On Convergence of the Unscented Kalman-Bucy Filter using Contraction
Theory

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Contraction theory entails a theoretical framework in which convergence of a nonlinear system can
be analysed differentially in an appropriate contraction metric. This paper is concerned with utilizing
stochastic contraction theory to conclude on exponential convergence of the Unscented Kalman-Bucy
Filter. The underlying process and measurement models of interest are Itô-type stochastic differential
equations. In particular, statistical linearisation techniques are employed in a virtual-actual systems
framework to establish deterministic contraction of the estimated expected mean of process values.
Under mild conditions of bounded process noise, we extend the results on deterministic contraction to
stochastic contraction of the estimated expected mean of process state. It follows that for regions of
contraction, a result on convergence, and thereby incremental stability, is concluded for the Unscented
Kalman-Bucy Filter. The theoretical concepts are illustrated in two case studies.

Keywords: stochastic contraction; Unscented Kalman-Bucy Filter; virtual-actual framework;
exponential convergence; statistical linearization

1. Introduction

The Kalman Filter (KF) is a well-established and widely used method for state estimation
due to its optimality, tractability and robustness. Furthermore, if the underlying process and
measurement models are linear, with additive Gaussian noise, then the KF is the optimal
Bayesian filter (Julier and Uhlmann 2004). If strict assumptions on linearity and Gaussian noise
are violated, which is often the case, then an alternative is to approximate the optimal Bayesian
filter (Sanjeev Arulampalam et al. 2002). Popular methods in how to perform approximation
include, but are not limited to; the Extended KF (EKF), Unscented KF (UKF) (Julier and
Uhlmann 2004) and variants thereof (Särkkä 2007), approximate grid-based methods, and
particle filters (see Sanjeev Arulampalam et al. (2002); Hu et al. (2008) and references therein).
For this work, however, we restrict our interest to the continuous-time formulation of the UKF.
In particular, we aim at extending the deterministic convergence analysis using contraction
theory, previously conducted for the EKF (Jouffroy and Fossen 2010; Bonnabel and Slotine
2012), to the continuous-time UKF.

The EKF is an approximate filter for nonlinear systems, based on first order linearisation.
The EKF, however, can produce unstable filters when local linearity is violated. Furthermore,
obtaining Jacobians by system linearisation are non-trivial in many applications (Julier and
The UKF is a recursive minimum mean-square-error estimator based on an approximate optimal Gaussian KF framework, that addresses some of the shortcomings associated with the EKF (Julier and Uhlmann 1997). The UKF employs deterministically sampled sigma-points (samples), which are propagated through the true nonlinear process and measurement models. Consequently, estimates of the posterior mean and covariance are accurate up to the second order (third order if the prior estimates are truly Gaussian) of the Taylor series expansion of any nonlinear system (Wan and Van der Merwe 2000).

Several results on stability and convergence for the EKF can be found (Ljung 1979; Boutayeb et al. 1997; Reif et al. 1998; Guo and Zhu 2002). However, despite the aforementioned benefits of the UKF over the EKF, limited results exist with respect to stability and convergence analysis for the UKF (Xiong et al. 2006). The stability analysis strategy proposed by Boutayeb et al. (1997), applied to the EKF, has been adopted for the analysis of the discrete-time UKF with intermittent observations (Li and Xia 2012) and continuous-time UKF (Xu et al. 2008) (also often referred to as the Unscented Kalman-Bucy Filter (UKBF); see for example Zhou et al. (2010); Särkkä (2007)). Xu et al. (2007) presents a stability result for a modified UKF in which the process and measurement noise for a nonlinear stochastic system are correlated.

In this work, contraction theory is employed to establish convergence, and thereby incremental stability, of the UKBF. Contraction theory analyses convergence differentially where the latter may imply generalizing the classical Krasovskii theorem or linear eigenvalue analysis (Lohmiller and Slotine 1998). A historical perspective on contraction theory is presented by Jouffroy (2005). Contraction theory for the deterministic case has been presented by Lohmiller and Slotine (1996, 1998, 2000). The extension of contraction theory for stochastic systems was presented by Pham et al. (2009), using time-varying, state-independent contraction metrics, and Pham and Slotine (2013), using time-varying, state-dependent contraction metrics. Contraction analysis has also been applied in a wide field of applications: two-link robot (Lohmiller and Slotine 1996; Jouffroy and Fossen 2010), continuous stirred-tank reactor (Lohmiller and Slotine 2000), ocean vehicles (Lohmiller and Slotine 1996; Jouffroy and Lottin 2002), Fitzhugh-Nagumo Oscillators (Pham et al. 2009), and ship manoeuvring (Jouffroy and Fossen 2010).

Contraction theory has been used in analysing contraction, implying exponential convergence, of the deterministic EKF (Jouffroy and Fossen 2010), and general output-feedback observers (Pham et al. 2009). However, no formal procedure in formulating a contraction metric for the UKF has been presented by either Jouffroy and Fossen (2010), or Pham et al. (2009). Maree et al. (2013) proposed a contraction metric, for which under restrictive observability assumptions, one can show convergence of the UKBF. As observed in Maree et al. (2013), a technical difficulty arises when one directly apply contraction theory to analyse the convergence of the UKBF. A reason for this technical difficulty is the fact that contraction theory utilizes the analytic Jacobian of a nonlinear system to analyse convergence differentially, while in contrast, the UKBF is based on a statistical linearisation framework.

A motivation for this work stems from the observation that contraction theory, originally introduced by Lohmiller and Slotine (1998), and extended by Pham et al. (2009); Pham and Slotine (2013), cannot be directly applied (in a similar simplistic manner as shown for the EKF Jouffroy and Fossen (2010)), to conclude on convergence of the UKBF. The aim of this work is to extend on the earlier work of Maree et al. (2013), by finding conditions under which contraction theory can be directly applied for convergence analysis of the UKBF, without the need of restrictive assumptions on system observability, and construction of complex contraction metrics. This work contributes to previous literature on contraction theory by incorporating results on statistical linearisation techniques (Gelb et al. 1974). In particular, statistical linearisation techniques enable us to evaluate statistical system Jacobians. These statistical Jacobians, in turn,
can be utilized in the context of contraction theory to analyze convergence of systems, based
on statistical linearization frameworks, differentially. Lastly, shown in the context of statistical
linearisation techniques, one can apply contraction theory with a general contraction metric,
previously proposed for the EKF (Jouffroy and Fossen 2010), for analysing convergence, and
thereby incremental stability, of the UKBF.

2. Preliminaries

Notation

The symbols R and I define the set of real and integer numbers, respectively. Let \( R^{n_r \times n_r} \) denote real valued matrices of \( n_r \)-rows and \( n_r \)-columns. The mean of a real-valued, time-varying variable \( x(t) \) is defined by the expectation operator \( E\{x(t)\} \). Covariance between jointly distributed real-valued time-varying variables \( x(t) \) and \( y(t) \) is expressed by the covariance operator \( \text{cov}\{x(t), y(t)\} \). The variance of the variable \( x(t) \) is expressed by \( \text{var}\{x(t)\} \). Operator \( \text{tr}\{\cdot\} \) defines the trace of a square matrix. Let \( \|\cdot\| \) and \( \|\cdot\|^F \) denote the spectral- and Frobenius-norm operators for some matrix argument. \( \sup\{\cdot\} \) is defined the supremum.

