Note

A note on the chromaticity of some 2-connected (n, n + 3)-graphs

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Abstract

Let \( P(G, \lambda) \) denote the chromatic polynomial of a graph \( G \). A graph \( G \) is chromatically unique if \( G \cong H \) for any graph \( H \) such that \( P(H, \lambda) = P(G, \lambda) \). This note corrects an error in the proof of the chromatic uniqueness of certain 2-connected graphs with \( n \) vertices and \( n + 3 \) edges.

Keywords: Chromatic polynomial; Chromatic uniqueness; Chromatically closed set; Chromatically equivalence class; Homeomorphism class

We consider only simple graphs in this note. Let \( P(G, \lambda) \) denote the chromatic polynomial of a graph \( G \). Two graphs \( G \) and \( H \) are chromatically equivalent (denoted by \( G \sim H \) ) if \( P(G, \lambda) = P(H, \lambda) \). The relation \( \sim \) is an equivalence relation on the class of graphs. We denote by \( \langle G \rangle \) the equivalence class determined by the graph \( G \) under \( \sim \). Thus \( G \) is chromatically unique iff \( \langle G \rangle = \{ G \} \) (see [1,3,5]).

Let \( \mathcal{P} \) be a set of graphs. It is clear that \( \mathcal{P} \subseteq \bigcup_{G \in \mathcal{P}} \langle G \rangle \). If \( \bigcup_{G \in \mathcal{P}} \langle G \rangle = \mathcal{P} \), then \( \mathcal{P} \) is called chromatically closed. Observe that a chromatically closed set \( \mathcal{P} \) is the union of some chromatic equivalence classes.

Let \( G_1(s, k, t) \) and \( G_2(\eta, \varepsilon) \) denote the graphs shown in Fig. 1. Let

\[
\mathcal{P} = \{ G_1(s, k, t) | s + t \geq 3, k + t \geq 3 \} \cup \{ G_2(\eta, \varepsilon) | \eta \geq 2, \varepsilon \geq 2 \}.
\]

Koh and Teo [2] showed that \( \mathcal{P} \) is chromatically closed and each graph in \( \mathcal{P} \) is chromatically unique. Very recently, Li and Feng [4] discovered that \( G_1(2, 2, 1) \), \( G_1(1, 1, 3) \) and \( G_2(2, 2) \) are chromatically equivalent. Thus, the result that each graph in \( \mathcal{P} \) is

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Fig. 1.

chromatically unique is not correct. In this note we determine the chromatic equivalence class for each graph in \( \mathcal{P} \).

We have carefully checked the proof in [2] on the chromatic closedness of \( \mathcal{P} \). We believe that their proof is correct.

**Lemma 1.** The set \( \mathcal{P} \) is chromatically closed.

The results in the note will be based on Lemma 1. Thus to determine the chromatically equivalence class of any graph \( G \in \mathcal{P} \), we need only search graphs \( H \) in the set \( \mathcal{P} \) such that \( H \sim G \). To do this, we shall compare \( P(H, \lambda) \) with \( P(G, \lambda) \) for each \( H \in \mathcal{P} \) to see if \( H \sim G \). Read and Whitehead [6] obtained some excellent results on determining the chromatic polynomials of homeomorphism classes of graphs by their reduced multigraphs and chain lengths. It is clear that \( G_1(s, k, t) \) and \( G_2(\eta, \epsilon) \) are homeomorphs of \( W_5 \), the wheel on 5 vertices. Thus we can obtain \( P(G_1(s, k, t), \lambda) \) and \( P(G_2(\eta, \epsilon), \lambda) \) by using the result of (6.3) on page 350 of [6].

We now cite some concepts and terminologies in [6]. By the ‘suppression’ of a vertex \( x \) of degree 2 in a graph \( G \) we shall mean the following operation: we delete the vertex \( x \) (deleting also the edges incident with it) and join, by a new edge, the vertices previously adjacent to \( x \). If, starting with a graph \( G \), we successively suppress vertices of degree 2 until this operation is no longer possible, we arrive at a graph having no vertices of degree 2. We shall call this graph the ‘reduced multigraph’ of \( G \), and denote it by \( M(G) \), or just \( M \).

The chain polynomial \( \text{Ch}(G, \omega) \) is defined to be

\[
\text{Ch}(G, \omega) = (-1)^q \lambda^{q-p} P(G, \lambda),
\]

where \( \omega = 1 - \lambda \), \( q \) and \( p \) are the edge number and vertex number of \( G \), respectively.

Observe that \( M(G_1(s, k, t)) \) and \( M(G_2(\eta, \epsilon)) \) are \( W_5 \), the wheel of 5 vertices. By the result of (6.3) in page 350 of [6], the following two results are obtained.

