THE ELASTIC DIELECTRIC

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Of central importance to the problem of the determination of the stress distribution within an elastic dielectric is a paper by Slepian,[1] which is usually quoted in any discussion of the topic. Slepian based his analysis on the expression

$$\mathbf{F} = \oiint_S \epsilon_0 [\mathbf{E}(\mathbf{E}\cdot\mathbf{n}) - {}^1/_2 E^2\mathbf{n}]dS \tag{1}$$

for the force on a dielectric in an electrical field $\mathbf{E}$ and bounded by a surface S with unit normal $\mathbf{n}$. He used an operational "definition" of stress and concluded that the compensating mechanical forces which must be introduced operationally are not derivable from a tensor. It is suggested here that Slepian's analysis is essentially correct and that the difficulty arises because of the choice of an operationally "defined" stress. This choice is inconsistent with the existence of an electrical surface stress—which is familiar, in the magnetic analogue, in studies of the form effect—and it is argued here that the Euler-Cauchy definition of stress is the appropriate one.

The Definition of Stress.—In authoritative works on continuum mechanics stress is introduced by means of the stress hypothesis of Euler and Cauchy,[2] that is, by asserting that, acting upon any imagined closed geometrical surface σ within the body, there exists a field of stress vectors $\mathbf{t}$ which has an equivalent effect to the (interparticle) forces exerted by the material outside σ upon the material within. For a dielectric material the interaprticle (i.e., intermolecular) forces are partly long-range in character and they may therefore contribute not only to $\mathbf{t}$ but also to $\mathbf{f}$, the body force per unit volume. For the present purpose, however, the important point to note is that σ is an imagined geometrical surface and not a physical surface of separation within the material.

An alternative procedure is to use the operational definition of stress in which it is imagined that a physical cut is made in the material along an internal element of surface $d\boldsymbol{\sigma} = \mathbf{n}d\sigma$. If means are then introduced for keeping the strains in the material on both sides of the cut the same as they were before the cut was made, then the force introduced by these means is $\mathbf{t}'d\sigma$, where $\mathbf{t}'$ is the operationally defined stress vector. In adopting this operational definition, Slepian commented: "It is not assumed that the cut and the introduced means do not disturb the microstructure and micromechanics of the material. For example, in the case of a fluid the cut and means would cause molecules to be reflected which would otherwise pass through the geometric element of surface $\mathbf{dS}$. It is assumed, however, that in spite of the change in the micromechanics, there is no change in the observable macromechanics."[1] It may also be noted there is a further element of idealization involved in that the cut is imagined to be of finite extent: in practice, as discussed later in this paper, it is only possible to measure the force on an element of volume when the element is completely separated from the rest of the body.

For an ordinary elastic material the stress acting at a physical surface of separation is zero: the stress vector that must be introduced to maintain mechanical conditions is thus $\mathbf{t}'$ so that $\mathbf{t}' = \mathbf{t}$ and an operational definition of stress is possible. For a dielectric material, however, as indicated below, $\mathbf{t}' \neq \mathbf{t}$, although it should be remembered that it is a consideration of $\mathbf{t}'$ that leads to the conclusion that the compensating mechanical forces are not derivable from a tensor. It is therefore desirable to investigate the alternative interpretation of stresses in an elastic dielectric when the Euler-Cauchy definition of stress is used.

A well-known consequence of the existence of the stress vector $\mathbf{t}$—obtained by considering[2] the equilibrium of an interior, tetrahedrally shaped element as its volume tends to zero—is that the state of stress in the material can be characterized by a stress tensor, t_{ij}, which is related to $\mathbf{t}$ in rectangular Cartesian coordinates, x_i $(i = 1, 2, 3)$, by the equation

$$t_i = t_{ij}n_j, \tag{2}$$

where the customary summation convention operates on a repeated suffix. Since electrical conditions vary rapidly across the physical surface of the body, it is, in general, necessary to introduce a specifically electrical surface stress $\mathbf{T}$ as well as the specifically electrical body force density $\mathbf{f}$. At a free surface, $\mathbf{T}$ is equilibrated by $\mathbf{t}$, so that the appropriate boundary condition is