2.1 Problem Description

In the general stochastic estimation problem it is often the case that the structure and dynamics of the model are known only approximately due to dynamical modelling errors and imprecise knowledge on system parameters. The subsequent goal is to obtain the best possible state estimate, using measured process data. However, the measurements may often contain noise components due to observation errors. Moreover, stochastic excitation typically appears in the state dynamics. The optimal recursive estimation for this kind of systems is often termed optimal continuous-time filtering (Gelb et al. 1974; Simon 2006; Särkkä 2007). We consider the continuous time-varying, nonlinear system

\[
\begin{align*}
\dot{x}(t) &= f(x(t), t) + \sigma_x(t)w(t) \quad (1a) \\
\dot{z}(t) &= h(x(t), t) + \sigma_z(t)v(t) \quad (1b)
\end{align*}
\]

in which we define state vector \( x(t) \in R^{n_x} \), process model, \( f : R \mapsto R^{n_x} \) and measurement model \( h : R \mapsto R^{n_z} \). We assume that a unique solution to (1a), and a stationary, ergodic probability measure (1b), exist. Note that any deterministic input signal \( u(t) \in R^{n_u} \) is subsumed in the time-varying process dynamics \( f(x(t), t) \). Consider the process noise, \( w(t) \sim (0, Q_c(t)) \), and measurement noise, \( v(t) \sim (0, R_c(t)) \), to be independent white noise, with spectral density matrices \( Q_c(t) \) and \( R_c(t) \), respectively. Define \( \sigma_x(t) \in R^{n_x \times n_x} \) and \( \sigma_z(t) \in R^{n_z \times n_z} \) the process and measurement noise intensity diagonal matrices. For this work we assume that \( z(t) = \frac{dy(t)}{dt} \) is a differential measurement (Särkkä 2007). Then, the process and measurement models (1) can equivalently be formulated as Itô-type stochastic differential equations

\[
\begin{align*}
dx(t) &= f(x(t), t) \, dt + \sigma_x(t) \, dW_w(t) \quad (2a) \\
dy(t) &= h(x(t), t) \, dt + \sigma_z(t) \, dW_v(t) \quad (2b)
\end{align*}
\]

in which \( dW_w(t) := w(t) \, dt \) and \( dW_v(t) := v(t) \, dt \) are independent, standard Wiener processes with diagonal diffusion matrices \( Q_c(t) \) and \( R_c(t) \), respectively.

Assumption 1. There exists a positive scalar \( \gamma \) such that \( \gamma I < \sigma_x(t)Q_c(t)\sigma_x^T(t) \) holds uniformly.

The existence of \( \gamma \) in Assumption 1 follows directly for any diagonal diffusion matrix \( Q_c(t) \) being positive definite, and for any diagonal matrix \( \sigma_x(t) \), in which the diagonal entries are all
non-zero. The matrix $\sigma_x(t)$ (also referred to as the dispersion matrix (Särkkä 2007)), and diffusion matrix $Q_x(t)$, can be inferred using statistical inference techniques that estimate spectral characteristics of experimental process data. The latter, in turn, allows specifying $\gamma$ implicitly. Since statistical inference of spectral characteristics is a relatively complicated problem, with many pitfalls (Brown and Hwang 1997), and outside the scope of this paper, we refer the interested reader to Bendat and Piersol (2010) for a comprehensive coverage on this topic.

2.2 Matrix Framework for UKF

The formulation, and presentation of the discrete-time UKF is thoroughly documented (Julier and Uhlmann 1997; Wan and Van der Merwe 2000; Julier and Uhlmann 2004; Simon 2006), and thus will not be covered in-depth here. Prior to presenting the UKBF, we adopt a matrix framework, as proposed by Särkkä (2007), to formulate the respective sigma-points and weights in a compact manner. Weighted averages and covariances are subsequently expressed using the matrix framework.

The intuition behind the UKF is that it is easier to approximate an arbitrary Gaussian distribution of system states, than some arbitrary nonlinear function, or transformation (Julier and Uhlmann 1997). Therefore, the UKF employs deterministically drawn sigma-points, which once propagated through the process and measurement models, aid in estimating the sample mean and variance of the actual process states. Suppose the actual process mean and variance for (1) are defined as $\bar{x}(t) := \mathbf{E}\{x(t)\}$ and $P(t) := \text{var}\{x(t)\}$ respectively. We assume an actual Gaussian process distribution of the form $x(t) \sim \mathcal{N}(\bar{x}(t), P(t))$. Suppose at time $t$ the estimated process mean and variance is defined by $\hat{x}(t)$ and $\hat{P}(t)$ respectively. Then, for the purpose of estimation we proceed in sampling $2n_x + 1$ sigma-points, $\hat{\mathbf{x}}_i(t) \in \mathbb{R}^{n_x}$, deterministically from the Gaussian distribution $\mathcal{N}(\hat{x}(t), \hat{P}(t))$, i.e.,

\[
\hat{\mathbf{x}}_{-i}(t) := \hat{x}(t) - \sqrt{n_x + \kappa}[L(t)]_i
\]

\[
\hat{\mathbf{x}}_0(t) := \hat{x}(t)
\]

\[
\hat{\mathbf{x}}_i(t) := \hat{x}(t) + \sqrt{n_x + \kappa}[L(t)]_i
\]

in which $\kappa \in \mathbb{R}$ is a secondary scaling parameter, usually being set to zero (Simon 2006), and $[L(t)]_i$ is the $i$-th column of the matrix $L(t)$ for all $i \in \mathbb{I}_{1:n_x}$. From the Cholesky decomposition $\hat{P}(t) = L(t)L(t)^T$, we have matrix $L(t)$ being a lower triangular matrix with non-negative diagonal entries, and $L(t)^T$ being the conjugate transpose thereof. To present the sigma-points (3) in a compact manner, we introduce the sigma-point matrix $\hat{\mathbf{X}}(t) \in \mathbb{R}^{n_x \times (2n_x+1)}$ defined as

\[
\hat{\mathbf{X}}(t) := [\hat{\mathbf{x}}_{-n_x}(t), \cdots, \hat{\mathbf{x}}_0(t), \cdots, \hat{\mathbf{x}}_{n_x}(t)]
\]  

**Remark 2.1.** Let $\mathbf{X}(t) \in \mathbb{R}^{n_x \times (2n_x+1)}$ be any general, real-valued sigma-point matrix defined in a similar fashion as in (4). Then, for compact notation in subsequent text we will adopt the short-hand notation $f(\mathbf{X}(t), t) \in \mathbb{R}^{n_x \times (2n_x+1)}$ and $h(\mathbf{X}(t), t) \in \mathbb{R}^{n_x \times (2n_x+1)}$ to express a matrix formulation with $i \in \mathbb{I}_{-n_x:n_x}$ column entries, where in the former, $f(\hat{\mathbf{x}}_i(t), t)$ defines the $i$-th column entry for $f(\mathbf{X}(t), t)$; in the latter, $h(\hat{\mathbf{x}}_i(t), t)$ defines the $i$-th column entry for $h(\mathbf{X}(t), t)$.

Define $w_i$ as the corresponding weight to the $i$-th sigma-point $\hat{\mathbf{x}}_i(t)$

\[
w_0 = \frac{\kappa}{n_x + \kappa}; \quad w_i = w_{-i} = \frac{1}{2(n_x + \kappa)}, \quad \forall i \in \mathbb{I}_{1:n_x}
\]

Next, let $\mathbf{W} \in \mathbb{R}^{2n_x+1}$ define the vector of weights $w_i$ for all $i \in \mathbb{I}_{-n_x:n_x}$ i.e., $\mathbf{W} :=
bounded, i.e., there exists two positive definite scalars $p_1, p_2$.