**Lemma 2.** For positive integers \( s, k \) and \( t \), we have

\[
\text{Ch}(G_1(s, k, t), \omega) = \omega(\omega + 1)(\omega - 1)^2((\omega + 1)\omega^{s+k+t} - 2\omega') - (\omega + 1)(\omega^k + \omega') + \omega + 3).
\]
Lemma 3. For positive integers \(\eta\) and \(\varepsilon\), we have
\[
\text{Ch}(G_2(\eta, \varepsilon), \omega) = \omega(\omega + 1)(\omega - 1)^2((\omega + 1)\omega^{\eta+\varepsilon+1} - (\omega + 1)(\omega^\eta + \omega^\varepsilon) - \omega + 3).
\]

By making use of Lemmas 2 and 3, we compare \(P(G, \lambda)\) with \(P(H, \lambda)\) for any \(G, H \in \mathcal{P}\). There are three cases to consider, which we present in the following three results.

Lemma 4. Let \(s, k, t, s', k', t'\) be positive integers with \(s \geq k\), \(s' \geq k'\) and \(t \geq t'\). Then \(G_1(s, k, t) \sim G_1(s', k', t')\) iff either
(i) \(s' = s\), \(k' = k\) and \(t' = t\), or
(ii) \(s = k = t - 2 = s' - 1 = k' - 1 = t'\).

Proof. By the definition, we have
\[
P(G_1(s, k, t), \lambda) = (-1)^{s+k+t+1}\lambda^{-3}\text{Ch}(G_1(s, k, t), \omega),
\]
where \(\omega = 1 - \lambda\).

It is straightforward to check the sufficiency by Lemma 2. We prove the converse. Assume that \(P(G_1(s, k, t), \lambda) = P(G_1(s', k', t'), \lambda)\). Then \(G_1(s, k, t)\) and \(G_1(s', k', t')\) have the same order, implying that \(s + k + t = s' + k' + t'\). Thus we have
\[
\text{Ch}(G_1(s, k, t), \omega) = \text{Ch}(G_1(s', k', t'), \omega).
\]
By Lemma 2, we have
\[
(\omega + 1)(\omega^k + \omega^t) + 2\omega^s = (\omega + 1)(\omega^{k'} + \omega^{t'}) + 2\omega^{s'},
\]
for all real \(\omega\). Let \(D(\omega)\) be the difference of the left-hand side and right-hand side of the above identity. Thus \(D(\omega) = 0\) for all real \(\omega\). As \(D(-1) = 2(-1)^t - 2(-1)^{t'}\), \(t - t'\) is even. Hence,
\[
D(\omega) = (\omega + 1)\left(w^k + w^s - w^{k'} - w^{s'}\right) - 2w^{s'}(1 + \omega) \sum_{i=0}^{t-t'-1} (-1)^i w^i.
\]
Let \(D(\omega) = (1 + \omega)F(\omega)\). We have \(F(\omega) = 0\) for all real \(\omega\). Observe that
\[
F(-1) = (-1)^k + (-1)^t - (-1)^{k'} - (-1)^{t'} - 2(-1)^{t'}(t - t') = 0.
\]
Thus \(2(t - t') \leq 4\), implying that \(t = t'\) or \(t = t' + 2\).
If \(t = t'\), then
\[
F(\omega) = w^k + w^s - w^{k'} - w^{s'} = 0
\]
for all real numbers \(\omega\). Since \(s \geq k\) and \(s' \geq k'\), we have \(s' = s\) and \(k' = k\).
If \( t = t' + 2 \), we have
\[
F(\omega) = \omega^k + \omega^s + 2\omega^{t' + 1} - 2w^{t'} - w^{k'} - w^{s'} = 0
\]
for all real numbers \( w \). Thus \( k' = s' = t' + 1 \) and \( t' = k = s \).

This completes the proof. \( \square \)

**Lemma 5.** Let \( s, k, t, \eta \) and \( \varepsilon \) be positive integers with \( s \geq k \) and \( \eta \geq \varepsilon \). Then \( G_1(s, k, t) \sim G_2(\eta, \varepsilon) \) iff either

(i) \( t = 1 \), \( s = \eta \) and \( k = \varepsilon \) or

(ii) \( t = 3, k = s = 1 \) and \( \eta = \varepsilon = 2 \).

**Proof.** By definition,
\[
P(G_1(s, k, t), \lambda) = (-1)^{s+k+t+1}\lambda^{-3} \text{Ch}(G_1(s, k, t), \omega)
\]
and
\[
P(G_2(\eta, \varepsilon), \lambda) = (-1)^{\eta+\varepsilon}\lambda^{-3} \text{Ch}(G_2(\eta, \varepsilon), \omega),
\]
where \( \omega = 1 - \lambda \).

