# ALGORITHMS FOR PERMUTATION STATISTICS

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**ABSTRACT OF THE DISSERTATION**

## Algorithms for Permutation Statistics

**by Andrew Michael Baxter**

**Dissertation Director: Doron Zeilberger**

Two sequences $u, v$ of $n$ positive integers are *order isomorphic* if $u_i < u_j$ if and only if $v_i < v_j$ for all pairs $(i, j) \in \{1, 2, \ldots, n\}^2$. A permutation $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathcal{S}_n$ is said to *contain* $\sigma \in \mathcal{S}_k$ *as a pattern* if there is some $k$-tuple $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $\pi(i_1)\pi(i_2)\cdots\pi(i_k)$ is order isomorphic to $\sigma$. The subsequence $\pi(i_1)\pi(i_2)\cdots\pi(i_k)$ is called a *copy* of $\sigma$. This notion of pattern containment is generalized to include adjacency restrictions, i.e., conditions that demand $i_x + 1 = i_{x+1}$ for certain $x \in \{1, 2, \ldots, k-1\}$.

A *permutation statistic* is a function $f\colon \bigcup_n \mathcal{S}_n \to \mathbb{C}$. The primary permutation statistics studied in this work are written in terms of the number of copies of a given pattern or patterns. The central concern of this thesis is to compute answers to problems of the following type: "Given patterns $\sigma^{(1)}, \ldots, \sigma^{(t)}$ and nonnegative numbers $k_1, k_2, \ldots, k_t$, how many permutations in $\mathcal{S}_n$ have $k_i$ copies of $\sigma^{(i)}$ for each $i$?" The techniques which apply will depend on the nature of the patterns $\sigma^{(i)}$, as well as whether or not all $k_i = 0$.

## Acknowledgements

This work would not have been completed without the aid of numerous individuals.

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Many thanks go to committee members Neil Sloane, Jim Lepowsky, and Vladimir Retakh for their comments on improving this thesis.

I would like to thank my collaborator Lara Pudwell, both for her guidance on the subtleties of constructing enumeration schemes, as well as pointing me toward several engaging research conferences. I also want to thank my collaborator Aaron Jaggard, who has taught me many useful lessons on the research process and the careful preparation of manuscripts.

Previous iterations of chapters 2, 4, and 7 have each undergone some peer-review process, and I would like to thank those anonymous and non-anonymous referees for their time and comments.

Numerous colleagues have helped improve this work through their careful reading, most notably Elizabeth Kupin, Lara Pudwell, and Vince Vatter. Most of the figures appearing in this work would not be present if not for the suggestions and original works by these three readers.

# Dedication

This work is dedicated to my wife, Kristen. I could not have completed this thesis without her love, patience, and encouragement.

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# Introduction

## 1.1 Permutations and Patterns

The fundamental objects of this work are permutations. Let $\mathcal{S}_n$ $(n \geq 0)$ denote the symmetric group of permutations of the set $[n] = \{1, 2 \ldots, n\}$. We will typically view these objects in one-line notation, i.e. a list where the permutation $\pi \in \mathcal{S}_n$ is written $\pi = \pi_1\pi_2 \cdots \pi_n$. We will sometimes write $\pi(1)\pi(2) \cdots \pi(n)$ to avoid excessive subscripts or to highlight the nature of permutations as bijections $\pi : [n] \to [n]$. Let $|\pi|$ denote the length of $\pi$. Generally we will ignore algebraic properties such as cycle structure and instead view $\pi$ as a finite sequence of $n$ letters chosen from $[n]$ without repetition. Thus it is reasonable to include the "empty permutation" $\epsilon$ which has length zero, and the set $\mathcal{S}_0 = \{\epsilon\}$.

From time to time it will be advantageous to take a more geometric viewpoint, and in this case we may graph $\pi \in \mathcal{S}_n$ by plotting the points $(i, \pi_i)$. For example, the permutation 2573641 has the graph shown in Figure 1.1.

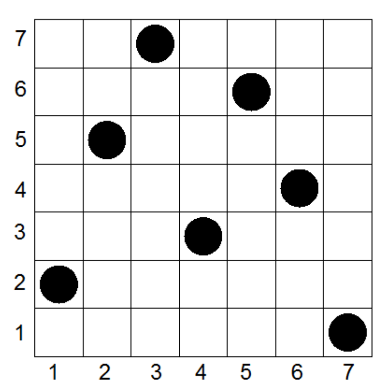


Figure 1.1: Graph of $\pi = 2573641$

We choose to graph $\pi$ in a way resembling a non-attacking rook placement on a

$n \times n$ chessboard in anticipation of ideas introduced in Chapter 6.

**Symmetry**

The graph of a permutation in an $n \times n$ square suggests that permutations are subject to the symmetries of the square. In particular we define three reflection operations on $\pi \in \mathcal{S}_n$. The *reverse* of $\pi$, $\pi^r = \pi_n \pi_{n-1} \cdots \pi_2 \pi_1$, corresponds to reflecting the graph across a vertical axis. For example $52431^r = 13425$. The *complement* of $\pi$, $\pi^c = (n+1-\pi_1)(n+1-\pi_2)\cdots(n+1-\pi_{n-1})(n+1-\pi_n)$, corresponds to reflection across a horizontal axis. For example $52431^c = 14235$. We can also consider the reflection of the square along the southwest-northeast diagonal. This corresponds to taking the inverse of the permutation considered as a function, that is $\pi^{-1}(i) = j$ iff $\pi(j) = i$. For example, $31542^{-1} = 25143$. Of course any other symmetry of the square can be realized by composing two or more of these maps (in fact, just $\cdot^r$ and $\cdot^{-1}$ will suffice).

For a set of permutations $S$, we will define the following sets:

$$S^r = \{\pi^r : \pi \in S\}$$

$$S^c = \{\pi^c : \pi \in S\}$$

$$S^{-1} = \{\pi^{-1} : \pi \in S\}.$$

### 1.1.1 Classical Pattern Avoidance

For a finite sequence of numbers $w = w_1 w_2 \cdots w_n$, we define the *reduction* $\text{red}(w)$ to be the finite sequence obtained by replacing the $i^{th}$ smallest letter(s) of $w$ with $i$. For example $\text{red}(839183) = 324132$. If $\text{red}(u) = \text{red}(w)$, we say that $u$ and $w$ are *order-isomorphic* and write $u \sim w$.

We say that $\pi \in \mathcal{S}_n$ *contains* $\sigma \in \mathcal{S}_k$ *as a [classical] pattern* if there is some $k$-tuple $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $\text{red}(\pi(i_1)\pi(i_2)\cdots\pi(i_k)) = \sigma$. The subsequence $\pi(i_1)\pi(i_2)\cdots\pi(i_k)$ is called a *copy* or *occurrence* of $\sigma$. If $\pi$ does not contain $\sigma$, then $\pi$ is said to *avoid* $\sigma$. Hence we see that $\pi = 34512$ contains the pattern 231 as formed by the letters $\pi_1\pi_3\pi_4 = 351$, but exhaustive checking shows $\pi$ avoids 132. The subset of

$\mathcal{S}_n$ consisting of permutations avoiding $\sigma$ is denoted $\mathcal{S}_n(\sigma)$. For a set of patterns $B$, $\pi$ is said to avoid $B$ if $\pi$ avoids all $\sigma \in B$, and we denote the set of $B$-avoiding permutations by

$$\mathcal{S}_n(B) := \bigcap_{\sigma \in B} \mathcal{S}_n(\sigma). \tag{1.1}$$

In this work we will be interested primarily in the cardinality of $\mathcal{S}_n(B)$, so let

$$s_n(B) := \#\mathcal{S}_n(B). \tag{1.2}$$

Observe that pattern containment and avoidance respect symmetries of the square as illustrated in the following lemma.

**Lemma 1.** *For permutation $\pi \in \mathcal{S}_n$ and pattern $\sigma \in \mathcal{S}_k$, the following are equivalent.*

*(a) $\pi$ contains $\sigma$.*

*(b) $\pi^r$ contains $\sigma^r$.*

*(c) $\pi^c$ contains $\sigma^c$.*

*(d) $\pi^{-1}$ contains $\sigma^{-1}$.*

This is advantageous since a method may lend itself better to explore $\mathcal{S}_n(B^r)$, say, rather than $\mathcal{S}_n(B)$ itself.

Some of the first work in determining sizes of sets $\mathcal{S}_n(B)$ was done by Simion and Schmidt in [81]. In this paper they give exact formulas for $s_n(B)$ for each $B \subseteq \mathcal{S}_3$, along with the first bijective proof that $s_n(123) = s_n(132)$. Since then, many more sets $\mathcal{S}_n(B)$ have been counted by a variety of methods. Section 1.3 outlines several of the most general methods. Notably, an exact formula for $s_n(1324)$ has been elusive for over twenty years.

The pattern containment relation suggests a poset viewpoint which motivates many of the related questions. While we will not take advantage of this structure explicitly, we take a moment here to outline the general structure. If $\pi$ contains $\sigma$, we denote this by $\pi \geq \sigma$, which forms a partial ordering on $\mathcal{S} = \bigcup_{n \geq 0} \mathcal{S}_n$. In this poset, a *permutation* *class* $\mathcal{S}(B) = \bigcup_{n\geq 0} \mathcal{S}_n(B)$ forms a downset[1] with basis $B$, since if $\pi$ avoids $B$ then so does any $\pi' \leq \pi$. There is great interest in studying other facets of this poset, e.g., there has been recent partial success computing the Möbius function in [24, 85]. This partial ordering does not translate naturally to the variations described below, with the exception of consecutive patterns.

### 1.1.2 Variations

The notion of avoidance given above is sometimes called *classical* pattern avoidance. In the past twenty years, however, several variations of this theme have appeared. We take the time to introduce them here, since many of the chapters that follow deal with these variations.

#### Consecutive Patterns

A permutation $\pi \in \mathcal{S}_n$ *consecutively contains* pattern $\sigma \in \mathcal{S}_k$ if there is an index $i$ such that $\text{red}(\pi_i\pi_{i+1}\cdots\pi_{i+k-1}) = \sigma$. Inversely, $\pi$ *avoids* $\sigma$ *consecutively* if $\pi$ does not consecutively contain $\sigma$. Hence $\pi = 35241$ avoids 132 consecutively, even though $\pi_1\pi_2\pi_4 \sim 132$ and so $\pi$ does not avoid 132 classically. Consecutive pattern avoidance is sometimes called *subword avoidance* or *subfactor avoidance.*

It is advantageous to think of "consecutive" as a quality of the pattern $\sigma$ rather than a description of the manner of containment. Hence we will describe permutations as containing/avoiding the *consecutive pattern* $\sigma$, and still use the notation $\mathcal{S}_n(\sigma)$ to denote those permutations avoiding the consecutive pattern $\sigma$. It will be clear from the context whether a given $\sigma$ is classical or consecutive.