Assumption 2.

Lemma 2.1 (Weighted Mean and Variance (Särkkä 2007)). Consider any time-varying state vector $x(t) \in \mathbb{R}^{n_x}$ with state distribution $x(t) \sim \mathcal{N}(\bar{x}(t), P(t))$. The corresponding deterministically sampled sigma-point matrix $X(t)$, evaluated similar to (4) for the given state distribution, completely captures the true mean and variance of the prior random variable $x(t)$, i.e., $E\{x(t)\} = X(t)W$ and $\text{cov}\{x(t), x(t)\} = X(t)WX(t)^T$.

2.3 Unscented Kalman-Bucy Filter Algorithm

The derivation of the UKBF follows along similar lines of deriving the continuous-time KF, also referred to as the Kalman-Bucy filter. The Kalman-Bucy filter derivation starts by taking the limit for the discrete-time KF, in which time step $\Delta t$ asymptotically approaches zero (Simon 2006). The UKBF is derived in similar fashion by applying the limit $\Delta t \to 0$ of the discrete-time UKF (Särkkä 2007). We adopt the matrix formulation of sigma-points and weights, previously introduced in Section 2.2, and derive the UKBF in the Appendix. Algorithm 2.1 summarizes the implementation steps.

Algorithm 2.1 (UKBF).

Step 1: (Initialize) Consider the initial estimated process state $\hat{x}(t_0)$ with Gaussian distribution $\mathcal{N}(\bar{x}(t_0), \hat{P}(t_0))$. Then, to initialize the UKBF at time $t = t_0$, sample $2n_x + 1$ initial sigma-points, and define the initial sigma-point matrix $\hat{X}(t_0) = [\hat{X}_{-n_x}(t_0), \ldots, \hat{X}_0(t_0), \ldots, \hat{X}_{n_x}(t_0)]$ in which $\hat{X}_i(t_0)$ for all $i \in \mathbb{I}_{-n_x:n_x}$ is evaluated similar to (3).

Step 2: (Solve ODE with continuous measurements) The continuous UKBF observer gain $\hat{K}(t)$, and ordinary differential equations of the estimated mean value of process state, and variance, is evaluated for time $t \geq t_0$ by solving

\begin{align}
\hat{K}(t) &= \hat{X}(t)Wh^T(\hat{X}(t), t)[\sigma_z(t)R_c(t)\sigma_z^T(t)]^{-1} \\
\hat{z}(t) &= f(\hat{X}(t), t)W + \hat{K}(t)[z(t) - h(\hat{X}(t), t)W] \\
\hat{P}(t) &= \hat{X}(t)Wf^T(\hat{X}(t), t) + f(\hat{X}(t), t)WX^T(t) + \sigma_z(t)Q_c(t)\sigma_z^T(t) - \hat{K}(t)\sigma_z(t)R_c(t)\sigma_z^T(t)\hat{K}^T(t)
\end{align}

where we have formally defined the differential measurement $z(t) = dy(t)/dt$. The sigma-point matrix $\hat{X}(t)$ evolves according to (4).

Assumption 2. The matrix $\hat{P}(t)$, being the solution to (7c), is uniformly upper and lower bounded, i.e., there exists two positive definite scalars $p, \tilde{p}$ such that

\begin{align}
pI < \hat{P}(t) < \tilde{p}I
\end{align}

Remark 2.2. Similar assumptions to that of Assumption 2 (being non-trivial but standard) has been made for the solution of the Ricatti matrix differential equation as present in the EKF formulation (Reif et al. 1998; Jouffroy and Fossen 2010; Bonnabel and Slotine 2012).
3. On Convergence for UKBF

We are interested in applying contraction theory to analyse the convergence of the UKBF differentially, when applied as an observer for system (1). Stability is generally viewed relative to some nominal motion or equilibrium point. Contraction analysis does not require explicit attractors to be known. Instead, a system is stable, in the context of contraction analysis, if final time behaviour of the system is independent of the initial conditions; hence, all trajectories converge to some nominal motion (Lohmiller and Slotine 1998). In particular, we are concerned with analysing the convergence of the estimated process mean, \( \hat{x}(t) \), towards the actual expected process mean, \( \bar{x}(t) \); or equivalently, find a contraction region in which any initial estimated process mean, \( \hat{x}(t) \), contracts toward the actual expected process mean, \( \bar{x}(t) \). We proceed by observing that the equivalent Itô-type stochastic differential interpretation of the estimated expected mean value of process state (7b) can be formulated as

\[
d\hat{x}(t) = F(\hat{X}(t), t)dt + G(\hat{X}(t), t)dW(t)
\]

where we have the deterministic part defined as

\[
F(\hat{X}(t), t) := f(\hat{X}(t), t)W + K(t)Z(t) - h(\hat{X}(t), t)]W
\]

and stochastic part defined as

\[
G(\hat{X}(t), t) := K(t)\sigma_z(t)
\]

In (9) we have applied the definition of white noise, \( dW(t) = v(t)dt \). For expressions (10) and (11), we have applied \( z(t) = Z(t)W + \sigma_z(t)v(t) \) in which \( Z(t) = h(X(t), t) \). To ensure the existence and solution to (9), we assume the following standard assumptions on continuity and restriction on growth (Pham and Slotine 2013).

**Assumption 3** (Continuity and Boundedness).

**Lipschitz continuity**: There exists a constant \( K_1 > 0 \) such that for all \( t \geq 0 \) and \( \{\hat{X}(t), X(t)\} \in \mathbb{R}^{n_x \times (2n_x + 1)}, \)

\[
||F(\hat{X}(t), t) - F(X(t), t)|| + ||G(\hat{X}(t), t) - G(X(t), t)||_F \leq K_1||\hat{X}(t) - X(t)||_F
\]  

**Restriction on growth**: There exists a constant \( K_2 > 0 \) such that for all \( t \geq 0 \) and \( \{\hat{X}(t), X(t)\} \in \mathbb{R}^{n_x \times (2n_x + 1)}, \)

\[
||F(\hat{X}(t), t)||^2 + ||G(\hat{X}(t), t)||^2_F \leq K_2(1 + ||\hat{X}(t)||^2_F)
\]

Convergence analysis of the UKBF will first proceed for the deterministic case in which we will utilize theory on deterministic contraction. The latter implies we temporarily discard the influence of measurement noise in (1b) where it follows that \( z(t) = h(x(t), t) \); moreover, in light of Lemma 2.1 we have \( z(t) = Z(t)W \) in which \( Z(t) = h(X(t), t) \).