$$t_{ij}n_j = T_i. \tag{3}$$

Now the t_{ij} are independent of $\mathbf{n}$, so that $\mathbf{t}$ is a linear vector function of $\mathbf{n}$, whereas it may be shown[3] that $\mathbf{T}$ contains not only a linear vector function of $\mathbf{n}$ but also a cubic term $\mathbf{T}^* = (\mathbf{P}\cdot\mathbf{n})^2\mathbf{n}/2\epsilon_0$, where $\mathbf{P} = \mathbf{D} - \epsilon_0\mathbf{E}$ is the polarization. However, there is no inconsistency between equations (2) and (3): equation (2) holds at all interior points, at which $\mathbf{n}$ is variable since it specifies the orientation of an element of an imagined interior geometrical surface, whilst (3) holds only on the boundary, where $\mathbf{n}$ is not variable since it must correspond to the physical surface of the body. The fact that $\mathbf{T}$ contains the cubic term $\mathbf{T}^*$ does not prohibit the existence of a stress tensor, or a stress vector that is a linear vector function of $\mathbf{n}$, any more than it would if a stress that were cubic in $\mathbf{n}$ were applied mechanically to the surface. However, if the operational definition of stress is assumed to be applicable for a dielectric, then $\mathbf{t}'$, being the stress acting over a physical surface of separation, contains the cubic term $\mathbf{T}^*$. (In fact, $\mathbf{t}' = \mathbf{t} - \mathbf{T}$.) Equation (2) then relates a cubic vector function of $\mathbf{n}$ to a linear vector function of $\mathbf{n}$, from which it may be concluded, with Slepian, that there is no stress tensor t_{ij} corresponding to $\mathbf{t}'$. It is clear, therefore, that a choice must be made between the Euler-Cauchy stress vector $\mathbf{t}$ and the operationally defined stress vector $\mathbf{t}' = \mathbf{t} - \mathbf{T}$.

Discussion.—According to the present approach to the problem $\mathbf{t}$ is the stress acting across a *geometrical* interior surface and not across a physical surface of separation. It is therefore fundamentally incorrect to introduce a physical cut in order to calculate the force on an element of the uncut material, since a surface stress $\mathbf{T}$ is introduced thereby which is nonzero and must certainly contain the cubic term $\mathbf{T}^*$. Nor is this difficulty removed by proceeding to the limit as the cut thickness tends to zero: it is true that the fields in the material adjacent to the cut tend to their actual values in this mathematical process, but the contribution $\mathbf{T}^*$ to the stress vector remains, whereas it would disappear in the corresponding physical process of rejoining the material to remove the physical discontinuity. It should be noted that it is not possible to correct for the presence of **T** because, although the form of **T*** is known, the form of the linear vector function **T**–**T*** is ambiguous. This is, first, because the integrand in (1) may be augmented by any vector that integrates to zero over S and, second, because a corollary of the Gauss divergence theorem may be used to transfer part of the surface integral into a volume integral over V. Since $\mathbf{t}' = \mathbf{t} - \mathbf{T}$, and **T** is unknown, it is pointless to try to investigate the form of t_{ij} by performing experiments in which part of a material body is separated from the remainder, because the force on the separated part includes a contribution from the surface of separation which prevents discovery of what the force on the part would have been had the separation not been made.

The actual stress **t** cannot be equated to the operationally defined stress $\mathbf{t}' = \mathbf{t} - \mathbf{T}$, and if this fact is ignored, the presence in **T** of terms that are cubic in **n** results in two related, but unfortunate, consequences. The first is that the force on a volume element depends on its shape, as pointed out by Slepian,[1] so that a volume element on which no force acts can be divided into two parts on each of which a force acts. Furthermore, if a distinction is made between "electrical" and "elastic" forces, this implies that the "elastic" forces must also be shape-dependent. The shape dependence of the force on a volume element may be concealed by the use of a particular type of volume element—for example, a parallelopiped with faces normal to the coordinate axes. It may also be concealed by always taking[4] volume elements bounded by surfaces parallel or normal to the polarization. The second unfortunate consequence, again as noted by Slepian,[1] is that **t** is not derivable from a tensor, which is another way of saying that it is not a linear vector function of **n**. This is important because it means that it is impossible for an arbitrary, tetrahedrally shaped volume to be in equilibrium under the action of the corresponding set of (four) stress vectors as its volume tends to zero. The introduction of a cut to effect a physical separation has therefore altered the equilibrium conditions, for the volume must be in equilibrium (static or dynamic) before the cut is introduced. However, if the statement that the material is in a state of stress has any meaning, then this state of stress must be one in which internal volume elements of any shape and size can be held in equilibrium. In this sense the use of a continuum theory of elasticity *necessitates* the introduction of a stress vector which is a linear vector function of **n**.