Observe that consecutive patterns exhibit the same symmetries as permutations, except for taking inverses. If a permutation $\pi$ contains a consecutive pattern $\sigma$, it is not necessarily true that $\pi^{-1}$ contains $\sigma^{-1}$. Hence parts (a), (b), and (c) of Lemma 1 are equivalent when $\sigma$ is a consecutive pattern.

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[1] A downset, also known as an order ideal, for poset $(P, \geq)$ is a subset $I \subseteq P$ such that if $x \in I$ then $y \in I$ for every $y < x$.

Several well-known counting problems can be phrased in terms of consecutive pattern avoidance. Permutations whose longest increasing run is of length at most $k$ are exactly those which avoid the consecutive pattern $12\cdots k(k{+}1)$. In [6] André introduces "alternating" permutations: $\pi \in \mathcal{S}_n$ such that $\pi_{2i-1} < \pi_{2i}$ and $\pi_{2i} > \pi_{2i+1}$. Alternating permutations are precisely those which simultaneously avoid the consecutive patterns 123 and 321. Involutions are in bijection with permutations avoiding 123 and 132.

The first studies of consecutive pattern avoidance for its own sake can be found in [43], which develops tools to explore permutations avoiding single consecutive patterns via generating functions. Avoiding sets of consecutive patterns of length 3 is considered by Kitaev in [60]. In the past two years, consecutive patterns have seen a flurry of activity, such as [69, 59, 39, 37, 5]. Each of these papers develops a different tool to study consecutive pattern avoidance. Chapter 3 presents another method to count permutations avoiding consecutive patterns as part of a more general class of patterns, dashed patterns, which are presented in the next section. Consecutive patterns also appear in section 4.8 of Chapter 4, where we discuss the number of permutations avoiding a set $B$ with $k$ copies of a given consecutive pattern.

**Dashed Patterns**

Classical and consecutive patterns have a common generalization via *dashed patterns.* Dashed patterns resemble classical patterns, with the constraint that some of the $i_j$ must be consecutive. Formally, a dashed pattern is a pair $(\sigma, X)$ where $\sigma$ is a permutation in $\mathcal{S}_k$ and $X \subseteq \{0\} \cup [k]$ a set of "adjacencies." Permutation $\pi \in \mathcal{S}_n$ contains dashed pattern $(\sigma, X)$ if there is a sequence $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that the following three criteria are satisfied:

- $\mathrm{red}(\pi_{i_1}\pi_{i_2}\cdots\pi_{i_k}) = \sigma$.
- $i_{x+1} = i_x + 1$ for each $x \in X \setminus \{0, k\}$.
- $i_1 = 1$ if $0 \in X$ and $i_k = n$ if $k \in X$.

In practice we write $(\sigma, X)$ as a permutation with a dash between $\sigma_j$ and $\sigma_{j+1}$ if $j \notin X$ and a square bracket on the left or right of $\sigma$ if $0 \in X$ or $k \in X$, respectively. Thus we will often refer to "the dashed pattern $\sigma$" without explicitly referring to $X$. For example, $(1243, \{0, 3\})$ is written [1–2–43. The permutation 162534 contains 1–2–43 as witnessed by the subsequence 1253, but the subsequence 1254 is not an occurrence since the 5 and 4 are not adjacent. The classical pattern $\sigma$ is precisely the dashed pattern $(\sigma, \emptyset)$ since no adjacencies are required, while the consecutive pattern $\sigma$ is the dashed patterns $(\sigma, [k-1])$ since all internal adjacencies are required. We will use the same notation $\mathcal{S}_n(\sigma)$ to denote the set of permutations avoiding the dashed pattern $\sigma$, and similarly $\mathcal{S}_n(B)$ denotes those permutations avoiding every dashed pattern $\sigma \in B$.

Observe that a dashed pattern $(\sigma, X) \in \mathcal{S}_k \times \big(\{0\} \cup [k]\big)$ exhibits similar symmetries to those of permutations. The reverse is given by $(\sigma, X)^r = (\sigma^r, k - X)$ where $k - X = \{k - x : x \in X\}$. The complement is $(\sigma, X)^c = (\sigma^c, X)$. Parts (a), (b), and (c) of Lemma 1 are equivalent when $\sigma$ is a dashed pattern. A notion of inverses among dashed patterns requires the generalization discussed in the next section.

Dashed patterns were introduced as "generalized patterns" by Babson and Steingrímsson in [9] as a generalization of classical patterns. While they were introduced as part of a systematic search for Mahonian permutation statistics, they soon took on a life of their own spawning numerous papers, including [32, 42]. See Steingrímssson's survey for a fuller history [84]. They have been linked to many of the common combinatorial structures such as the Catalan and Bell numbers, as well several rarer or as-yet unseen structures.

This thesis demonstrates how tools from both consecutive and classical pattern avoidance may be used to explore dashed patterns. Chapter 2 uses the "cluster method" to compute the number of permutations in $\mathcal{S}_n$ which contain $k$ copies of a given dashed pattern of length 3. The cluster method was originally developed to consider questions similar to consecutive pattern avoidance. Chapter 3 outlines how to use "enumeration schemes" to compute $s_n(B)$ where $B$ is a set of dashed patterns. Enumeration schemes were originally developed for when $B$ contains only classical patterns.

**Bivincular Patterns**

Bivincular patterns were introduced in [19] as a further generalization of dashed patterns by making additional restrictions on what it means to contain a pattern. While bivincular patterns do not appear in this thesis, they provide a potential direction in which to generalize some of the results contained herein.

A bivincular pattern is a triple $(\sigma, X, Y)$ for $\sigma \in \mathcal{S}_k$, and $X, Y \subseteq \{0\} \cup [k]$. Permutation $\pi \in \mathcal{S}_n$ contains $(\sigma, X, Y)$ if there are indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that the following five criteria are satisfied:

- $\text{red}(\pi_{i_1}\pi_{i_2}\cdots\pi_{i_k}) = \sigma$.
- $i_{x+1} = i_x + 1$ for each $x \in X \setminus \{0, k\}$.
- $i_1 = 1$ if $0 \in X$ and $i_k = n$ if $k \in X$.
- Let $j_y$ be the $y^{th}$ smallest letter appearing in $\{\pi_{i_1}, \ldots, \pi_{i_k}\}$. Then $j_{y+1} = j_y + 1$ for each $y \in Y \setminus \{0, k\}$.
- $j_1 = 1$ if $0 \in Y$ and $j_k = n$ if $k \in Y$.

Observe that the first three criterion are identical to the containment criteria for containing the dashed pattern $(\sigma, X)$, and so $\pi$ contains dashed pattern $(\sigma, X)$ if and only if $\pi$ contains the bivincular $(\pi, X, \emptyset)$. The fourth criterion requires the letters forming the copy of $\sigma$ to be consecutive values. The fifth criterion requires the pattern to use the smallest or largest letters in $\pi$. For example, the subsequence 1253 in the permutation 162534 is a copy of $(1243, \{3\}, \{1, 2\})$ but not $(1243, \{3\}, \{3\})$ since the third and fourth smallest letters (3 and 5) do not have contiguous values.

As above, $\pi$ avoids $(\sigma, X, Y)$ if $\pi$ does not contain $(\sigma, X, Y)$ and we denote the set of $(\sigma, X, Y)$-avoiders by $\mathcal{S}_n((\sigma, X, Y))$.

Observe that a bivincular pattern $(\sigma, X, Y)$ of length $k$ exhibits similar symmetries to those of permutations. The reverse is given by $(\sigma, X, Y)^r = (\sigma^r, k - X, Y)$ and the complement is $(\sigma, X, Y)^c = (\sigma^c, X, k - Y)$, where again $k - A = \{k - a : a \in A\}$.

The inverse[2] is given by $(\sigma, X, Y)^{-1} = (\sigma^{-1}, Y, X)$. Lemma 1 holds for any bivincular pattern $\sigma = (\sigma, X, Y)$.

Bousquét-Melou et al. introduced bivincular patterns in [19] in order to better study $(\mathbf{2}+\mathbf{2})$-free posets and ascent sequences. In particular they found a bijection (via ascent sequences) between $\mathcal{S}_n((231, \{1\}, \{1\}))$ and unlabeled $(\mathbf{2}+\mathbf{2})$-free posets on $n$ elements. Parviainen made the first systematic attempt at enumeration by computing $s_n((\sigma, X, Y))$ for most $\sigma \in \mathcal{S}_2 \cup \mathcal{S}_3$, $X, Y \subseteq \{0, 1, 2, 3\}$ in [72].

### Barred Patterns

Barred pattern avoidance is a different variation on classical pattern avoidance, where $\pi$ is only allowed to contain $\sigma$ as part of some larger pattern $\sigma'$. These appear only in section 4.5 of Chapter 4 to connect the work of that chapter to the work in [75].

Define $\bar{S_k} := S_k \times A$ for $A \subseteq [k]$, where $(\sigma, A) \in \bar{S_k}$ is usually written $\sigma_1 \cdots \sigma_k$ such that $\sigma_i$ has a bar over it iff $i \in A$. For example, $(132, \{1\}) = \bar{1}32$. Let $\underline{\sigma}$ be the permutation formed by reducing the subpermutation of $\sigma$ formed by the unbarred letters, and let $\bar{\sigma}$ denote the permutation formed by $\sigma$ disregarding bars. A permutation $\pi$ is said to avoid the barred pattern $\sigma$ if every copy of the (classical) pattern $\underline{\sigma}$ may be extended to form a copy of the (classical) pattern $\bar{\sigma}$. For example, $\pi$ avoids $\sigma = \bar{1}32$ if every occurrence of $\underline{\sigma} = \text{red}(32) = 21$ is part of a larger $\bar{\sigma} = 132$, i.e. any pair of letters $\pi_i > \pi_j$ for $i < j$ is preceded by some letter less than $\pi_j$. Hence 132 avoids $\{\bar{1}32\}$, while 321 does not. Observe that if $\pi$ avoids $\underline{\sigma}$ then it automatically avoids the barred pattern $\sigma$. As before, we denote by $\mathcal{S}_n(B)$ the set of $B$-avoiding permutations for a set of barred patterns $B$.

