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On a question of T. Sheil-Small regarding valency of harmonic maps

Dedicated to Bogdan Bojarski on the occasion of his 80th birthday

ABSTRACT. The aim of this work is to answer positively a more general question than the following which is due to T. Sheil-Small: Does the harmonic extension in the open unit disc of a mapping f from the unit circle into itself of the form $f(e^{it}) = e^{i\varphi(t)}$, $0 \leq t \leq 2\pi$, where φ is a continuously non-decreasing function that satisfies $\varphi(2\pi) - \varphi(0) = 2N\pi$, assume every value finitely many times in the disc?

Introduction. Let $\mathbb{D}$ and $\mathbb{T}$ be the open unit disc and the unit circle respectively, and let N be a positive integer. An ***N*-valent quasi-homeomorphism from the unit circle into itself** is a circle mapping $f : \mathbb{T} \to \mathbb{T}$ of the form $f(e^{it}) = e^{i\varphi(t)}$, $0 \leq t \leq 2\pi$, where φ is a non-decreasing function that satisfies $\varphi(2\pi) - \varphi(0) = 2N\pi$. It can be seen that every such quasi-homeomorphism is a pointwise limit of a sequence of circle mappings $f_n : \mathbb{T} \to \mathbb{T}$ of the form $f_n(e^{it}) = e^{i\varphi_n(t)}$, $0 \leq t \leq 2\pi$, where φ_n is a continuously strictly increasing function that satisfies $\varphi_n(2\pi) - \varphi_n(0) = 2N\pi$.

A **1-valent quasi-homeomorphism** is referred to as **quasi-homeomorphism.**

The celebrated Radó–Kneser–Choquet Theorem can be stated as follows.

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Theorem A (Radó–Kneser–Choquet Theorem [3, pp. 29–34])**.** *Suppose that F is the harmonic extension in $\mathbb{D}$ of a quasi-homeomorphism f from the unit circle into itself. Then F is univalent.*

In an attempt to generalize this theorem to 2-valent quasi-homeomorphisms f from the unit circle into itself, it was suggested that the respective functions F are at most 4-valent. However, examples presented in [2] have shown that some of these mappings could be 6-valent or 8-valent. Furthermore, the construction procedure used in the paper suggested the possibility of finding a 2-valent quasi-homeomorphism from the unit circle into itself whose harmonic extension in $\mathbb{D}$ assumes a predetermined finite valency. But this remains an open problem.

In a personal communication with the first author about a decade ago, T. Sheil-Small raised the following question:

> *If F is the harmonic extension in $\mathbb{D}$ of a mapping f of the form $f(e^{it}) = e^{i\varphi(t)}$, $0 \le t \le 2\pi$, where φ is a continuously non-decreasing function that satisfies $\varphi(2\pi) - \varphi(0) = 2N\pi$, then does F assume every value finitely many times in the disc?*

In this note, we show that the answer to this question is positive. In fact, a more general result is shown henceforth to be true.

For a function $F : \mathbb{D} \to \mathbb{C}$ and a point $\zeta \in \mathbb{T}$, let $C(F, \zeta)$ and $C(F, \mathbb{T})$ denote the cluster sets of F at ζ and on $\mathbb{T}$ respectively.

The result of the note can be stated as follows.

Theorem 1. *Suppose that F is the harmonic extension in $\mathbb{D}$ of an N-valent quasi-homeomorphism f from the unit circle into itself that takes on three distinct values. Then F takes on every point in $\mathbb{D} \setminus C(F, \mathbb{T})$ finitely many times.*

As a consequence we have:

Corollary 2. *Suppose that F is the harmonic extension in the open unit disc of a mapping f of the form $f(e^{it}) = e^{i\varphi(t)}$, $0 \le t \le 2\pi$, where φ is a continuously non-decreasing function that satisfies $\varphi(2\pi) - \varphi(0) = 2N\pi$. Then F takes on every point in $\mathbb{D}$ finitely many times.*

Before embarking on the proof of Theorem 1, we define an *algebraic curve* as a connected component of the preimage of a straight line or circle under an analytic function.

Proof of Theorem 1. Write $F = u + iv$, where u and v are the real and imaginary parts of F. Suppose that there exist a point $\omega \in \mathbb{D} \setminus C(F, \mathbb{T})$ and a set S of countably infinitely many distinct values $z_n \in \mathbb{D}, n = 1, 2, \ldots$ such that $F(z_n) = \omega$ for all n; note that $|z_n| \le \rho < 1$ for some ρ since $\omega \notin C(F, \mathbb{T})$. Let $\omega = u_0 + iv_0$ for $u_0, v_0 \in \mathbb{R}$; then $u(z_n) = u_0$ and $v(z_n) = v_0$ for all n.

Consider the level set $u = u_0$; note that this is a set-union of mutually disjoint algebraic curves. Suppose that each of these curves carries a finite subset of S of the points z_n. Then these curves are countably infinite and may be denoted by $C_n, n = 1, 2, \ldots$. Label one of the points of $S \cap C_n$ by ζ_n for every n. Since $|\zeta_n| \leq \rho < 1$ for all n, there exists a subsequence (ζ_{n_k}) of (ζ_n) that converges to a point ζ. Evidently, $|\zeta| \leq \rho < 1$, $F(\zeta) = \omega$ and ζ belongs to some level curve $C : u = u_0$. This yields a contradiction since near ζ the curve C fails to be isolated from the level curves C_n.

