) = 0, \ \sup_{t \in \mathbb{R}} F(1,t) = +\infty$$

and

$$\sup_{0.2 \leq |t| \leq 1} F(2,t) = 0, \ \sup_{|t| \leq 1} F(2,t) = \sup_{t \in \mathbb{R}} F(2,t) = 0.0001,$$

which implies conditions (6.2) and (6.3).

Thus, by Theorem 6, there exists $\lambda^* > 0$ such that the system (9) has at least three solutions.

Next, a brief discussion about the main hypotheses of Theorem 6 is in order: indeed, while the condition (6.1) is a standard coercivity assumption, conditions (6.2) and (6.3) are rather unusual; hence, it is a natural question whether such assumptions can be removed or weakened.

The answer is, in general, negative, as the following examples will show.

Example 8 Consider the system

$$(10) \quad \begin{cases} -\Delta x(1) + \Delta x(0) = -\lambda x(1) \\ -\Delta x(2) + \Delta x(1) = \lambda \\ x(0) = x(3) = 0 \end{cases},$$

which is of the type (P_λ) with $T = p = 2$ and

$$f(1,t) = -t, \ f(2,t) = 1.$$

We have then

$$F(1,t) = -\frac{t^2}{2}, \ F(2,t) = t,$$

Note that in this case we have $c_1 = \left(\frac{3}{2}\right)^{\frac{1}{2}}$ and $c_2 = 6^{\frac{1}{2}}$.

It is easily seen that the condition (6.1) is fulfilled; besides, for arbitrary $0 < r < s$ we have

$$\sup_{\frac{r}{c_2} \le |t| \le \frac{s}{c_1}} F(1,t) = -\frac{r^2}{2c_2^2}, \ \sup_{|t| \le \frac{s}{c_1}} F(1,t) = \sup_{t \in \mathbb{R}} F(1,t) = 0$$

and

$$\sup_{\frac{r}{c_2} \le |t| \le \frac{s}{c_1}} F(2,t) = \sup_{|t| \le \frac{s}{c_1}} F(2,t) = \frac{s}{c_1}, \ \sup_{t \in \mathbb{R}} F(1,t) = +\infty,$$

so condition (6.2) is satisfied while (6.3) is not.

Now, direct computation shows that for $\lambda = -\frac{3}{2}$ the system (10) admits no solutions, while for $\lambda \neq -\frac{3}{2}$ (in particular, for every $\lambda > 0$) it has exactly one solution given by

$$x(1) = \frac{\lambda}{3 + 2\lambda}, \ x(2) = \frac{2\lambda + \lambda^2}{3 + 2\lambda};$$

thus, the thesis of Theorem 6 does not hold.

Example 9 Let $p, p_1 > 1$ be real numbers and consider the system

$$(11) \quad \begin{cases} -\varphi_p(\Delta x(1)) + \varphi_p(\Delta x(0)) = -\lambda \varphi_{p_1}(x(1)) \\ -\varphi_p(\Delta x(2)) + \varphi_p(\Delta x(1)) = 0 \\ x(0) = x(3) = 0 \end{cases},$$

which is of the type (P_λ) for $T = 2$ and

$$f(1,t) = -\varphi_{p_1}(t), \ f(2,t) = 0.$$

We have then

$$F(1,t) = -\frac{|t|^{p_1}}{p_1}, \ F(2,t) = 0.$$

Note that in this case

$$c_1 = (1 - 2^{1-p})^{\frac{1}{p}}, \ c_2 = (2 + 2^p)^{\frac{1}{p}}.$$

As above, condition (6.1) is satisfied; besides, for arbitrary $0 < r < s$ we have

$$\sup_{\frac{r}{c_2} \le |t| \le \frac{s}{c_1}} F(1,t) = -\frac{r^{p_1}}{p_1 c_2^{p_1}}, \ \sup_{|t| \le \frac{s}{c_1}} F(1,t) = \sup_{t \in \mathbb{R}} F(1,t) = 0$$

and obviously

$$\sup_{\frac{r}{c_2} \le |t| \le \frac{s}{c_1}} F(2,t) = \sup_{|t| \le \frac{s}{c_1}} F(2,t) = \sup_{t \in \mathbb{R}} F(2,t) = 0,$$

so condition (6.3) is satisfied while (6.2) is not.