While dashed patterns provided a common generalization for both consecutive and classical patterns, barred patterns can be seen as a generalization that spans both (classical) pattern avoidance and containment: $\mathcal{S}_n((\sigma, \emptyset))$ is exactly the set of permutations which avoid $\sigma$, while $\mathcal{S}_n((\sigma, [k]))$ is exactly the set of permutations which contain $\sigma$.

There are natural notions of reversal, complement, and inverse for barred patterns,

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[2] At this point the term "inverse" is short-hand for the reflection across the southwest-northeast diagonal.

so long as the bars move in the appropriate ways. Formally, $(\sigma, A)^r = (\sigma^r, k - A)$, $(\sigma, A)^c = (\sigma^c, A)$, and $(\sigma, A)^{-1} = (\sigma, A')$ where $\sigma_i \in A'$ for each $i \in A$.

There is no reason that one cannot form barred bivincular patterns for a high degree of generality. In fact Brändén and Claesson introduce *mesh patterns* in [21] to provide a common generalization.

Barred patterns first occurred in the characterization of 2-stack-sortable permutations in [93] (see 1.1.5, below). Recently barred patterns have appeared in a variety of other contexts, such as in [97, 25, 19, 18].

**Pattern Avoidance by Words**

The definition of reduction above is well-defined even for what might be called "permutations with repeated letters," i.e. words. Hence the above definition of pattern containment/avoidance serve equally well for permutations and patterns with repeated letters. For example, the permutations $\mathcal{S}_n$ can be characterized as the set of words in $[n]^n$ which avoid the classical pattern 11. Questions of enumerating words avoiding permutations first appeared in [22]. The methods of Chapter 2 not only compute the number of *permutations* containing $k$ copies of certain dashed patterns, but also the number of *words* with $k$ copies of those dashed patterns. Section 4.4 of Chapter 4 presents a method to compute the number of words with $k$ inversions avoiding a set of classical patterns, in connection with the work in [76].

**Pattern Avoidance by Special Permutations**

Just as we can generalize permutations, we can also restrict our focus to specific classes of permutations and ask the analogous questions. Looking at these smaller pieces of $\mathcal{S}_n$ helps to understand the bigger picture. Chapter 6 of the current work restricts attention to the alternating subgroup of $\mathcal{S}_n$, i.e., the even permutations. Even pattern-avoiding permutations were also studied in [65, 67, 3]. The techniques of Chapter 4 also provide a method to compute the number of even pattern-avoiding permutations for certain pattern sets $B$.

Such questions follow a tradition which traces back to the seminal work in [81], where Simion and Schmidt compute the number of pattern-avoiding involutions[3] of length $n$ for several pattern sets $B \subseteq \mathcal{S}_3$. This avenue of inquiry is pursued further in [50, 57].

Similar work has been done on pattern-avoiding derangements, which are permutations with no fixed points[4]. In particular, work in [78, 77, 41] focuses on counting elements of $\mathcal{S}_n(B)$ according to the number of fixed points.

### 1.1.3 Wilf-Equivalence

Exact formulas for $s_n(B)$ are often difficult to determine, but that does not mean that we cannot compare sets $B$ and $B'$ according to the relative sizes of $s_n(B)$ and $s_n(B')$. Most work has focused on avoiding singleton sets, and so we restrict ourselves to this case. Two permutations $\sigma$ and $\tau$ are said to be *Wilf-equivalent* if, for all $n \geq 0$, $s_n(\sigma) = s_n(\tau)$; we denote Wilf-equivalence by $\sigma \equiv \tau$. The *Wilf classification* of a set of patterns is the set of equivalence classes under the Wilf-equivalence relation, and this classification is a central line of inquiry in pattern avoidance.

As Lemma 1 implies, the symmetries of the square yield several trivial Wilf-equivalences. Since $\pi$ avoids $\sigma$ if and only if $\pi^r$ avoids $\sigma^r$, the reversal map provides a bijection $\mathcal{S}_n(\sigma) \leftrightarrow \mathcal{S}_n(\sigma^r)$. Hence we see $\sigma \equiv \sigma^r$. Similarly, $\sigma \equiv \sigma^c$ and $\sigma \equiv \sigma^{-1}$. Further, these equivalences hold for sets of patterns, even sets of patterns of mixed types, so we see that $s_n(B) = s_n(B^r) = s_n(B^c) = s_n(B^{-1})$.

There are also non-trivial Wilf-equivalences, however. The first observed equivalence was $123 \equiv 132$, shown via a counting argument in [62] and via a bijection in [81]. With the trivial equivalences this implies $\sigma \equiv \tau$ for any $\sigma, \tau \in \mathcal{S}_3$. For classical patterns of length four there are three distinct equivalence classes, as compared to the nine classes implied by trivial equivalences. The most general result regarding Wilf-equivalence can each be phrased in terms of prefix-manipulation. Define the *direct sum* of permutations

[3] Also called self-conjugate permutations, these are $\pi$ for which $\pi = \pi^{-1}$

[4] Combinatorially, a fixed point of permutation $\pi$ is a letter $\pi_i = i$

$\alpha \in \mathcal{S}_k$ and $\beta \in \mathcal{S}_\ell$ to be the length-$(k+\ell)$ permutation

$$\alpha \oplus \beta := \alpha_1\alpha_2\cdots\alpha_k(\beta_1+k+1)(\beta_2+k+1)\cdots(\beta_\ell+k+1) \tag{1.3}$$

This is most easily seen geometrically as placing the graph of $\beta$ above and to the right of the graph of $\alpha$. See Figure 1.2 which illustrates $312 \oplus 2413 = 3125746$. We are

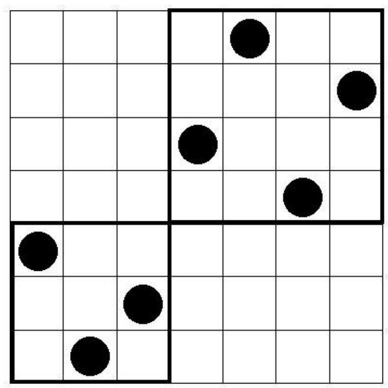

Figure 1.2: The direct sum $312 \oplus 2413 = 3125746$

now ready to state the three most broadest Wilf-equivalence results known for classical patterns.

**Theorem 2** (Backelin, West, Xin, [10])**.** *Let $J_t = t(t-1)\cdots 21$ be the decreasing permutation. Then for any $\beta \in \mathcal{S}_k$,*

$$J_t \oplus \beta \equiv J_{t-1} \oplus 1 \oplus \beta \tag{1.4}$$

Since the increasing permutation $I_t = 12\cdots(t-1)t = 1 \oplus 1 \oplus \cdots \oplus 1$, we may apply Theorem 2 $t-1$ times to get the following "prefix-reversal" result.

**Corollary 3.** *Let $I_t = 12\cdots(t-1)t$ be the increasing permutation. Then $J_t \oplus \beta \equiv I_t \oplus \beta$.*

Using similar techniques, Stankova and West proved the following

**Theorem 4** (Stankova, West, [82])**.** *For any $\beta \in \mathcal{S}_k$, $312 \oplus \beta$ and $231 \oplus \beta$.*

Proofs for these theorems will be discussed in depth in Chapter 6.

These two theorems explain all non-trivial Wilf-equivalences for classical patterns of length at most $k = 7$.

**Variations**

Most of the attention in Wilf-equivalence results has been given to classical patterns, but the same questions apply for variations such as consecutive patterns, dashed patterns, and bivincular patterns. The work of Parviainen in [72] focuses on the Wilf-classification of bivincular patterns of length at most 3, and this represents the sum of the knowledge for these objects. Claesson does the the analogous work for dashed pattern of length at most 3 in [32]. The first steps toward general results were taken in [42, 61], and these are mentioned in more depth in Chapter 3.

There are a several general Wilf-equivalence results known for consecutive patterns. Since they are not referenced in the sequel, we will direct the reader to [69, 59, 5].

Little has been done for Wilf-classification of barred patterns. Callan classifies patterns of length 4 with one bar in [27]. Pudwell classifies all barred patterns of length at most 4 and partially classifies barred patterns of length 5 in [75].

Following the discussion regarding pattern avoidance by special classes of permutations, one can ask about variant forms of Wilf-equivalence over subsets of the permutations. In particular, $\sigma$ and $\tau$ are *involution-Wilf-equivalent* if $\#\mathcal{I}_n(\sigma) = \#\mathcal{I}_n(\tau)$ for all $n \geq 0$, and we denote this by $\sigma \equiv_{\mathcal{I}_n} \tau$. Proving patterns $\sigma \equiv_{\mathcal{I}_n} \tau$ is a central focus of several papers, including [20, 38, 56]. It still remains open whether $\sigma \equiv_{\mathcal{I}_n} \tau$ implies $\sigma \equiv \tau$, although the converse is certainly false (e.g., $\#\mathcal{I}_n(123) = 3$ while $\#\mathcal{I}_n(231) = 4$, even though $123 \equiv 231$).

In Chapter 6 we consider Wilf-equivalence over the even permutations, which we call *even-Wilf-equivalence*. In particular we present an analogue of Theorem 2 for even-Wilf-equivalence.

### 1.1.4 Asymptotics

Knuth proved in [62] that for classical pattern 132, $s_n(132) = \frac{1}{n+1}\binom{2n}{n}$, the Catalan numbers. From this it is quickly seen that $s_n(132) \leq 4^n$ for all $n$. Combined with other evidence, this lead to the long-standing Stanley-Wilf Conjecture in 1980 which was finally proven nearly twenty-five years later in [68]:

**Theorem 5** (Marcus, Tardos, [68]). *For any (classical) permutation pattern $\sigma$, there is some constant $c_\sigma$ such that for all $n \geq 0$:*

$$s_n(\sigma) \leq {c_\sigma}^n.$$

Marcus and Tardos proved this result for $c_\sigma \leq 15^{2k^4\binom{k^2}{k}}$, where $k$ is the length of $\sigma$. Clearly this is a great overestimate, since this gives the base $c_{132} = 15^{27216}$ while Knuth's result shows that $c_{132} = 4$ would suffice. Since the proof of the Stanley-Wilf conjecture, work has been done to determine the value of the *growth rate* $\lim_{n\to\infty} \sqrt[n]{s_n(B)}$, even when $s_n(B)$ cannot be enumerated exactly. See, for example, the work of Albert et al. on $s_n(1324)$ in [2]. Work in [58, 88, 92] focuses on determining which values $c$ are growth rates for some $B$. These attempts include classifications of sets $B$ so that $s_n(B)$ has polynomial, rather than exponential, growth. It should be noted that the limits $\lim_{n\to\infty} \sqrt[n]{s_n(B)}$ are not known to exist for general $B$, although Arratia proves in [7] that the limit exists for singleton sets $B$.