### 3.1 Deterministic Contraction

In this section we introduce contraction theory to establish contraction in the deterministic case for the estimated expected mean process state dynamics (7b). However, we will first present results on statistical linearisation, prevalent for the discussion on deterministic contraction of the UKBF.
3.1.1 Statistical Linearisation

Some approximate optimal Bayesian recursive filters, such as the EKF, employ an analytic Jacobian of the underlying system, to represent the system in some local region. In this case, contraction theory can be applied to establish contraction, and therefore convergence, for the EKF (Jouffroy and Fossen 2010; Bonnabel and Slotine 2012). The latter is due to the fact that contraction theory analyses contraction properties of infinitesimal small displacements along the trajectories of the system, using the analytic Jacobian of the underlying system at all points in time (Lohmiller and Slotine 1996). The UKBF, in contrast to the EKF, employs a general deterministic sampling framework to calculate Gaussian approximations to the optimal Bayesian trajectories of the system, using the analytic Jacobian at all points in time (Van der Merwe 2004). Hence, avoiding the evaluation of system Jacobians. The latter implies that the application of contraction theory to analyse convergence of the UKBF may prove more complex than for the EKF. To proceed in using contraction theory for analysing convergence of the UKBF, we first present some results on statistical linearisation techniques. We seek a linear approximation for process and measurement vector functions $f(\hat{x}(t), t)$ and $h(\hat{x}(t), t)$

\[
\begin{align*}
    f(\hat{x}(t), t) &= A\hat{x}(t) + b + \varepsilon_f \\
    h(\hat{x}(t), t) &= C\hat{x}(t) + d + \varepsilon_h
\end{align*}
\] (14a)

where $\hat{x}(t)$ is the solution to (7b). We desire to choose the coefficients $\{A, b\}$ and $\{C, d\}$ such that the approximation errors $\varepsilon_f$ and $\varepsilon_h$ are minimized on average. Any method that incorporates the statistical properties of $\hat{x}(t)$ to accomplish the latter is thought of as a statistical linearisation technique (Gelb et al. 1974). In other words, statistical linearisation techniques find the statistical interpretation of $\{A, b\}$ and $\{C, d\}$ such that approximation (14) takes into account the prior distributions of $\hat{x}(t)$, which in turn performs better in the statistical average sense compared to the first order Taylor series truncation (Van der Merwe 2004). In essence, we are interested in evaluating the minimum of the variance of linearisation errors (Gelb et al. 1974)

\[
\begin{align*}
    \sigma_{\varepsilon_f}^2 &= \mathbf{E}\{\varepsilon_f^T \varepsilon_f\} \\
    \sigma_{\varepsilon_h}^2 &= \mathbf{E}\{\varepsilon_h^T \varepsilon_h\}
\end{align*}
\] (15a)

Consider $\hat{P}(t)$, being the solution to (7c), which denotes the estimated variance of $\hat{x}(t)$. Also, let $\hat{P}_{\hat{x}f^T(\hat{x})}^T$ and $\hat{P}_{\hat{x}h^T(\hat{x})}^T$ be the covariance terms $\mathbf{cov}\{\hat{x}(t), f^T(\hat{x}(t), t)\}$ and $\mathbf{cov}\{\hat{x}(t), h^T(\hat{x}(t), t)\}$, evaluated for a weighted, unscented transform-based statistical framework, respectively. Then, in Lemma 3.1, we show that the coefficients $\{A, C\}$ can be chosen in terms of $\hat{P}(t)$, $\hat{P}_{\hat{x}f^T(\hat{x})}^T$ and $\hat{P}_{\hat{x}h^T(\hat{x})}^T$.

**Lemma 3.1.** The generalized minimum mean square error approximations to (14a) and (14b) are obtained for the coefficients $\{A, b\}$ and $\{C, d\}$, given by

\[
\begin{align*}
    A := \hat{P}_{\hat{x}f^T(\hat{x})}^T \hat{P}^{-1} ;
    b := \mathbf{E}\{f(\hat{x}(t), t)\} - A \mathbf{E}\{\hat{x}(t)\}
\end{align*}
\] (16a)

\[
\begin{align*}
    C := \hat{P}_{\hat{x}h^T(\hat{x})}^T \hat{P}^{-1} ;
    d := \mathbf{E}\{h(\hat{x}(t), t)\} - C \mathbf{E}\{\hat{x}(t)\}
\end{align*}
\] (16b)

It follows that the approximation errors $\varepsilon_f$ and $\varepsilon_h$ are minimized in a statistical average sense.

**Proof.** Substitute expressions (14a) and (14b), solved with respect to the linearisation errors, into (15a) and (15b). Next, evaluate the partial derivative of (15a) with respect to $b$, and set the derivative to zero to obtain $\mathbf{E}\{(f(\hat{x}(t), t) - A\hat{x}(t) - b)\} = 0$ which is achieved by $b = \mathbf{E}\{f(\hat{x}(t), t)\} - A \mathbf{E}\{\hat{x}(t)\}$. Similarly, the minimum of (15b) is characterized by $d =$
\[ \mathbf{E}\{h(\hat{x}(t), t)\} - C \mathbf{E}\{\hat{x}(t)\}. \] 

Next, we substitute \( b \) into \( \varepsilon_f \), and take the partial derivative of \( \sigma^2_{\varepsilon_f} \) with respect to \( A \)

\[
\frac{\partial \sigma^2_{\varepsilon_f}}{\partial A} = \mathbf{E}\{(f(\hat{x}) - A\hat{x} - b)(-\hat{x} + \mathbf{E}\{\hat{x}\})^T\} = A \mathbf{E}\{(\hat{x} - \mathbf{E}\{\hat{x}\})(\hat{x} - \mathbf{E}\{\hat{x}\})^T\} - \mathbf{E}\{(f(\hat{x}) - \mathbf{E}\{f(\hat{x}(t), t)\})(\hat{x} - \mathbf{E}\{\hat{x}\})^T\} = A\hat{P} - \hat{P}_T\varepsilon_f(\hat{x}) = 0 \tag{17a}
\]

\[
\hat{P}_T\varepsilon_f(\hat{x}) = 0 \tag{17b}
\]

That is, (15a) is minimized for \( A = \hat{P}_T\varepsilon_f(\hat{x})\hat{P}^{-1} \). Similarly, (15b) is minimized for \( C = \hat{P}_T\varepsilon_{\varepsilon_f}(\hat{x})\hat{P}^{-1} \).

**Remark 3.1.** In Lemma 3.1, the choice of coefficients \( \{A, C\} \), in light of a unscented transform-based statistical framework (see also Lemma 2.1), implies the relations \( \hat{P} = \hat{X}(t)\hat{W}\hat{X}^T(t) \) (see also Appendix), \( \hat{P}_T\varepsilon_f(\hat{x}) = f(\hat{X}(t), t)\hat{W}\hat{X}^T(t) \) and \( \hat{P}_T\varepsilon_{\varepsilon_f}(\hat{x}) = h(\hat{X}(t), t)\hat{W}\hat{X}^T(t) \) to hold.

### 3.1.2 Virtual-Actual System Framework

To proceed in the deterministic contraction analysis for convergence, we employ a virtual-actual system framework (Jouffroy and Fossen 2010) in conjunction with contraction analysis theory (Lohmiller and Slotine 1998).

**Definition 1 (Virtual-Actual framework).** Define the following virtual-actual system framework

\[
\hat{\chi}(t) = f(\chi(t), t) + \mathbf{K}(t)[z(t) - h(\chi(t), t)] =: \tilde{f}(\chi(t), t) \tag{18}
\]

Since we are interested in analysing the convergence of the UKBF in a statistical average sense (convergence analysis of the estimated expected mean of process state, \( \bar{x}(t) \)), it is imperative to interpret the virtual-actual system framework (18) with respect to its statistical expectation.

**Lemma 3.2 (Virtual-Actual System Interpretation).** Consider the expected mean of the virtual-actual system (18). Then, the expected mean of the actual system (1a), and deterministic part of the expected mean of process dynamics (9), i.e., (10), are particular solutions of the expected mean of the virtual-actual system (18), given \( \chi(t) \) is either replaced with \( \bar{x}(t) \) or \( \hat{x}(t) \), respectively.

**Proof.** For the actual system (1a) we have that since \( \mathbf{E}\{w(t)\} = 0 \), the expected mean of the actual process is simply the process itself. Next, if we substitute \( \chi(t) \) with \( \bar{x}(t) \) and evaluate the expected mean of (18), then, by incorporating the results of Lemma 2.1 and using the property of \( \mathbf{E}\{Z(t)\} = \hat{Z}(t)\hat{W} \), we subsequently evaluate the deterministic part of (9).