Although the force on a geometrically demarcated volume can be given a meaning as the force on all the particles contained in the volume, it might be thought that an operational definition would be more satisfactory. Indeed, a familiar presentation of scientific topics is in terms of an operational system in which basic quantities that are susceptible of direct measurement are connected by laws which are to be tested by experiment. This idea is attractive in a pedagogical treatment but its limitations are well known and have been stated by Southwell.[5] Briefly, the objection is that it rests on a circular argument, for no experiment can be interpreted without recourse to ideas in themselves part of the theory under examination.

Apart from this general criticism of the exclusive use of operationally defined quantities, there is a particular objection that can be made to the operational definition of stress, namely that the proposed operation cannot be performed. It is possible to introduce a physical surface of separation within the material but, in practice, a force can only be measured on an element of volume that is *completely* separated from the rest of the body. But if the element is completely separated, the force on it is given by (1), where **E** is the field within the cut that isolates the element. It has already been seen that it is not possible (because of the ambiguity in **T**–**T***) to proceed from (1) to a determination of the stress acting on an element of surface, and, therefore, the proposed definition must fail. To enquire how the proposed operational definition has actually been used[6, 7] to derive expressions for stresses is therefore instructive.

Since the cut is not actually made, nor is the force measured, the proposed experiment is a gedankenexperiment, and the stress must be determined by considering conditions at an element of surface and performing a calculation. This, in turn, implies that some sort of model must be used (e.g., volume and surface distributions of charge for a dielectric and of pole and/or current for a ferromagnetic material), and a scrutiny of the references cited above reveals that this is, in fact, the case. Now the models correctly reproduce electromagnetic conditions in the cut and they must therefore predict **F** correctly from (1). But how do they yield values $\mathbf{t}'$? The answer is that they do not: What is predicted is not $\mathbf{t}' = \mathbf{t} - \mathbf{T}$, but **T** itself, corresponding to the conclusion that $\mathbf{t} = 0$. This reveals an intrinsic defect in the use of these models. The prediction that **t** is zero is correct for the model because the model is unstable: no mechanical means have been specified whereby, for example, the positions of the charges in the distribution are stabilized. However, the conclusion that $\mathbf{t} = 0$ in the actual material is incorrect—just as it would be in a purely mechanical situation—because no account has been taken of extrinsic body force densities (**f**) and surface tractions operating elsewhere, although they contribute to **t** at the point under consideration. Nor is it surprising that the model and the actual material differ in this respect. In the actual material the force between two adjacent molecules can always be regarded, wholly or in part, as a surface stress acting across a surface separating them; but from two adjacent regions in a continuous distribution of, for example, charge there can be successively subtracted without limit regions which are not adjacent and which, therefore, do not interact via a surface stress.

From what has been said above, it may be concluded that, on balance, the operational definition of stress should be rejected and that the stress vector should be identified with the stress acting over an imaginary geometrical surface. The ultimate justification for this decision must rest on its utility. Although any theory based on quantities not operationally defined must raise legitimate doubts, it should be noted that what is really being attempted here is the extension of the concept of stress—which is familiar in elasticity—into an area in which an operational definition is not possible. Strictly speaking, an operational definition is not possible in ordinary elasticity either, but in a purely mechanical situation there is no reason to believe that **T** is other than zero. It should also be noted that the use of the concept of stress in ordinary elasticity draws considerable support from the fact that experimental measurements of deformation are in conformity with the existence of a linear constitutive relationship (Hooke's law) between stress and strain. Similarly, the utility of the Euler-Cauchy definition of stress cannot be fully tested for an elastic dielectric without reference to predictions of strain. It is much more difficult to determine the appropriate constitutive relationship for a polarized dielectric, but some progress is currently being made[8-10] in this direction.

Summary.—It is argued that the Euler-Cauchy definition of stress should be used in preference to attempting an operational definition when considering stresses in an elastic dielectric. This suggestion resolves a number of outstanding difficulties and permits a reappraisal of the role of (electrical) models in the determination of stresses in such materials.

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[6] Brown, W. F., *Amer. J. Phys.*, **19**, pp. 290, 333 (1951).

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