When we move to consecutive pattern avoidance, these exponential bounds no longer hold. For example Elizalde proved in [43] that for the consecutive pattern 123 $\frac{s_n(123)}{n!} \sim \gamma(\rho)^n$ as $n \to \infty$ where $\gamma = e^{3\sqrt{3}\pi}$ and $\rho = 3\sqrt{3}/(2\pi)$. This result, and other analogous results for other consecutive patterns, lead Warlimount to conjecture that for any consecutive pattern $\sigma$, there are constants $\gamma_\sigma$ and $\rho_\sigma$ such that

$$\frac{s_n(\sigma)}{n!} \sim \gamma_\sigma(\rho_\sigma)^n$$

This conjecture was recently proven true by Ehrenborg et al. in [39] by using integral operators and spectral theory.

The present will not concern with asymptotic enumeration, but we do consider asymptotic properties of certain permutation statistics in Chapter 7.

### 1.1.5 Applications and Influences

Several areas outside of combinatorics have directed the development of the study of permutation patterns. The two most influential areas are computer science via sorting analysis, and algebraic geometry via Schubert calculus.

**Analysis of Sorting Machines**

While MacMahon enumerated some classes of pattern-avoiding permutation, the subject blossomed following Knuth's introduction of stack-sortability in *The Art of Computer Programming, Vol. 1* [62]. A *stack* is a first-in last-out data structure. Permutation $\pi \in \mathcal{S}_n$ is *stack-sortable* if we may "sort" it into the increasing permutation $\pi = 12 \cdots n$ by passing it through a stack exactly once. At each stage we may either move the leftmost letter of what remains of the input permutation onto the stack, or move the topmost letter in the stack to the right-hand side of the output permutation. The smaller letter is always moved. See Figure 1.3 for how to stack-sort 1423 and 3142.

| Input | Stack | Output |
|---|---|---|
| 1423 | | |
| 423 | 1 | |
| 423 | | 1 |
| 23 | 4 | 1 |
| 3 | 24 | 1 |
| 3 | 4 | 12 |
| | 34 | 12 |
| | 4 | 123 |
| | | 1234 |

| Input | Stack | Output |
|---|---|---|
| 3142 | | |
| 142 | 3 | |
| 42 | 13 | |
| 42 | 3 | 1 |
| 42 | | 13 |
| 2 | 4 | 13 |
| | 24 | 13 |
| | 4 | 132 |
| | | 1324 |

Figure 1.3: Illustrating $s(1423) = 1234$ and $s(3142) = 1324$. Letters are added to the left side of the stack, so the "topmost" letter appears as the leftmost.

If $s(\pi)$ is the output permutation from performing this stack-sorting algorithm on $\pi \in \mathcal{S}_n$, it is natural to ask about $s^{-1}(12 \cdots n)$, i.e., the stack-sortable permutations. Knuth settles this in [62]:

**Theorem 6** (Knuth, [62])**.** *Permutation $\pi \in \mathcal{S}_n$ is stack-sortable (i.e., $s(\pi) = 12 \cdots n$) if and only if $\pi$ (classically) avoids 231.*

Following this result, Tarjan and others started considering other sorting machines composed of stacks, queues, and deques in series and parallel. The language of pattern avoidance became invaluable, as illustrated by the following results of Tarjan.

**Theorem 7** (Tarjan, [87])**.**

1. *Permutation $\pi \in \mathcal{S}_n$ is sortable by $m$ queues in parallel if and only if $\pi$ (classically) avoids $(m+1)m \cdots 1$.*

2. *Permutation* $\pi \in \mathcal{S}_n$ *is sortable by* $m$ *stacks in parallel if and only if* $\pi$ *(classically) avoids* $23 \cdots (m+2)1$

In his Ph.D. thesis [93], West considers the question of which $\pi \in \mathcal{S}_n$ have $s(s(\pi)) = 12 \cdots n$, dubbing these 2-stack-sortable.[5] His classification was the first appearance of the barred patterns introduced in Section 1.1.2.

**Theorem 8** (West, [93]). *Permutation* $\pi \in \mathcal{S}_n$ *is 2-stack-sortable if and only if* $\pi \in \mathcal{S}_n(2341, 3\bar{5}241)$

West also conjectured a formula to count 2-stack-sortable permutations, at which point Zeilberger used a computer to verify West's conjectured formula for $s_n(2341, 3\bar{5}241)$ by solving a degree-9 functional equation (which was discovered via computer).

**Theorem 9** (Zeilberger, [99]).

$$s_n(2341, 3\bar{5}241) = \frac{2(3n)!}{(n+1)!(2n+1)!}$$

Observe $s_n(2341, 3\bar{5}241)$ is OEIS sequence A000139 [52].

A full characterization of $t$-stack-sortable permutations, those where $s^t(\pi) = 12 \cdots n$, for $t \geq 3$ has proven elusive. See section 8.2 of [16] for more details on sorting results.

### Schubert Varieties

At the end of the nineteenth century Schubert developed some powerful techniques for counting sets of points, lines, planes, and hyperplanes satisfying given intersection criteria. His theory was largely based on intuition, but of such merit that Hilbert made his fifteenth problem the search for a rigorous foundation to what became known as *Schubert calculus*. Schubert varieties are one product of this search.

Let $G$ be a semisimple algebraic group with Borel subgroup $B$, and let $P$ be a standard parabolic subgroup of $G$. Then the homogeneous space $X = G/P$ consists of finitely many $B$-orbits that may be parameterized by certain elements of the Weyl

[5] Note that this is different from sorting with two stacks in series. For example, one can sort 3241 with two stacks while $s(s(3241)) = 2134$. Atkinson et al. address sorting with two stacks in series in [8].

group $W$. The Schubert variety $X_w$ is the closure of the $B$-orbit associated to $w \in W$. The properties of $X_w$ can then be described in terms of properties of $w$. Of particular interest to us, let $G = GL_n(\mathbb{C})$ and let $B$ be the subgroup of upper-triangular matrices. Then the Weyl group is $W = \mathcal{S}_n$, so the Schubert varieties are indexed by permutations. In [63] Lakshmibai and Sandhya characterize the *smooth* varieties in terms of pattern-avoidance properties of their indexing permutations. In particular:

**Theorem 10** (Lakshmibai and Sandhya, [63])**.** *Let $G = GL_n(\mathbb{C})$ and $B$ be the subgroup of upper-triangular matrices. Then the Schubert variety indexed by $\pi$ is smooth if and only if $\pi$ (classically) avoids 3412 and 4231.*

Other properties have been similarly characterized in terms of pattern-avoidance. For example, Bousquet-Mélou and Butler prove in [18] that the variety $X_\pi$ is "locally factorial" if and only if $\pi$ avoids 1324 and $21\bar{3}54$.

## 1.2 Permutation Statistics

The study of permutation statistics forms another facet of the study of permutations. Formally, a permutation statistic is a function $f : \bigcup \mathcal{S}_n \to \mathbb{C}$.[6] For example, one of the most well-studied statistics is the *descent number* $\mathrm{des}(\pi) = \#\{i : \pi_i > \pi_{i+1}\}$, so $\mathrm{des}(23154) = 2$ given by the down-steps 31 and 54.

The *weight enumerator* of a set of permutations $S$ according to weight $q^{f(\pi)}$ is simply $\sum_{\pi \in S} q^{f(\pi)}$ for indeterminate $q$. The resulting formal power series (or polynomial if $S$ is finite) is sometimes called the *distribution* of $f$ over $S$. The weight enumerators for $\mathcal{S}_n$ with weight $q^{\mathrm{des}(\pi)}$ is $A_n(q) = \sum_{\pi \in \mathcal{S}_n} q^{\mathrm{des}(\pi)}$. The polynomials $q\,A_n(q)$ are called the *Eulerian polynomials* and are a well-studied sequence of polynomials. Determining distributions such as this forms a central question in the study of permutation statistics.

Perhaps the most-studied permutation statistic is the *inversion number*, which is the minimum number of adjacent transpositions required to "sort" $\pi$ into the identity permutation $12 \cdots n$. Combinatorially, $\textsc{inv}(\pi) = \#\{(i, j) : i < j, \pi_i > \pi_j\}$. For example, $\textsc{inv}(23154) = 3$, formed by the pairs 21, 31 and 54. Netto introduced the inversion

[6] Typically $f$ has range $\mathbb{N}$, the nonnegative integers.

number in [70], where he also shows that the weight enumerator is given by

$$F_n(q) = \sum_{\pi \in \mathcal{S}_n} q^{\text{INV}(\pi)} = \mathbf{n}_q!,$$

where $\mathbf{n}_q!$ is the well-known $q$-factorial: $(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1})$. This is easy enough to see, since we may construct any permutation in $\mathcal{S}_n$ by inserting $n$ into a permutation in $\mathcal{S}_{n-1}$. The insertion such that $n$ becomes the $i^{th}$ letter will create $n-i$ new inversions, and so we see that

$$\sum_{\pi \in \mathcal{S}_n} q^{\text{INV}(\pi)} = (q^{n-1} + q^{n-2} + \cdots + q^0) \sum_{\pi \in \mathcal{S}_{n-1}} q^{\text{INV}(\pi)}.$$

In [64] MacMahon introduced a statistic which came to be known as the *major index*.[7] The major index is defined as $\text{MAJ}(\pi) = \sum_{i:\pi_i > \pi_i+1} i$. MacMahon proved that the distribution of MAJ over $\mathcal{S}_n$ is also given by the $q$-factorials, i.e.,

$$F_n(q) = \sum_{\pi \in \mathcal{S}_n} q^{\text{MAJ}(\pi)} = \mathbf{n}_q!$$

Hence we see that INV and MAJ are *equidistributed* over $\mathcal{S}_n$ for any $n$. A statistic which has $\mathbf{n}_q!$ is called *Mahonian*, and in the past years many other finding Mahonian statistics have surfaced such as those in [35, 47, 51, 80]. Babson and Steingrímsson introduced dashed patterns in [9] as an attempt to systematize and unify these statistics.