Using contraction theory to analyse the contraction properties of the virtual system (18) requires the definition of a virtual displacement operator, \( \delta_f\chi(t) \), which is an infinitesimal small displacement at fixed time for system \( \tilde{f}(\chi(t), t) \) (Lohmiller and Slotine 1998). Assuming that (18) is continuously differentiable, then the first variation of (18) is

\[
\delta_f\chi(t) = \frac{\partial \tilde{f}}{\partial \chi}(t) =: \tilde{A}\delta_f\chi(t) \tag{19}
\]

in which \( \tilde{A} := A_f - \mathbf{K}(t)C_f \), and matrices \( A_f \) and \( C_f \) are the Jacobian matrices of \( f(x(t), t) \) and

---

1For brevity sake, we will omit time-varying variable \( t \) for the remainder of this proof.
\[ h(x(t), t), \text{ respectively, i.e.,} \]
\[ A_J := \frac{\partial f(x(t), t)}{\partial x(t)} \bigg|_{x(t) = \hat{x}(t)}; \quad C_J := \frac{\partial h(x(t), t)}{\partial x(t)} \bigg|_{x(t) = \hat{x}(t)} \]  

(20)

Using the optimal choice for coefficients \( \{A, C\} \) in Lemma 3.1, define \( \tilde{A}_s := A - K(t)C \) such that for the matrix linearisation difference, \( \Delta A := A - \tilde{A}_s \), we can express the relation

\[ \tilde{A} = \tilde{A}_s + \Delta A \]  

(21)

between the statistical and analytic Jacobian of (18).

**Assumption 4.** The analytic-statistical Jacobian difference \( \Delta A \) is uniformly bounded

\[ \|\Delta A\| \leq \gamma \left( 1 - \epsilon \right) \]

(22)

in which \( \epsilon \) is some sufficiently small positive scalar, and \( \gamma \) and \( p \) are from Assumptions 1 and 2, respectively.

Let \( \theta(\chi(t), t) \) be a square, state-dependent and time-varying transformation matrix. Next, consider the state transformation \( \delta f\eta(t) = \theta(t)\delta f\chi(t) \). We define a uniformly positive definite, symmetric, continuously differential metric \( M(\chi(t), t) := \theta^T(t)\theta(t) \), such that we can express the following transformed, virtual displacement distance

\[ \delta^T f\eta(t)\delta f\eta(t) = \delta^T_f\chi(t)M(t)\delta f\chi(t) \]

We are interested in finding a metric \( M(t) \), and positive definite scalar \( \beta_M \), such that for the velocity of the virtual displacement,

\[ \delta^T f\chi(t) \left( \tilde{A}^T(t)M(t) + \dot{M}(t) + M(t)\tilde{A} \right) \delta f\chi(t) \]

(23)

we can conclude exponential convergence to a nominal trajectory in the regions of

\[ \tilde{A}^T(t)M(t) + \dot{M}(t) + M(t)\tilde{A} \leq -\beta_M M(t), \quad \beta_M > 0 \]  

(24)

**Definition 2** (Contraction metric). Define the contraction metric \( M(t) := \tilde{P}^{-1}(t) \) being a positive definite, symmetric matrix in which \( \tilde{P}(t) \) is the solution to (7c). From Assumption 2, it is clear that \( M(t) \) is bounded,

\[ \frac{1}{p}I < M(t) < \frac{1}{\bar{p}}I \]  

(25)

For the choice of contraction metric in Definition 2, we proceed next in presenting a deterministic contraction result.

**Lemma 3.3.** Suppose Assumptions 1, 2 and 4 hold. Consider the contraction metric as per Definition 2, to be used for contraction analysis. Then, there exists a positive definite scalar \( \beta_M \) such that the virtual-actual system (18) contracts in the regions of

\[ \tilde{A}^T(t)M(t) + \dot{M}(t) + M(t)\tilde{A} \leq -\beta_M M(t), \quad \beta_M > 0 \]  

(26)
Proof. Simple algebraic manipulation for the left-hand side of relation (26) reveals

\[
\begin{align*}
\hat{A}^T M(t) + \dot{M}(t) + M(t)\dot{A} &= M(t) \left[ \hat{A} \dot{P}(t) - \dot{P}(t) + \dot{P}(t) \hat{A}^T \right] M(t) \\
&= M(t) \left[ \hat{A} \dot{P}(t) - \dot{P}(t) + \dot{P}(t) \hat{A}^T + \Delta \hat{A} \dot{P}(t) + \dot{P}(t) \Delta \hat{A}^T \right] M(t) \\
&= M(t) \left[ \hat{A} \dot{P}(t) - \dot{P}(t) + \dot{P}(t) \overline{A}^T - \mathcal{K}(t) C \dot{P}(t) - \dot{P}(t) C^T \mathcal{K}(t) + \Delta \hat{A} \dot{P}(t) + \dot{P}(t) \Delta \hat{A}^T \right] M(t) \\
&= M(t) \left[ \hat{A} \dot{P}(t) - \dot{P}(t) + \dot{P}(t) \overline{A}^T - \mathcal{K}(t) C \dot{P}(t) - \dot{P}(t) C^T \mathcal{K}(t) + \Delta \hat{A} \dot{P}(t) + \dot{P}(t) \Delta \hat{A}^T \right] M(t)
\end{align*}
\]

(27)

where from Lemma 2.1, we have applied the relation \( \dot{M}(t) = -M(t) \dot{P}(t) M(t) \). From Lemmata 2.1 and 3.1 we have that (27) evaluates

\[
M(t) \left[ f(\chi(t), t) W^T(\dot{\chi}(t)) + \dot{\chi}(t) W \dot{f}(\chi(t), t) - \mathcal{K}(t) C \dot{P}(t) - \dot{P}(t) C^T \mathcal{K}(t) \right] M(t)
\]

(28)

where from Lemma 2.1, we have applied the relation \( \dot{P}(t) = f(\chi(t), t) W \dot{\chi}(t) \). Relation (28) can further be simplified by incorporating the UKBF variance dynamics (7c). Hence, (28) evaluates

\[
M(t) \left[ \mathcal{K}(t) \sigma_z(t) R_c(t) \sigma_z^T(t) \mathcal{K}(t) - \sigma_z(t) Q_c(t) \sigma_z^T(t) - \mathcal{K}(t) C \dot{P}(t) - \dot{P}(t) C^T \mathcal{K}(t) \right] M(t)
\]

(29)

Next, for an intermediate calculation, multiply the UKBF Kalman gain (7a), from right, with \( \sigma_z(t) R_c(t) \sigma_z^T(t) \mathcal{K}(t) \), which gives

\[
\mathcal{K}(t) \sigma_z(t) R_c(t) \sigma_z^T(t) \mathcal{K}(t) = \chi(t) W^T(\chi(t), t) \mathcal{K}(t)
\]

(30)

By applying the relation \( \dot{P}(t) = h(\chi(t), t) W \dot{\chi}(t) \) in the context of Lemma 2.1, it follows that (30) can be expressed as

\[
\mathcal{K}(t) \sigma_z(t) R_c(t) \sigma_z^T(t) \mathcal{K}(t) = \dot{P}(t) C^T \mathcal{K}(t)
\]

(31)