In order to study the solution set of (11), we observe that the inverse mapping of φ_p is φ_q, where $q = \dfrac{p}{p-1}$; hence, from the second equation of (11) we get

$$x(2) - x(1) = \varphi_q(\varphi_p(-x(2))) = -x(2),$$

so

$$x(2) = \frac{x(1)}{2}$$

and from the first equation of (11)

$$-(1+2^{1-p})\varphi_p(x(1)) = \lambda \varphi_{p_1}(x(1)). \tag{12}$$

We remark that $x(1) = 0$ always solves (12), then we distinguish three cases:

- if $p_1 < p$, from (12) we deduce that for $\lambda \geq 0$ (11) admits only the zero solution, while for $\lambda < 0$ it has also two nontrivial solutions given by

  $$x(1) = \pm \left(-\frac{\lambda}{1+2^{1-p}}\right)^{\frac{1}{p-p_1}}, \; x(2) = \pm \frac{1}{2}\left(-\frac{\lambda}{1+2^{1-p}}\right)^{\frac{1}{p-p_1}}$$

  whose norms tend to $+\infty$ as $\lambda \to -\infty$;

- if $p_1 = p$, the system has a unique (negative) eigenvalue $\tilde{\lambda} = -(1+2^{1-p})$ such that for $\lambda = \tilde{\lambda}$ (11) admits infinitely many solutions given by $x(1) = h$, $x(2) = \dfrac{h}{2}$ for every $h \in \mathbb{R}$, while for $\lambda \neq \tilde{\lambda}$ (11) admits only the zero solution;

- if $p_1 > p$, from (12) we deduce that for $\lambda \geq 0$ (11) admits only the zero solution, while for $\lambda < 0$ it has also two nontrivial solutions given by

  $$x(1) = \pm \left(-\frac{1+2^{1-p}}{\lambda}\right)^{\frac{1}{p_1-p}}, \; x(2) = \pm \frac{1}{2}\left(-\frac{1+2^{1-p}}{\lambda}\right)^{\frac{1}{p_1-p}}$$

  whose norms tend to 0 as $\lambda \to -\infty$.

In any case, for every $\lambda > 0$ the system (11) has only the zero solution, so the thesis of Theorem 6 does not hold.

Remark 10 In [12], Ricceri posed a question which we can rephrase as follows: can we find X, J, $\bar{x}$, p, r, s as in Theorem 3, satisfying the assumptions (3.1), (3.2) and (3.3), such that there exists a *unique* $\lambda^* > 0$ for which the functional

$$x \mapsto \frac{\|x - \bar{x}\|^p}{p} + \lambda J(x)$$

admits at least three critical points?

The problem is well motivated (see [12], Remarks 1 and 3) but still unsolved: hopefully, our extension of Ricceri's result from the case $p = 2$ to arbitrary $p > 1$ could make it easier to find a solution (for instance in the framework of Section 3).

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References

[1] R.P. Agarwal, *Difference equations and inequalities*, Marcel Dekker Inc. (2000).

[2] R.P. Agarwal, K. Perera, D. O'Regan, *Multiple positive solutions of singular and nonsingular discrete problems via variational methods*, Nonlinear Anal. **58** (2004) 69–73.

[3] R.P. Agarwal, K. Perera, D. O'Regan, *Multiple positive solutions of singular discrete p–Laplacian problems via variational methods*, Adv. Difference Equ. **2005** (2005) 93–99.

[4] X. Cai, Z. Guo, J. Yu, *Periodic solutions of a class of nonlinear difference equations via critical point method*, Computers and Mathematics with Applications **52** (2006) 1639–1647.

[5] X. Cai, J. Yu, *Existence theorems for second–order discrete boundary value problems*, J. Math. Anal. Appl. **320** (2006) 649–661.

[6] F. Faraci, A. Iannizzotto, *Multiplicity theorems for discrete boundary value problems*, Æquationes Math. **74** (2007) 111–118.

[7] Z. Guo, M. Ma, *Homoclinic orbits and subharmonics for nonlinear second order difference equations*, Nonlinear Anal. **67** (2007) 1737–1745.

[8] L. Jiang, Z. Zhou, *Three solutions to Dirichlet boundary value problems for p–Laplacian difference equations*, Adv. Difference Equ. **2008** (2008) Article ID 345916, 10 pages.

[9] V. Lakshmikantham, D. Trigiante, *Theory of difference equations: Numerical methods and applications*, Academic Press (1988).

[10] M. Mihăilescu, V. Rădulescu, S. Tersian, *Eigenvalue problems for anisotropic discrete boundary value problems*, to appear on Journal of Difference Equations and Applications.

[11] B.Ricceri, *Well–posedness of constrained minimization problems via saddle–points*, J. Global Optim. **40** (2008) 389–397.

[12] B. Ricceri, *A multiplicity theorem in $\mathbb{R}^n$*, J. Convex Anal., to appear.

[13] M. Struwe, *Variational methods. Applications to nonlinear partial differential equations and Hamiltonian systems*, Springer–Verlag (2000).

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