The equidistribution result between inversion number and major index has been refined significantly. In particular [45] Foata shows (bijectively) that INV and MAJ are equidistributed over $\{\pi \in \mathcal{S}_n : \pi_n = i\}$ for any $i$. We will make use of this equidistribution in Chapter 7.

One may also consider the simultaneous distribution of pairs of statistics (as well as triples, 4-tuples, and so on). For example, the joint distributions of the major index and descent number over $\mathcal{S}_n$ is a well-studied sequence of polynomials, called *Euler-Mahonian*:

$$A_n(q,t) := \sum_{\pi \in \mathcal{S}_n} q^{\text{MAJ}(\pi)}\, t^{\text{des}(\pi)}$$

---

[7] MacMahon introduced this as the *greater index*, which was translated into French as *l'indice majeur*. This was translated back into English as the current term, and we are left with an allusion to MacMahon's military rank.

Questions regarding equidistribution of pairs or triples of statistics are often considered, for example in [48]. The joint distribution of $(\textsc{inv}, \textsc{maj})$ over $\mathcal{S}_n$ is the major focus of Chapter 7. The techniques of Chapters 2 and 4 will also include methods to compute joint distributions.

Pattern functions are a general class of permutation statistics, which we study in chapters 2 and 4. Let $\sigma$ be a pattern and define the pattern function $\sigma : \bigcup_{n\geq 0} \mathcal{S}_n \to \mathbb{N}$ mapping $\pi$ to the number of copies of $\sigma$ appearing in $\pi$. While $\sigma$ may be any variation of "pattern," when considering pattern functions in this work we will restrict ourselves to dashed patterns, writing classical patterns with all dashes in place. For example, the descent number $\mathrm{des}(\pi) = (21)(\pi)$ while inversion number is given by $\textsc{inv}(\pi) = (2\text{–}1)(\pi)$. It is clear that pattern functions generalize pattern-avoidance in a natural way, since $\mathcal{S}_n(\sigma)$ is simply the subset of $\mathcal{S}_n$ for which $\sigma(\pi) = 0$.

It is sometimes advantageous to consider linear combinations of pattern functions. For example, the major index can be written:

$$\textsc{maj}(\pi) = (3\text{–}21)(\pi) + (2\text{–}31)(\pi) + (1\text{–}32)(\pi) + (21)(\pi).$$

In [9], Babson and Steingrímsson considered all linear combinations of dashed pattern functions of length 3, narrowing this to construct eleven Mahonian statistics, although many were equivalent to statistics which were previously studied in the literature. They left three more apparently Mahonian statistics as conjectures, which were quickly swept up in [48, 46]. In connection with the current thesis, Zeilberger used the Umbral Transfer Matrix Method discussed in 5 to prove several of these conjectures in [48]. In [21] Bränden and Claesson use linear combinations of mesh patterns to replicate other permutation statistics.

The distribution of pattern functions and linear combinations thereof is the primary focus of Chapters 2 and 4.

## 1.3 General Enumeration Methods

Among the literature there exist several arguments which have had widespread effectiveness in enumerating $\mathcal{S}_n(B)$ for many different sets $B$. In this section we will outline four of them: generating trees, substitution decomposition, insertion encoding, and enumeration schemes. These four techniques are compared and contrasted in [91]. Since chapters 4 and 5 focus primarily on enumeration schemes, this topic will be presented in the most detail. The methods in sections 1.3.1, 1.3.2, and 1.3.3 are presented for context and do not enter into the main body of the current thesis.

All of these methods were developed for the case of classical avoidance, and so it will be convention that all patterns mentioned are classical. Generating trees have been used to consider variant notions of patterns, for example [94] presents a generating tree for $\mathcal{S}_n(321, 3\bar{1}42)$. Pudwell's enumeration schemes for barred patterns in [75] are summarized in Chapter 4, and we present enumeration schemes for dashed patterns in Chapter 3.

### 1.3.1 Generating Trees

In [31] Chung et al. introduced generating trees to explore Baxter permutations, which then featured prominently in West's work in [93, 94, 95]. The core notion is to construct a rooted labeled tree, called a *generating tree*, such that the nodes of the tree are in bijection with $\mathcal{S}_n(B)$. One then uses the structure of that tree to count $s_n(B)$.

For permutation $\pi \in \mathcal{S}_n$, a *child* of $\pi$ is any permutation which can be constructed by inserting $(n+1)$ into $\pi$. This lets one construct a rooted tree $T$ with nodes $\mathcal{S} = \bigcup_n \mathcal{S}_n$. Let $T(B)$ be the subtree corresponding to $\mathcal{S}(B) = \bigcup_n \mathcal{S}_n(B)$.

A *generating tree* is a rooted labeled tree such that the labels of a node uniquely determine the labels of its children. Thus one can define a generating tree by determining the label of the root and the rules for labeling the children of a vertex with a given label (called the *succession rules*). For example, the complete binary tree is defined by the root label (1) and the succession rule $(1) \rightsquigarrow (1)(1)$. These means that each node labeled (1) (which is all of them) has two children themselves each labeled (1).

To use generating trees to enumerate $\mathcal{S}_n(B)$, one must construct a generating tree so that the succession rules create a tree isomorphic to $T(B)$. Although one is not restricted to integer labels ($k$-tuples are more common), this process can require considerable ingenuity specific to the pattern set $B$. Once one has such a generating tree, the newly-discovered structure can give rise to a generating function such as done in [17].

In [90], Vatter gives a characterization of finite sets $B$ such that $T(B)$ is isomorphic to a finitely labeled generating tree. From there one can use the transfer matrix method described in [83] to show that such permutations have rational generating functions. In particular, Vatter shows that if the set $B$ is finite and contains both a child of $12\cdots k$ for some $k$ and a child of $\ell(\ell-1)\cdots 1$ for some $\ell$, then $T(B)$ is isomorphic to a finitely labeled generating tree. He also provides a Maple package (FinLabel) which constructs the generating tree and consequent rational generating function. As explained in [91], such sets $B$ are also guaranteed to have finite enumeration schemes.

### 1.3.2 Substitution Decomposition

An *interval* of $\pi \in \mathcal{S}_n$ is a set of contiguous indices in $\pi$ which map to a contiguous set of values in $[n]$. For example, 4657 is an interval of $\pi = 2465713$. Obviously each $\pi \in \mathcal{S}_n$ has $n$ trivial intervals of length 1 and one trivial interval of length $n$, but a $\pi$ is *simple* if there are no non-trivial intervals.

Any permutation can be decomposed into its intervals, providing a recursive structure to the set of permutations where the simple permutations serve as atomic elements. In [1] Albert and Atkinson show:

**Theorem 11.** *If* $\bigcup_{n\geq 0} \mathcal{S}_n(B)$ *contains only finitely many simple permutations, then* $s_n(B)$ *has an algebraic generating function.*

To get a flavor of the decomposition argument, consider the simplest case of the 132-avoiding permutations. Any $\pi \in \mathcal{S}_n(132)$ may be decomposed into $\pi' \, n \, \pi''$. Since $\pi$ avoids 132, each letter in $\pi'$ must be greater than each letter in $\pi''$. Furthermore, $\mathrm{red}(\pi')$ and $\mathrm{red}(\pi'')$ are also 132-avoiding permutations, so we get the relation $s_n(132) = \sum_{i=1}^{n} s_{i-1}(132)s_{n-i}(132)$. For generating function $C(x) = \sum_{n\geq 0} s_n(132)x^n$, the relation gets us the functional equation $C(x) = 1 + x\,C(x)^2$ and so we see that $C(x)$ is algebraic.

### 1.3.3 Insertion Encoding

The insertion encoding, introduced in [4], is a way to bring developments in formal language theory to bear on the enumeration problem for pattern-avoiding permutations. Each permutation is built up from a single open slot, denoted $\diamond$, using a sequence of "moves" chosen from the following list of four. We start at stage 0 with $\diamond$, and at stage $n$ we insert $n$ according to one the following moves:

$\mathbf{f}_j$ Fill the $j^{th}$ slot with $n$, i.e. replace $\diamond$ with $n$.

$\boldsymbol{\ell}_j$ Insert $n$ to the left of the $j^{th}$ slot, i.e. replace $\diamond$ with $n\diamond$.

$\mathbf{r}_j$ Insert $n$ to the right of the $j^{th}$ slot, i.e. replace $\diamond$ with $\diamond n$.

$\mathbf{m}_j$ Insert $n$ into the middle of the $j^{th}$ slot to split the slot in two, i.e. replace $\diamond$ with $\diamond n \diamond$.

The process ends when the last slot has been filled, and we list the steps as a word to be read left to right. For example, 31254 is constructed as follows:

| Stage | Move type | Permutation so far |
|---|---|---|
| 0 | — | $\diamond$ |
| 1 | $\mathbf{m}_1$ | $\diamond 1 \diamond$ |
| 2 | $\boldsymbol{\ell}_2$ | $\diamond 12 \diamond$ |
| 3 | $\mathbf{f}_1$ | $312\diamond$ |
| 4 | $\mathbf{r}_1$ | $312 \diamond 4$ |
| 5 | $\mathbf{f}_1$ | 31254 |

We record this series of moves as a word, and so 31254 has the "insertion encoding" $\mathbf{m}_1\boldsymbol{\ell}_2\mathbf{f}_1\mathbf{r}_1\mathbf{f}_1$.

These insertion encodings form a formal language, and pattern-avoidance properties of a permutation class correspond to restrictions on their insertion encodings. When the insertion encodings for a given permutation class form a regular language, that class has a rational generating function. In [4] Albert et al. show that the only restriction to regularity are permutations $\pi$ such that $\max \pi_{2i+1} < \min \pi_{2i}$ or $\min \pi_{2i+1} > \max \pi_{2i}$. They call these *vertical alternations* and prove the following theorem.

**Theorem 12** (Albert, Linton, Ruškuc, [4])**.** *For finite $B$, the insertion encodings for permutation class $\mathcal{S}(B)$ form a regular language if and only if $\mathcal{S}(B)$ contains finitely many vertical alternations.*

In [89], Vatter provides an alternate classification of classes whose insertion encodings form a regular language. He also provides the Maple package InsEnc which, given set $B$, checks whether the insertion encodings for $\mathcal{S}(B)$ form a regular language and if so will compute the (rational) generating function.