Substituting relation (31), and conjugate transpose thereof, into (29), and applying Assumption 4, implies that (27) evaluates

\[
\begin{align*}
\hat{A}^T M(t) + \dot{M}(t) + M(t)\dot{A} &= M(t) \left[ -\mathcal{K}(t) \sigma_z(t) R_c(t) \sigma_z^T(t) \mathcal{K}(t) - \sigma_z(t) Q_c(t) \sigma_z^T(t) + \Delta \hat{A} \dot{P}(t) + \dot{P}(t) \Delta \hat{A}^T \right] M(t) \\
&= M(t) \left[ -\mathcal{K}(t) \sigma_z(t) R_c(t) \sigma_z^T(t) \mathcal{K}(t) - \sigma_z(t) Q_c(t) \sigma_z^T(t) + \Delta \hat{A} \dot{P}(t) + \dot{P}(t) \Delta \hat{A}^T - \gamma I + \frac{\gamma(1-\varepsilon)}{2\rho} I \bar{p} + \rho I \frac{2(1-\varepsilon)}{2\rho} \right] M(t) \\
&= -M(t) \gamma \epsilon I \mathcal{M}(t)
\end{align*}
\]

In light of Definition 2 we conclude there exists a positive scalar \( \beta_M = \frac{\gamma}{\rho} \) such that

\[
-M(t) \gamma \epsilon I \mathcal{M}(t) < -\beta_M M(t) < 0
\]

is satisfied for all \( t \).
**Theorem 3.4** (Deterministic contraction). There exists a positive definite scalar $\beta_M$ such that the deterministic process dynamics (10) will contract with contraction rate of $\beta_M$.

Proof. Lemma 3.3 concludes contraction for the virtual-actual system (18) and existence of a positive definite scalar $\beta_M$. Hence, exponential convergence to a single trajectory in the regions of (26) is implied. From Lemma 3.2 we have that the deterministic process dynamics, (10) is a particular solution to the virtual-actual system (18). Hence, the deterministic interpretation of (9) inherits the contraction properties of the virtual-actual system (18) with a contraction rate $\beta_M$ as established by Lemma 3.3.

3.2 Stochastic Contraction

Consider the general Itô-type stochastic differential equation

$$\dot{a} = F(a, t) \, dt + G(a, t) \, dW_a(t)$$

(33)

where $F(a, t) : \mathbb{R} \to \mathbb{R}^{n_a}$ is a deterministic system function, $G(a, t) \in \mathbb{R}^{n_a \times n_a}$ a noise-intensity matrix, and $dW_a(t)$ a standard Wiener process. Since $M(t)$ (as defined by Definition 2) is uniformly positive definite, there exists a continuously differential curve $\Gamma : [0, 1] \to \mathbb{R}^n$, with the boundary conditions $\Gamma(0) = a$ and $\Gamma(1) = b$, such that the minimum local distance between $a$ and $b$ is (Pham and Slotine 2013)

$$d_{M(t)}^2(a, b) = \int_0^1 (\frac{\partial \Gamma}{\partial \tau} (\tau))^T M(\Gamma(\tau)) \left( \frac{\partial \Gamma}{\partial \tau} (\tau) \right) d\tau$$

We next state a stochastic contraction theorem for the general Itô process (33).

**Theorem 3.5** (Stochastic contraction (Pham and Slotine 2013)). Consider the Itô-type stochastic differential system (33). Assume there exists a possibly state-dependent metric $M(t) := \theta(t)^T \theta(t)$ with lower bound $\lambda$, and a positive definite scalar $\beta_M$, which verify the conditions

$$\frac{\partial F^T}{\partial a} M(t) + \dot{M}(t) + M(t) \left( \frac{\partial F}{\partial a} \right) \leq -\beta_M M(t)$$

(C1)

and

$$\text{tr}\{G^T(a, t)M(t)G(a, t)\} \leq c$$

(C2)

uniformly for some constant $c$. Also, assume two given trajectories $a(t)$, $b(t)$, with initial conditions $a(0) = \xi$ and $b(0) = \xi'$, being independent of noise. Then, for all $t \geq t_0$

$$E\left\{\|\Delta a(t)\|^2 \right\} \leq \frac{c}{\lambda \beta_M} + \frac{1}{\lambda} E\left\{d_{M(0)}^2(\xi, \xi')\right\} e^{-2\beta_M t}$$

(34)

in which $\Delta a(t) := a(t) - b(t)$.

**Remark 3.2.** Theorem 3.5 states the necessary conditions (C1) and (C2) under which an Itô-type stochastic differential system (33) will contract in the state-dependent metric $M(t)$ with contraction rate $\beta_M$. If the metric $M(t)$ is in fact state-independent, then the bound (34) is optimal (Pham and Slotine 2013) and equivalent to that stipulated by Pham et al. (2009).
In the context of stochastic contraction analysis for the UKBF, we have satisfied (C1) previously in Lemma 3.3. Next, we proceed in analysing (C2) in the context of the UKBF, by means of Lemma 3.6.

**Lemma 3.6.** Suppose Assumptions 2 and 3 hold. Then, the exists a positive scalar $c$ such that

$$
\text{tr}\{\sigma_z^T(t)K^T(t)M(t)K(t)\sigma_z(t)\} \leq c
$$

**Proof.** From Assumption 3 it is clear there exists a constant $K_2 > 0$ such that

$$
\text{tr}\{\sigma_z^T(t)K^T(t)M(t)K(t)\sigma_z(t)\} \leq K_2(1 + ||\dot{X}||^2_F)
$$

From Assumption 2, in conjunction with Definition 2, it is clear that (35) holds if a positive scalar $c = \sup\left\{\frac{K_2}{\bar{p}}(1 + ||\dot{X}||^2_F)\right\}$ exists. Existence of $c$ is implied by boundedness of $\dot{X}(t)$, which holds since $\dot{X}(t)$ is constructed (4) from $\dot{P}(t)$, which is bounded by Assumption 2. ■

The main contribution for this work is presented next in the form of a contraction result for the UKBF.

**Theorem 3.7 (Contraction of UKBF).** Given an actual, noisy process (1) and corresponding UKBF observer (7), suppose Assumptions 1-4 hold. Then, the estimated expected mean of the process (1a), as observed by the UKBF, will contract in the contraction metric $M(t)$, as per Definition 2, towards a bounded region (34), with contraction rate $\beta_M$.

**Proof.** The deterministic part of the Itô-type stochastic differential equation (10) contracts in the metric $M(t)$ by Theorem 3.4 which verifies condition (C1). From Assumption 3, Lemma 3.6 verifies condition (C2). Hence, it follows that all the conditions given in Theorem 3.5 for stochastic contraction are met, which concludes the stochastic contraction of (9). ■

The contraction result of the UKBF in Theorem 3.7 allows us to conclude convergence for all initial trajectories of the estimated expected mean of process state to some nominal motion. As a consequence, it allows us to conclude on incremental stability, where stability in the context of contraction analysis, is analysed differentially (Lohmiller and Slotine 1998).