It follows that the level set $u = u_0$ is a disjoint union of finitely many algebraic curves of which one, say C, carries countably infinitely many points z_n that we denote by $\zeta_1, \zeta_2, \ldots$. Observe the following:

(1) C never encloses a Jordan domain in $\mathbb{D}$ because of the maximum principle for harmonic functions;
(2) C is a union of analytic Jordan arcs γ that are mutually disjoint except possibly for a common critical point of u;
(3) Every γ clusters in $\mathbb{T}$.

Suppose that some arc γ accumulates on a non-degenerate subarc $J \subset \mathbb{T}$; denote the interior of J by J°. Let $\eta \in J^\circ$. Note that in every direction towards η from $\mathbb{D}$ there exists a sequence of points in γ converging to η on which u attains the value u_0. This entails by a result of Schwarz [1, Theorem 23] that u is continuous and is identically u_0 on J°.

Let g be the analytic completion on u. By the reflection principle, g is analytic on J°. Fix $\eta \in J^\circ$. It is immediate that $g([0, \eta])$ is an arc that meets the vertical line $L : u = u_0$ in the (u, v)-plane at countably infinitely many points that are away from infinity. Since both arcs $g([0, \eta])$ and L are analytic, $g([0, \eta]) \subset L$, see [4, Theorem 7.19, pp. 241–244], and equivalently $u = u_0$ on $[0, \eta]$. But η is an arbitrary point of J°; hence u is identically u_0 on the open circular sector with vertex at the origin and subtending J° and consequently on $\mathbb{D}$, which yields a contradiction.

Thus every Jordan arc γ terminates in every direction at a point in $\mathbb{T}$. We contend that every γ is a crosscut of $\mathbb{D}$. For suppose otherwise, then some γ is a loop with a unique point $\eta \in \overline{\gamma} \cap \mathbb{T}$. If $G \subset \mathbb{D}$ is the bounded region enclosed by γ, then, because u is a bounded harmonic function, the limit

$$\lim_{z \to \eta} u(z) = u_0 \text{ through values } z \in \overline{G}.$$

We infer, by the maximum principle, that u is identically u_0 in G and consequently in $\mathbb{D}$, which gives a contradiction. This proves our claim.

Suppose now that γ terminates at two distinct points $\alpha, \beta \in \mathbb{T}$, and let $\gamma' \subset C$ be a crosscut of $\mathbb{D}$ similar to γ. It is immediate that γ' can not terminate at both α, β. In fact, γ' can neither terminate at α nor at β. For suppose γ' terminates at α; then, since C is connected, there exists a continuum that meets both γ and γ'. But then $K \cup \gamma \cup \gamma'$ bounds a Jordan subdomain K of $\mathbb{D}$, which gives a contradiction.

It follows at once that $\overline{\gamma}$ and $\overline{\gamma'}$ are either disjoint or cross at a singleton in $\mathbb{D}$; namely a critical point of u. Thus $\overline{C}$ is a tree whose vertices are the critical points of u and the terminal points of the arcs γ. We show that this tree is finite. Suppose otherwise, then the crosscuts γ comprising C are countably infinite, and consequently the same are the endpoints of C. The latter points subdivide $\mathbb{T}$ into countably infinitely many subarcs λ. Let λ_1 and λ_2 be two of these arcs that share a common terminal point ν, and let G_1 and G_2 be the Jordan domains bounded by $\overline{C} \cup \lambda_1$ and $\overline{C} \cup \lambda_2$ respectively. Note that G_1 and G_2 have a common boundary arc, denoted by $\delta \subset C$, with an endpoint at ν. Evidently, $g(\delta)$ is a line segment of the vertical line $L : u = u_0$. Note that g, like u, has no critical points in the interior of δ since g and u share these points, and that $u(z) \neq u_0$ and $u(z') \neq u_0$ for all $z \in G_1$ and $z \in G_2$ or else $C \cap (G_1 \cup G_2)$ is nonempty.

It follows that $g(G_1)$ and $g(G_2)$ lie on different sides of L. But by the hypotheses on f, $u - u_0$ cannot change the sign more than N times. This implies at once that the number of arcs λ is at most $2N$; thus the number of crosscuts γ comprising C is at most N.

We conclude that some crosscut γ, denoted by Γ, contains infinitely countably many points ζ_n. We may assume without loss of generality that $\zeta_n \in \Gamma$ for all $n = 1, 2, \ldots$.

On the other hand, by undergoing the same discussion on v instead of u we conclude that there exists a crosscut Γ' that is contained in the level set $v = v(z_0)$ and contains infinitely countably many of the points $\zeta_n \in \Gamma$, $n = 1, 2, \ldots$. Since every $|\zeta_n| \leq \rho < 1$ $n = 1, 2, \ldots$ and the arcs Γ and Γ' are analytic, Γ and Γ' coincide.

Suppose now that $\xi \in \mathbb{T}$ is a terminal point of Γ (or Γ'). Then

$$u(z) \to u_0 \quad \text{for} \quad z \in \Gamma \quad z \to \xi;$$

hence $u_0 \in C(u, \xi)$. By the same token we conclude that $v_0 \in C(v, \xi)$. Therefore, $\omega \in C(F, \mathbb{T})$ and we have a contradiction to our original assumption. This completes the proof of Theorem 1.

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