As a special case of Theorem 12, observe that any class with a finitely labeled generating tree contains finitely many vertical alternations. Hence their corresponding insertion encodings form a regular language. This was first noted in [91].

### 1.3.4 Enumeration Schemes

Enumeration schemes are succinct encodings for a family of recurrence relations enumerating a family of sets. The enumerated sets are subsets of $S_n(B)$ determined by prefixes.

For pattern $p \in \mathcal{S}_k$, let $\mathcal{S}_n(B)[p]$ be the set of permutations $\pi \in \mathcal{S}_n(B)$ such that $\text{red}(\pi_1\pi_2\ldots\pi_k) = p$. We call $p$ the *prefix pattern*. To refine further, let $w \in [n]^k$ and define $\mathcal{S}_n(B)[p; w]$ to be those permutations in $\pi \in \mathcal{S}_n(B)[p]$ such that $\pi_1\pi_2\cdots\pi_k = w$. For example,

$$\mathcal{S}_5(123)[21; 53] = \{53142, 53214, 53241, 53412, 53421\}.$$

Since we are interested in enumeration, it will be handy to have the notation $s_n(B)[p] = \#\mathcal{S}_n(B)[p]$ and $s_n(B)[p; w] = \#\mathcal{S}_n(B)[p; w]$.

By looking at the prefix of a permutation, one can identify likely "trouble spots" where forbidden patterns may appear. For example, suppose we wish to avoid the pattern 123. Then the presence of the pattern 12 in the prefix indicates the potential for the whole permutation to contain a 123 pattern. Vatter uses the symmetric notion of partitioning $\mathcal{S}_n(B)$ according to the pattern formed by the smallest $k$ letters in $\pi \in \mathcal{S}_n(B)$ and their relative positions in [91]. Thus Vatter's schemes for $\mathcal{S}_n(B^{-1})$ are equivalent to our schemes for $\mathcal{S}_n(B)$.

Enumeration schemes take a divide-and-conquer approach to enumeration. We define the *child* of a permutation $p \in \mathcal{S}_k$ to be any permutation $p' \in \mathcal{S}_{k+1}$ such that $\mathrm{red}(p'_1 p'_2 \cdots p'_k) = p$.[8] Any $\mathcal{S}_n(B)[p]$ for $p \in \mathcal{S}_k$ may be partitioned into the family of sets $\mathcal{S}_n(B)[p']$ for each of its children $p' \in S_{k+1}(B)[p]$. The sets indexed by these children are then counted as described below, and their sizes are totaled to obtain $s_n(B)[p]$. In the end we have counted $s_n(B)$, since $\mathcal{S}_n(B) = \mathcal{S}_n(B)[\epsilon] = \mathcal{S}_n(B)[1]$, where $\epsilon$ is the empty (i.e., length 0) permutation.

For $p \in S_k$ a set $\mathcal{S}_n B[p]$ fits into one of three cases:

(1) If $n = k$, then $\mathcal{S}_n(B)[p]$ is either $\{p\}$ or $\emptyset$, depending on whether $p$ avoids $B$.

(2) For each $w \in [n]^k$ such that $\mathrm{red}(w) = p$, one of the following happens:

  (2a) $\mathcal{S}_n(B)[p; w]$ is empty, so $s_n(B)[p; w] = 0$

  (2b) $\mathcal{S}_n(B)[p; w]$ is in bijection with some other $\mathcal{S}_{\hat{n}}(B)[\hat{p}; \hat{w}]$ for $\hat{n} < n$, so it follows that $s_n(B)[p; w] = s_{\hat{n}}(B)[\hat{p}; \hat{w}]$.

(3) $\mathcal{S}_n(B)[p]$ must be partitioned further, so $s_n(B)[p] = \sum_{p' \in \mathcal{S}_{k+1}(B)[p]} s_n(B)[p']$.

Case (1) provides the base cases for our recurrence. If case (2) applies, then we will use it preferentially over case (3). If case (2) does not apply, we must divide $\mathcal{S}_n(B)[p]$ as in case (3). Determining whether case (2) applies makes use of *gap vector criteria* to test (2a) and *reversible deletions* to form the bijection in (2b). These concepts are outlined in the following subsections.

### Gap Vectors

The motivation for gap vectors lies in the idea of "vertical space" (in the sense of the graph of a permutation) in a prefix set $C$. Sometimes the difference between the values of letters in the prefix is so great that a forbidden pattern *must* appear. To make this more precise, we follow our example above and compute $s_n(123)[12]$. Observe that $\mathcal{S}_n(123)[12; w_1 w_2]$ is empty if $w_1 < w_2 < n$, since otherwise if $\pi \in \mathcal{S}_n(123)[12; w_1 w_2]$

[8] This is distinct from, but similar to, the notion of a child discussed above in generating trees. In Vatter's schemes the definitions coincide.

then $\pi_i = n$ for some $i \geq 3$ and so $w_1 w_2 n$ forms a 123 pattern. Since the possibility for any $\pi_i > w_2$ for $i \geq 3$ prohibits the formation of a 123-avoiding permutation, we must restrict the space above $w_2$.

To formalize this, consider $\mathcal{S}_n(B)[p; w]$ and let $c_i$ be the $i^{th}$ smallest letter in $w$. Let $c_0 = 0$ and $c_{k+1} = n + 1$, and form the $(k+1)$-vector $\vec{g} = \vec{g}(n, w)$ so that the $i^{th}$ component is $g_i = c_i - c_{i-1} - 1$. Note that $g_i$ counts the number of letters for any $\pi \in \mathcal{S}_n(B)[p; w]$ which lie strictly between $c_{i-1}$ and $c_i$, i.e. the number of letters $\pi_j$ following the prefix $(j > k)$ and $c_{i-1} \leq \pi_j \leq c_i$. We call $\vec{g}$ the *spacing vector* for $w$.

In the example above, if $\vec{g}(n, w) \geq \langle 0, 0, 1 \rangle$ in the product order of $\mathbb{N}^3$ (i.e., component-wise), then $\mathcal{S}_n(123)[12; w] = \emptyset$. We call $\langle 0, 0, 1 \rangle$ a gap vector for the prefix 12. More generally we may make the following definition:

**Definition 13.** *Given a set of forbidden patterns $B$ and prefix $p$, then $\vec{v}$ is a* gap vector for prefix $p$ with respect to $B$ *if, for all $n$, $\mathcal{S}_n(B)[p; w] = \emptyset$ for any $w$ such that $\vec{g}(n, w) \geq \vec{v}$. When this happens, we say that $w$* satisfies *the* gap vector criterion *for $\vec{v}$.*

Hence $\vec{v} = \langle 0, 0, 1 \rangle$ is a gap vector for $p = 12$ with respect to $B = \{123\}$, and any prefix set $w = w_1 w_2$ with $w_1 < w_2 < n$ satisfies the gap vector condition for $v$. It should be noted that this definition reverses the satisfy/fail terminology of [91], but matches that of [102, 74, 76, 75].

Observe that gap vectors for a given prefix $p \in \mathcal{S}_k$ form an upper order ideal in $\mathbb{N}^{k+1}$, since if $\vec{v}$ is a gap vector so is any $\vec{v'} \geq \vec{v}$. Hence it suffices to determine only the minimal elements (which form a basis). For details on the discovery of gap vectors, and automating the process, see [91, 102] and Chapter 3.

To construct an enumeration scheme, we compute basis gap vectors for prefix $p$. To test case (2a) above for a given set $\mathcal{S}_n(B)[p; w]$, we compute $\vec{g}(n, w)$ and compare it to each of the computed basis gap vectors. Note that this provides a sufficient but not necessary criterion for determining whether $\mathcal{S}_n(B)[p; w] = \emptyset$, especially if the process for computing basis gap vectors is not exhaustive.[9]

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[9]Vatter provides a variant notion of gap vectors in [91] which allows an exhaustive algorithm for finding the basis gap vectors. The presentation in [102] discards exhaustiveness for the sake of speeding computation.

**Reversible Deletability**

When $w$ fails the gap vector criterion for all gap vectors $\vec{v}$, we must rely on bijections with previously-computed $\mathcal{S}_{\hat{n}}(B)[\hat{p};\hat{w}]$. To continue our example above, consider $\mathcal{S}_n(123)[12;w_1n]$. Here $w_1n$ fails all gap vector criteria, because $\langle 0,0,1\rangle$ forms the basis for the ideal of gap vectors and $\vec{g}(n,w_1n) = \langle w_1-1, n-w_1-1, 0\rangle$. However, any $\pi \in \mathcal{S}_n(123)[12;w_1n]$ has $\pi_2 = n$, so we may use the map the map $d_2 :\ \pi_1\pi_2\ldots\pi_n \mapsto \mathrm{red}(\pi_1\pi_3\ldots\pi_n)$ to form a bijection $\mathcal{S}_n(123)[12;w_1n] \to \mathcal{S}_{n-1}(123)[1;w_1]$. The deletion of a letter always preserves pattern-avoidance properties when considering classical patterns, but reversing the process by adding a letter has the potential for creating a forbidden pattern. Here, however, an $n$ at the second index cannot possibly create a 123, so we may safely reverse the deletion.

Define the deletion $d_r : \pi \mapsto \mathrm{red}(\pi_1\ldots\pi_{r-1}\pi_{r+1}\ldots\pi_n)$, that is, the permutation obtained by omitting the $r^{th}$ letter of $\pi$ and reducing. Similarly, define $d_r(w)$ to be the word obtained by deleting the $r^{th}$ letter and decrementing by one each letter $w_i > w_r$. For example $d_3(3642) = 352$. As discussed above, if $\pi \in \mathcal{S}_n(B)[p;w]$ then $d_r(\pi) \in \mathcal{S}_{n-1}(B)[d_r(p);d_r(w)]$, since deleting a letter cannot cause the appearance of a pattern. Sometimes this operation is invertible, however, as illustrated above. This brings us to the following definition:

**Definition 14.** *The index $r$ is* reversibly deletable with respect to $p$ and $B$ *if the map*

$$d_r : \mathcal{S}_n(B)[p;w] \to \mathcal{S}_{n-1}(B)[d_r(p);d_r(w)]$$

*is a bijection for any $w$ such that $\mathcal{S}_n(B)[p;w] \neq \emptyset$.*

Vatter uses the term *$ES^+$-reducible* for the same concept in [91].