**Corollary 3.8 (Incremental stability of UKBF).**

The estimated mean of the process state, $\hat{x}(t)$, exponentially converge to a bounded region with the expected mean of the nominal process state, $\bar{x}(t)$, being in its interior, i.e.,

$$
\lim_{t \to \infty} E\left\{||\hat{x}(t) - \bar{x}(t)||^2\right\} \leq \frac{c\bar{p}^2}{\gamma \bar{p}}
$$

**Proof.** With respect to Theorem 3.5, (34), we evaluate $\lambda = \frac{1}{\bar{p}}$, using (25). Next, for the positive definite scalar $\beta_M = \frac{c\bar{p}}{\gamma \bar{p}}$ defined in Lemma 3.3, the upper bound in (36) follows as one evaluate the limit $t \to \infty$ of (34). Hence, a direct consequence of Theorem 3.7 is: (i) For a noisy system ($c > 0$): Exponentially asymptotic convergence of $\hat{x}(t)$ to a bounded region with $\bar{x}(t)$ in interior. (ii.) For vanishing noise at some local region of convergence ($c = 0$): Exponentially asymptotic convergence of $\hat{x}(t)$ to $\bar{x}(t)$. ■
4. Examples

4.1 Numerical Case

We consider an isothermal CSTR (Lee and Bailey 1980), \( A \rightarrow B, \ r = kc_A^n \), in which for the reaction rate \( r \), \( k \) is the rate constant and \( n \) is the reaction order. The material balance for this isothermal CSTR process is

\[
\frac{dc_A(t)}{dt} = \frac{1}{\tau} (c_{Af}(t) - c_A(t)) - kc_A^n(t) + w(t)
\]

with process measurement,

\[
z(t) = -\frac{1}{2\pi} (c_{Af}(t) - c_A(t)) + \frac{1}{2} kc_A^{n+1}(t) + v(t)
\]

in which \( w(t) \) and \( v(t) \) are additive process and measurement noise with a spectral densities of \( Q_c = 1 \) and \( R_c = 1 \) and noise intensity matrices \( \sigma_x = 0.75 \) and \( \sigma_z = 1 \), respectively. We have \( c_A(t) \) is the molar \( A \) concentration, \( c_{Af}(t) \) a manipulated feed rate of \( A \) concentration and parameters \( \tau = 10, k = 1.2, n = 2 \). We assume a sinusoidal manipulated feed rate, \( c_{Af}(t) = 100(1 - \sin(2\pi t)) \).

We are interested in implementing the UKBF as an observer to the CSTR process. That is, we want to estimate the expected mean of the molar \( A \) concentration, \( \bar{x}(t) := c_A(t) \), when stochastic excitation appears on the process dynamics, and for noisy measurements.

4.1.1 Discussion

The simulation was performed over a time period of 5s using a 4-th order stochastic Runge-Kutta method (Kasdin 1995) with time steps \( \Delta t = 5 \times 10^{-4} \). For initial conditions \( \hat{x}(0) = 0, \ P(0) = 1 \), Figure 1.a depicts the convergence of UKBF estimated expected mean of process state, \( \hat{x}(t) \), to the actual expected mean of the process, \( \bar{x}(t) \). Figure 1.b illustrates the corresponding convergence of estimated sigma-points to the actual deterministically sampled sigma-points (only for illustration purposes shown). Figure 1.c illustrates the validity of the contraction condition \( (C1) \) as established in Lemma 3.3 (note Figure 1.c depicts the expected average of the Left-Hand-Side (LHS) of \( (C1) \)). Estimation is done for noisy process measurements illustrated in Figure 1.d.
4.2 Analytic Case

For an analytic case study, we consider the estimation of a slightly modified version of the Ornstein-Uhlenbeck (OU) process, previously presented by Kowalczuk and Domzalski (2011). The process of interest describes the movement of a particle, with the dynamics

\[ \dot{v}(t) = \kappa (\theta - v(t)) + w(t) =: f(v(t), t) \quad (37a) \]
\[ z(t) = \alpha v(t) + r(t) =: h(v(t), t) \quad (37b) \]

in which \( v(t) \) defines the velocity of the particle. We have \( w(t) \) and \( r(t) \) being independent white process and measurement noise, respectively. Constant scalars \( \kappa \) and \( \theta \) define the rate of the velocity-mean reversion, and long-term mean of the velocity, respectively. We consider the differential measurement \((37b)\), in which the scalar \( \alpha \) is a constant. We apply the UKBF as an observer to \((37)\). The primary interest is to establish convergence of the estimated particle velocity, \( \hat{v}(t) \), towards the actual particle velocity, \( v(t) \), on average. We first analyse deterministic contraction of the applied UKBF in the context of Lemma 3.3. Consider the analytic Jacobians of \((37a)\) and \((37b)\) (in a deterministic setting). Then, from \((20)\), and using the UKBF Kalman gain \( K(t) \), define the analytic Jacobian system matrix as \( A := -\kappa + K(t)\alpha \). For the statistical Jacobian evaluation of \((37a)\) and \((37b)\), we apply Lemma 3.1. By inspection, (similar computations as in Lemma 3.1) it can be shown that the coefficient \( b \) should be chosen as \( b = \kappa \theta - \kappa E\{v(t)\} - A E\{v(t)\} \). Given the choice of \( b \), computations similar to that of \((17)\), reveals

\[
\frac{\partial^2 E}{\partial A} = E\{(f(v) - Av - E\{f(v)\}) + E\{Av\}(-v + E\{v\})\}
= E\{\kappa (\theta - v) - Av - E\{\kappa (\theta - v)\} + E\{Av\}(-v + E\{v\})\}
= -\kappa E\{(v - E\{v\})(-v + E\{v\}) + A E\{(-v + E\{v\})(-v + E\{v\})\}
= \kappa P + AP = 0 \quad (38)
\]

That is, \((15a)\) is minimized for \( A = -\kappa \). Similarly, \((15b)\) is minimized for \( C = \alpha \). If we use the coefficients \( \{A, C\} \), in conjunction with the UKBF Kalman gain \( K(t) \), and define the statistical Jacobian system matrix, \( \hat{A} = A + K(t)C \), then it is clear that \( \hat{A} \equiv \hat{A} \). It follows from \((21)\) that \( \Delta A = 0 \). From the latter result, analysis complexity in Lemma 3.3, is considerably reduced, i.e.,

\[
\hat{A}^T M(t) + \hat{M}(t) + M(t) \hat{A} = M(t) \left[ \hat{A} \hat{P}(t) - \hat{P}(t) + \hat{P}(t) \hat{A}^T \right] M(t)
\]

\[
\begin{align*}
(\text{Lemma 3.1}) & \Rightarrow M(t) \left[ \hat{P}(t) - \hat{P}(t) + \hat{P}(t) \hat{A} \hat{M}(t) \hat{A}^T \right] M(t) \\
& \Rightarrow M(t) \left[ \hat{P}(t) - \hat{P}(t) + \hat{P}(t) \hat{M}(t) \hat{A} \hat{M}(t) \hat{A}^T \right] M(t)
\end{align*}
\]

If we apply the relation \( \hat{P}(t) = f(\chi(t), t) W \chi^T(t) \) to \((39)\) (see Lemma 3.1 and Remark 3.1), then, it follows that

\[
\hat{A}^T M(t) + \hat{M}(t) + M(t) \hat{A} = M(t) \left[ -\kappa(t) \sigma_x(t) R_c(t) \sigma_x^T(t) \kappa^T(t) - \sigma_x(t) Q_c(t) \sigma_x^T(t) \right] M(t)
\leq -M(t) \gamma IM(t) < -\beta_M M(t)
\]

which concludes deterministic contraction with a contraction rate of \( \beta_M = \frac{\gamma}{2} \). Under standard assumptions (in particular Assumption 3) one can proceed in applying Corollary 3.8 to conclude stochastic convergence of the UKBF, as applied to the process \((37)\).
5. Concluding Remarks

This work presents results on incremental stability for the UKBF given a process with stochastic noise excitation on process dynamics, and noisy measurements. Incremental stability for the UKBF is established by analysing convergence differentially by employing contraction theory. In particular, deterministic contraction is first shown for a virtual-actual systems framework. Under mild conditions of stochastic system growth, stochastic contraction is subsequently concluded for the estimated expected process dynamics (7b), where the latter is a particular solution of the virtual-actual systems framework. Convergence, being a consequential characteristic of a contractive systems (Lohmiller and Slotine 1998), enables us to conclude with results on incremental stability for the UKBF.