It is straightforward to check the sufficiency by Lemmas 2 and 3. Now assume that \( P(G_1(s, k, t), \lambda) = P(G_2(\eta, \varepsilon), \lambda) \). Then \( G_1(s, k, t) \) and \( G_2(\eta, \varepsilon) \) have the same order, implying that \( s + k + t = \eta + \varepsilon + 1 \). By Lemmas 2 and 3, we have
\[
-2\omega + 2\omega^{t'} + (\omega + 1)(\omega^k + \omega^s) - (\omega + 1)(\omega^\eta + \omega^\varepsilon) = 0
\]
for all real numbers \( \omega \). Let \( \mu(\omega) \) denote the left-hand side of this identity. Observe that \( \mu(-1) = 2 + 2(-1)^t \). Thus \( t \) is odd.

If \( t = 1 \), we have
\[
\mu(\omega) = (\omega + 1)(\omega^k + \omega^s - \omega^\eta - \omega^\varepsilon) = 0
\]
for all real numbers \( \omega \), implying that \( s = \eta \) and \( k = \varepsilon \), since \( s \geq k \) and \( \eta \geq \varepsilon \).

Now assume that \( t \geq 3 \) is odd. Then
\[
\mu(\omega) = (\omega + 1) \left( \omega^k + \omega^s - \omega^\eta - \omega^\varepsilon + 2\sum_{i=1}^{t-1} (-1)^i \omega^i \right).
\]
Let \( \mu(\omega) = (\omega + 1)v(\omega) \). Since \( v(\omega) \) is the zero polynomial, \( v(-1) = 0 \) in particular. Thus
\[
(-1)^k + (-1)^s - (-1)^\eta - (-1)^\varepsilon + 2(t - 1) = 0.
\]
Hence \( t \leq 3 \), implying that \( t = 3 \). Therefore
\[
v(\omega) = \omega^k + \omega^s - \omega^\eta - \omega^\varepsilon + 2\omega^2 - 2\omega = 0
\]
for all real numbers \( \omega \), implying that \( k = s = 1 \) and \( \eta = \varepsilon = 2 \).

This completes the proof. \( \square \)
**Lemma 6.** Let $\eta, \varepsilon, \eta'$ and $\varepsilon'$ be positive integers with $\eta \geq \varepsilon$ and $\eta' \geq \varepsilon'$. Then $G_2(\eta, \varepsilon) \sim G_2(\eta', \varepsilon')$ iff $\eta = \eta'$ and $\varepsilon = \varepsilon'$.

**Proof.** By definition,

$$P(G_2(\eta, \varepsilon), \lambda) = (-1)^{\eta + \varepsilon - 3} \text{Ch}(G_2(\eta, \varepsilon), \omega),$$

where $\omega = 1 - \lambda$. Thus the sufficiency can be verified easily by Lemma 3. We prove the converse. Assume that $G_2(\eta, \varepsilon) \sim G_2(\eta', \varepsilon')$. Then $G_2(\eta, \varepsilon)$ and $G_2(\eta', \varepsilon')$ have the same order, implying that $\eta + \varepsilon = \eta' + \varepsilon'$. We also have $\text{Ch}(G_2(\eta, \varepsilon), \omega) = \text{Ch}(G_2(\eta', \varepsilon'), \omega)$. By Lemma 3, we have

$$\omega^\eta + \omega^\varepsilon = \omega'^\eta + \omega'^\varepsilon'$$

for all real numbers $\omega$. Since $\eta \geq \varepsilon$ and $\eta' \geq \varepsilon'$, we have $\eta = \eta'$ and $\varepsilon = \varepsilon'$.

By Lemmas 1 and 4–6, we finally arrive at the following result.

**Theorem.** Let $s, k$ and $t$ be positive integers. Then

(i) $\{G_1(2,2,1), G_1(1,1,3), G_2(2,2)\}$ is a chromatic equivalence class;

(ii) $\{G_1(s,k,1), G_2(s,k)\}$ is a chromatic equivalence class if $s \geq 2$, $k \geq 2$ and $s + k \geq 4$;

(iii) $\{G_1(s,s,s+2), G_1(s+1,s+1,s)\}$ is a chromatic equivalence class if $s \geq 2$;

(iv) $G_1(s,k,t)$ is chromatically unique if $t \geq 2$, $(s,k,t) \neq (a,a,a+2)$ and $(s,k,t) \neq (a+1,a+1,a)$ for all $a \geq 1$.

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**References**