Note that if $r < s$ are both reversibly deletable with respect to $p$ and $B$, then the composition $d_r(d_s(\pi))$, which we may consider as the simultaneous deletion at both $r$ and $s$, is still invertible. For a set $R$ with $r_1 = \min R$, recursively define $d_R(\pi) = d_{r_1}(d_{R\setminus r_1}(\pi))$, and so $d_R(\pi)$ is the permutation obtained by omitting $\pi_r$ for each $r \in R$ and reducing. Similarly for the prefix word $w$ let $d_R(w) = d_{r_1}(d_{R\setminus r_1}(w))$. For example, $d_{\{3,4\}}(3642) = 24$. Making the deletions from right to left eases notation.

We say $R$ is *reversibly deletable with respect to* $p$ *and* $B$ if the map $d_R : \mathcal{S}_n(B)[p;w] \to \mathcal{S}_{n-|R|}(B)[d_R(p); d_R(w)]$ is a bijection for any $w$ failing all gap vector criteria for $p$ with respect to $B$. Note that the empty set $R = \emptyset$ is always reversibly deletable: we are interested in finding *non-empty* reversibly deletable sets when they exist. Also observe that if $r_1, r_2, \ldots, r_t$ are each reversibly deletable individually, then $R = \{r_1, r_2, \ldots, r_t\}$ is reversibly deletable as well. The converse is also true for the case of classical patterns, but it is not true when $B$ contains dashed patterns as discussed in Chapter 3.

Vatter showed that identifying reversibly deletable indices is a finite process and thus subject to a computer search. Automating this process is discussed in [91, 100, 102] as well as Chapter 3. Careful examination of these deletion maps allows for the primary results of Chapter 4. Inverting these deletions to create insertions allows for the primary results of Chapter 5.

When we construct enumeration schemes, we try to discover reversibly deletable sets for prefixes $p$ to construct the bijections in case (2b) above. Again this is only a sufficient condition for proving such bijections exist.

**Formal Definition of Enumeration Schemes**

Formally, an enumeration scheme $E$ for $\mathcal{S}_n(B)$ is a set of triples $(p, G_p, R_p)$, where $p \in \mathcal{S}_k$ is the prefix pattern, $G_p$ is a basis for a set of gap vectors associated with $p$, and $R_p$ is a reversibly deletable set with respect to $p$. Furthermore, the following criteria must hold:

1. $(\epsilon, \emptyset, \emptyset) \in E$.

2. If $(p, G_p, R_p) \in E$ and $R_p = \emptyset$, then $(p', G_{p'}, R_{p'}) \in E$ for every child $p'$ of $p$.

3. If $(p, G_p, R_p) \in E$ and $R_p \neq \emptyset$, then $(\hat{p}, G_{\hat{p}}, R_{\hat{p}}) \in E$ for $\hat{p} = d_{R_p}(p)$

One can then "read" the enumeration scheme $E$ to compute $\mathcal{S}_n(B)[p;w]$ according to the following rules:

1. If $w$ passes the gap vector criteria for some $v \in G_p$, then $\mathcal{S}_n(B)[p;w] = \emptyset$.

2. For each prefix word $w$ such that $\mathrm{red}(w) = p$ and which fails the gap criteria for all $\vec{v} \in G$, we have the bijection:

$$d_{R_p} : \mathcal{S}_n(B)[p; w] \to \mathcal{S}_{n-|R_p|}(B)[d_{R_p}(p); d_{R_p}(w)]$$

(i.e. $R_p$ is a reversibly deletable set of indices).

When combined with the obvious initial conditions (e.g. if $p = w$ avoids $B$ then $\mathcal{S}_n(B)[p; w] = \{p\}$), the enumeration scheme presents the recurrences for $s_n(B)[p; w]$, and hence $s_n(B)$.

To illustrate, consider the enumeration scheme for $\mathcal{S}_n(123)$:

$$\{(\epsilon, \emptyset, \emptyset), (1, \emptyset, \emptyset), (12, \{\langle 0, 0, 1\rangle\}, \{2\}), (21, \emptyset, \{1\})\} \tag{1.5}$$

Since $R_\epsilon = \emptyset$, the first condition above requires the presence of $(1, G_1, R_1)$. Starting with the pattern 1 yields no additional information, so $R_1 = \emptyset$ and thus explaining the presence of $(12, G_{12}, R_{12})$ and $(21, G_{21}, R_{21})$. As discussed above, $\{\langle 0, 0, 1\rangle\}$ forms a basis for the gap vectors for 12, and whenever $w$ fails this gap vector criteria the second letter is reversibly deletable. For the fourth entry in the scheme, suppose that $\pi \in \mathcal{S}_n(\emptyset)[21]$ contains a 123 pattern involving the first letter, say $\pi_1 < \pi_i < \pi_j$ for $i < j$. Then since $\pi_2 < \pi_1$, we see that $\pi_2 < \pi_i < \pi_j$ is another 123 pattern. Therefore $\pi_1$ cannot be the deciding factor for whether $\pi$ contains 123. Hence the index 1 is reversibly deletable, so $R_{21} = \{1\}$.

Enumeration schemes exhibit a tree-like structure. The empty prefix $\epsilon$ serves as the root, and the children of each prefix are drawn as children in the rooted tree. When a prefix has nontrivial gap vector criteria, we list those basis vectors below it. When prefix $p$ has a non-empty reversibly deletable set $R$, we draw a dashed arrow from $p$ to $d_R(p)$ labeled with "$d_R$". See Figure 1.4 for an example.

If $|E|$ is finite, we say that $B$ admits a finite enumeration scheme. A finite enumeration scheme gives us a polynomial-time algorithm to compute $s_n(B)[p; w]$. We construct the system of recurrences based on the partitions and bijections above, along with base cases as given by the gap vector criteria and the trivial cases when $\mathcal{S}_n(B)[p] = \{p\}$ or

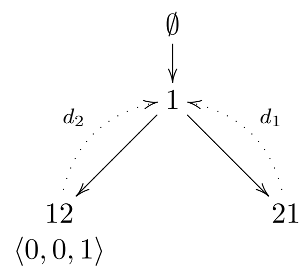


Figure 1.4: Tree representation of the enumeration scheme for $\mathcal{S}_n(123)$

∅. For example, the above enumeration scheme in (1.5) translates into the following system of recurrences:

$$\begin{aligned}
s_n(123) &= s_n(123)[\epsilon] \\
&= s_n(123)[1] \text{when } n > 0 \\
&= \sum_{i=1}^{n} s_n(123)[1; i] \\
s_n(123)[1; i] &= \sum_{j=1}^{i-1} s_n(123)[21; ij] + \sum_{j=i+1}^{n} s_n(123)[12; ij] \\
s_n(123)[12; ij] &= \begin{cases} 0 & \text{if } n - j \geq 1 \\ s_n(123)[1; i] & \text{otherwise} \end{cases} \\
s_n(123)[21; ij] &= s_n(123)[1; j]
\end{aligned}$$

Simplifying the above, we get:

$$s_n(123)[1; i] = \sum_{j=1}^{i} s_{n-1}(123)[1; j] \tag{1.6}$$

One can then evaluate this recurrence to identify the closed form $s_n(123) = \frac{1}{n+1}\binom{2n}{n}$, the Catalan numbers.

The length of the longest prefix $p$ appearing in $E$ is the called the *depth* of $E$. Not every set $B$ admits a finite enumeration scheme, the simplest example being the classical pattern 231. Let $E_{231}$ be the scheme for $\mathcal{S}_n(231)$, and let $J_t = t(t-1)\cdots 21$ be the decreasing permutation of length $t$. It can be shown that for any $t$ there are no gap vectors for $J_t$ and no non-empty reversibly deletable sets. Hence $E_{231}$ contains the triple $(J_t, \emptyset, \emptyset)$ for all $t \geq 1$ and hence is infinite. It should be noted, however, that the enumeration scheme for $\mathcal{S}_n(132)$ is finite (of depth 2) and $s_n(231) = s_n(132)$ by symmetry.

In general, if $B$ admits an enumeration scheme $E_B$ of depth $d$ then its set of complements $B^c = \{\sigma^c : \sigma \in B\}$ also admits an enumeration scheme $E_{B^c}$ of depth $d$. In fact, one can say $(p, G, R) \in E_B$ if and only if $(p^c, G^r, R) \in E_{B^c}$ where $G^r = \{\langle g_{k+1}, g_k, \ldots, g_1\rangle : \langle g_1, g_2, \ldots, g_{k+1}\rangle \in G\}$. This follows directly from the definitions given above and is left to the reader. The analogous statements regarding $B^r = \{\sigma^r : \sigma \in B\}$ do not hold and so $B$ may not have a finite scheme while $B^r$ does, as exhibited by $B = \{231\}$.

Due to their automatability, enumeration schemes provide a powerful tool to enumerate a set $\mathcal{S}_n(B)$ for a given $B$. The discovery of simple generating tree rules requires considerable ingenuity. On the other hand, to use enumeration schemes one needs only to load the requisite Maple packages, enter the pattern set $B$ and search parameters delimiting maximum depth and the maximum size of gap vectors to test, and wait for the computer to search for a scheme. If a scheme exists, then one can compute $s_n(B)$ quickly, getting results for, say, $n \leq 40$ in under five minutes on a personal computer.

The data produced by a scheme is then used as fodder to conjecture generating functions or explicit formulas in $n$, or the algorithm itself may be considered "the answer" (as defined by Wilf in [96]). Furthermore, we show in Chapter 5 how the scheme itself implies a system of functional equations for the generating functions $F_p(z) = \sum_{n\geq 0} s_n(B)[p]z^n$ for each prefix $p$ appearing in the scheme. We also show in Chapter 4 how to refine schemes to get information about the distribution of certain permutation statistics over $\mathcal{S}_n(B)$.

# Chapter 2

# The Cluster Method and Dashed Patterns

## 2.1 Introduction

In this chapter, we derive recurrences to compute weight enumerators counting the number of permutations of a multiset with $r$ occurrences of any dashed pattern of length 3 with one internal dash. The special case of permutations of $\{1, \ldots, n\}$ allows for some more specialized recurrences. Further the method can also compute the joint distributions for the number of occurrences of different patterns, yielding recurrences to compute certain pattern-based permutation statistics.