Contraction theory requires analytic Jacobians of the underlying system of interest to analyze convergence differentially. The UKBF, however, utilizes a unscented transform-based framework for computing approximations to the optimal Bayesian filter. It follows that Assumption 4 is required to accommodate the application of contraction theory (in a similar simplistic manner as shown for the EKF (Jouffroy and Fossen 2010)) for convergence analysis of the UKBF. Assumption 4 establishes bounds on the difference between an analytic and statistical Jacobian of the underlying system. As in the special case of Example 4.2, we note that for linear systems, the analytic and statistical Jacobian are equivalent; hence, one can apply contraction theory in a straightforward manner to analyze convergence of the UKBF. It is worth pointing out, however, that for the special class of systems (linear) in Example 4.2, one could have simply applied the Kalman-Bucy filter as observer. The latter is the optimal Bayesian filter for estimation, if also the additional restrictive requirement of Gaussian noise on the process and measurement models is satisfied.

A possible route for future work will be the extension of contraction theory to incorporate statistical system Jacobians, instead of analytic system Jacobians, to analyze convergence differentially in a statistical average sense. The latter, in addition, will allow the application of contraction theory for convergence analysis of systems in which: (i) the analytic Jacobian cannot be evaluated, (ii) the underlying system of interest incorporates unscented transforms in its formulation (without the prior requirement of Assumption 4). As noted by Gelb et al. (1974); Van der Merwe (2004), more accurate (on average) system Jacobians are evaluated, in general, when one apply statistical linearisation techniques. In the context of contraction theory, it may prove insightful to investigate the influence of statistical Jacobians with respect to contraction regions.

References


Appendix A. Derivation of UKBF

\textbf{Proof.} Consider the continuous-time, system model (1). First, discretize process model (1) with a time increment of $\Delta t$ and assume that for $\Delta t$ sufficiently small it holds that $w(t)\Delta t \sim \mathcal{N}(0, Q_c(t)\Delta t)$ and $v(t)\Delta t \sim \mathcal{N}(0, R_c(t)\Delta t)$. It follows that

$$
\begin{align*}
x(t + \Delta t) &= x(t) + f(x(t), t)\Delta t + \sigma_x(t)w(t)\Delta t + o(\Delta t) \\
\Delta y &= h(x(t + \Delta t), t + \Delta t)\Delta t + \sigma_z(t)v(t)\Delta t
\end{align*}$$

where $o(\Delta t)$ is a discretization error where it holds that $\frac{o(\Delta t)}{\Delta t} \to 0$ and $\Delta y = y(t + \Delta t) - y(t) \to \frac{dy(t)}{dt} = z(t)$ when $\Delta t \to 0$. For the estimated process state Gaussian distribution $\mathcal{N}(\hat{x}(t), \hat{P}(t))$, consider $\hat{X}(t)$ as defined in (4). Then, for a $\Delta t$ system evolution from $x(t)$ to $x(t + \Delta t)$,

$$\hat{X}(t + \Delta t) = \hat{X}(t) + f(\hat{X}(t), t)\Delta t + o(\Delta t)$$

express the predicted mean $\hat{x}^-(t + \Delta t) = \hat{X}(t + \Delta t)W$ and variance as

$$\hat{P}^-(t + \Delta t) = \hat{X}(t + \Delta t)W\hat{X}^T(t + \Delta t) + \sigma_x(t)Q_c(t)\sigma_x^T(t)\Delta t$$

Next, substituting $\hat{X}^-(t)$ into $\hat{X}^-(t)$ and using the relations $\hat{x}^-(t) = \hat{X}(t)W$ and $\hat{P}(t) = \hat{X}(t)W\hat{X}^T(t)$, it follows

$$\hat{x}^-(t + \Delta t) = \hat{x}^-(t) + f(\hat{X}(t), t)W\Delta t + o(\Delta t)$$

$$\hat{P}^-(t + \Delta t) = \hat{P}(t) + \hat{X}(t)Wf^T(\hat{X}(t), t)\Delta t + f(\hat{X}(t), t)W\hat{X}^T(t)\Delta t + \sigma_x(t)Q_c(t)\sigma_x^T(t)\Delta t$$

assuming we are neglecting higher order terms. We can write the measurement update equations, given $\hat{X}(t)$, as follows

$$\hat{X}(t + \Delta t) = \left[\hat{X}_{-n_z}(t + \Delta t), \cdots, \hat{X}_{n_z}(t + \Delta t)\right]$$

$$\hat{Z}(t + \Delta t) = h(\hat{X}^-(t + \Delta t), t + \Delta t)\Delta t$$

$$\hat{z}^-(t + \Delta t) = h(\hat{X}(t + \Delta t), t + \Delta t)\Delta t$$

$$\hat{P}_{zz}(t + \Delta t) = \hat{Z}(t + \Delta t)W\hat{Z}^T(t + \Delta t) + \sigma_z(t)R_c(t)\sigma_z^T(t)\Delta t$$

$$\hat{P}_{xz}(t + \Delta t) = \hat{X}^-(t + \Delta t)W\hat{Z}^T(t + \Delta t)$$

and the UKF update step as

$$\mathcal{K}(t + \Delta t) = \hat{P}_{xz}(t + \Delta t)\hat{P}_{zz}^{-1}(t + \Delta t)$$

$$\hat{x}(t + \Delta t) = \hat{x}^-(t + \Delta t) + \mathcal{K}(t + \Delta t)\left[\Delta y - \hat{z}^-(t + \Delta t)\right]$$

$$\hat{P}(t + \Delta t) = \hat{P}^-(t + \Delta t) - \mathcal{K}(t + \Delta t)\hat{P}_{xz}(t + \Delta t)\mathcal{K}^T(t + \Delta t)$$
Next, substitute (A6) into (A7), and retain only first-order terms, on which the Kalman gain update expression,

\[ K(t + \Delta t) = \hat{X}^-(t + \Delta t) W H^T(t + \Delta t) [\sigma_z(t) R_c(t) \sigma_z^T(t)]^{-1} + \frac{o(\Delta t)}{\Delta t} \] (A8)

estimated mean update expression, and,

\[ \dot{x}(t + \Delta t) - \dot{x}^-(t) = f(\hat{X}(t), t) W \Delta t + K(t + \Delta t) [\Delta y - h(\hat{X}^-(t + \Delta t), t + \Delta t) W \Delta t] \] (A9)

estimated variance update expression

\[ \hat{P}(t + \Delta t) - \hat{P}(t) = \dot{\hat{X}}(t) W f^T(t + \Delta t) + f(\hat{X}(t), t) W \dot{\hat{X}}^T(t) \Delta t + \sigma_z(t) Q_c(t) \sigma_z^T(t) \Delta t - K(t + \Delta t) \sigma_z(t) R_c(t) \sigma_z^T(t) \Delta t K^T(t + \Delta t) \Delta t \] (A10)

follows. The expressions for the optimal continuous-time filtering expressions of the UKBF in Algorithm 2.1 is subsequently evaluated when dividing expressions (A8)-(A10) by \( \Delta t \), taking the time-limit of \( \Delta t \rightarrow 0 \), and applying \( z(t) = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} \) from the differential measurement assumption.

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