Our principal tool will be the cluster method, as described by Noonan and Zeilberger in [71]. In its most basic form the cluster method counts the number of words of length $\ell$ in a given alphabet $A$ which avoid a certain set of forbidden subwords (i.e. contiguous blocks of letters). In that same article, Noonan and Zeilberger extend the method to find the number of words of length $\ell$ which contain a given number of those forbidden subwords (i.e., consecutive patterns). Unlike pattern-containment, these are literal occurrences, i.e., the word 2341 does not literally contain 123 even though it contains 123 as a pattern. The permutation statistics des, INV, and MAJ were implicitly treated with this method in [98].

Burstein and Mansour in [26] list the number of words of length $\ell$ in the alphabet $\{1, \ldots, n\}$ avoiding a given dashed pattern for each pattern of length at most 3. Claesson and Mansour in [33] provide a recurrence to compute the number of permutations with $r$ occurrences of a single dashed pattern of length 3 with one internal dash. They follow in [34] by counting permutations avoiding pairs of two such patterns. This chapter provides a synthesis of these works, proving a method to compute the number of permutations with $r_1$ copies of one pattern and $r_2$ copies of another. Also related to this chapter, explicit weight enumerators for the number of permutations with $r$ occurrences of the pattern (2–31) and $k$ descents are given in [86, 36].

In section 2.2 we outline the conventions regarding operations on words and recall definitions of generalized patterns. Section 2.3 describes how we apply the cluster method to derive recurrences for the distributions of generalized patterns on multiset permutations. Section 2.4 considers the special case of distributions of patterns over permutations in $\mathcal{S}_n$, adapting the recurrences found in Section 2.3.

## 2.2 Preliminaries

### 2.2.1 Operations on Words

For word $w \in [n]^\ell$, let $\nu_i(\pi)$ represent the number of copies of $i$ in $w$, and $\nu(w) := \langle \nu_1(w), \nu_2(w), \ldots \rangle \in \mathbb{N}^\mathbb{N}$. We call $\nu$ the *alphabet vector* for $w$. For a $\mathbf{m} \in \mathbb{N}^\mathbb{N}$ with finitely many nonzero entries, we may define $\mathcal{S}_\mathbf{m}$ to be all words $\pi$ so that $\nu(\pi) = \mathbf{m}$. In practice we omit the tail of zeros for such an alphabet vector. Let $n(\mathbf{m})$ be the index of the largest nonzero entry in $\mathbf{m}$, i.e., the largest letter appearing in a $w \in \mathcal{S}_\mathbf{m}$. We may define the reduction of an alphabet vector $\text{red}(\mathbf{m})$ to be the vector formed by removing internal zeroes from $\mathbf{m}$. For example $\text{red}(\langle 2,0,0,1,0,3,2,0,0,\ldots\rangle) = \langle 2,1,3,2,0,0,\ldots\rangle$. This is natural since if $w \in \mathcal{S}_\mathbf{m}$, then $\text{red}(w) \in \mathcal{S}_{\text{red}(\mathbf{m})}$. Because of the reduction operation, we will only need to consider alphabet vectors $\mathbf{m}$ such that $\sum_i m_i \geq n(\mathbf{m})$ so we will only consider words which are at least as long as their largest element.

For words $u = u_1 \cdots u_k$ and $v = v_1 \cdots v_\ell$, we denote the concatenation as $uv = u_1 \cdots u_k v_1 \cdots v_\ell$.

We may also define the reversal and complement operations on words $w \in \mathcal{S}_\mathbf{m}$. Let $w^r = w_\ell w_{\ell-1} \cdots w_1$ and $w^c = (n+1-w_1)(n+1-w_2)\cdots(n+1-w_\ell)$ where $n = n(\mathbf{m})$ is the largest letter in $w$. Note that $\nu(w^r) = \nu(w)$, but $\nu(w^c) = \mathbf{m'}$ where $\mathbf{m}'_i = n+1-m_i$ for any nonzero $m_i$ and $m'_i = 0$ if $m_i = 0$. Hence reversal provides a bijection from $\mathcal{S}_\mathbf{m}$ to itself while complementation provides a bijection from $\mathcal{S}_\mathbf{m}$ to $\mathcal{S}_{\mathbf{m}'}$. Since words can have repeated letters, inverses are not well-defined.

Since we will count the descents of a word, it will be useful to define the *descent set* $Des(w) := \{i : w_i > w_{i+1}\}$. This should not be confused with $\text{des}(w) := \#Des(w)$.

### 2.2.2 Dashed Pattern Functions

Dashed patterns are introduced in section 1.1.2 of Chapter 1. For a dashed pattern $\sigma$, we will denote the pattern function $\sigma : \mathcal{S}_{\mathbf{m}} \to \mathbb{N}$ to be the function where $\sigma(w)$ is the number of copies of $\sigma$ in $w$. For example, the pattern function $(1\text{–}23)(w)$ for word $w = w_1 w_2 \cdots w_\ell$ is the number of subsequences $w_i w_j w_{j+1}$ such that $i < j$ and $w_i < w_j < w_{j+1}$. For words we have additional (non-trivial) pattern functions with repeated letters such as $(1\text{–}12)(w) = \#\{(i,j) : i < j, w_i = w_j < w_{j+1}\}$. We will regularly use the symbols $\sigma$ and $\tau$ to represent pattern functions. Treating these as functions allows multi-pattern functions like $\sigma + 2\tau$, the number of occurrences of $\sigma$ plus twice the number of occurrences of $\tau$.

In this chapter, we will only explicitly consider those patterns with three letters and one dash located between the first and second letters, that is those of the shape $\sigma_1\text{–}\sigma_2\sigma_3$. These are called *type*-$(1,2)$ by Mansour in [66]. If $\sigma_2 > \sigma_3$, then we call such patterns *descent-based* and if $\sigma_2 < \sigma_3$ they are called *rise-based*. The reverse of a type-$(1,2)$ pattern is a type-$(2,1)$ pattern $\sigma_1\sigma_2\text{–}\sigma_3$. The complement of a descent-based statistic is a rise-based statistic. Hence it will suffice to discuss descent-based permutations of type-$(1,2)$, since any other length 3 patterns with one internal dash can be obtained from such a pattern via symmetries.

We are interested in the distribution of patterns over sets of words. For a pattern function $\sigma$ and set of words $\mathcal{S}$, the *distribution of* $\sigma$ *over* $\mathcal{S}$ is the weight enumerator

$$F_\sigma^{\mathcal{S}}(q) = \sum_{w \in \mathcal{S}} q^{\sigma(w)}. \tag{2.1}$$

For example, for pattern $\sigma = (1\text{–}32)$ and the set $\mathcal{S} = \mathcal{S}_{\langle 1,1,1,1\rangle} = \mathcal{S}_4$, there are three permutations of length 4 with two occurrences of (1–32), (1243, 2143, and 1432), and six with one occurrence (1324, 1423, 1342, 2431, 3142, and 4132). The remaining permutations avoid 1–32. Hence $F_\sigma^{\mathcal{S}_4}(q) = 15 + 6\,q + 3\,q^2$. We will use ideas from the cluster method to derive recurrences which quickly compute $F_\sigma^{\mathcal{S}_{\mathbf{m}}}(q)$. Two statistics

$\sigma, \tau$ are *equidistributed over* $\mathcal{S}$ if $F_\sigma^{\mathcal{S}}(q) = F_\tau^{\mathcal{S}}(q)$.

## 2.3 The Cluster-based Recurrence

We now adapt the cluster method to find the number of words with a given number of occurrences of a given pattern. Instead of counting the number of occurrences of a descent-based pattern, we instead consider the problem of weight-counting the descents, where the descent-weight is determined by the pattern. For example, the pattern function (1–32) can be defined in terms of weighted descents:

$$(1\text{–}32)(w) = \#\{(j,i) : j < i, w_j < w_{i+1} < w_i\} = \sum_{i \in Des(w)} \#\{j : j < i, w_j < w_{i+1}\}.$$

The *descent-weight with respect to* $\sigma = \sigma_1\text{–}\sigma_2\sigma_3$ for index $i$ of word $w$ is the number of copies of $\sigma$ in which $w_i w_{i+1}$ plays the part of $\sigma_2\sigma_3$. Denote this by $\sigma_d(w,i)$. Note that $\sigma_d(w,i) = 0$ if $i \notin Des(w)$. Above we see $(1\text{–}32)_d(w,i) = \#\{j : j < i, w_j < w_{i+1}\}$.

Thus we may manipulate the weight enumerator $F_\sigma(q)$ as follows:

$$\begin{aligned} F_\sigma(q) &= \sum_{w \in \mathcal{S}_\mathbf{m}} q^{\sigma(w)} \\ &= \sum_{w \in \mathcal{S}_\mathbf{m}} \prod_{i \in Des(w)} q^{\sigma_d(w,i)} \end{aligned}$$

We then employ the identity $x = (1 + (x-1))$ to get

$$F_\sigma(q) = \sum_{w \in \mathcal{S}_\mathbf{m}} \prod_{i \in Des(w)} (1 + (q^{\sigma_d(w,i)} - 1))$$

Now use the identity $\prod_{i \in T}(1 + x_i) = \sum_{S \subseteq T} \prod_{i \in S} x_i$ to expand the product:

$$F_\sigma(q) = \sum_{w \in \mathcal{S}_\mathbf{m}} \sum_{S \subseteq Des(w)} \prod_{i \in S} (q^{\sigma_d(w,i)} - 1),$$

Following the exposition in [71], we will re-interpret the double-sum as a single sum over pairs $(w, S)$. These pairs $(w, S)$ can be considered *marked words*, where some subset $S$ of the descents of $w$ are "marked" while others are left unmarked. Each marked descent $w_i w_{i+1}$ (represented by $i \in S$) has weight $(q^{\sigma_d(w,i)} - 1)$, and the weight of a marked word is the product of the weights of the marked descents. Unmarked descents do not contribute to the weight. Denote the weight of $(w, S)$ by $W(w, S)$.

We break here to provide a concrete example. Consider the word $w = 2637541$ and pattern $\sigma = (1–32)$. Then $\sigma_d(w,4) = 2$, since 275 and 375 are both occurrences of (1–32) which involve the descent at position 4. Similarly, $\sigma_d(w,2) = 1$, $\sigma_d(w,5) = 2$, and $\sigma_d(w,6) = 0$. Now consider the marked word $(w, \{2,5\})$, where we can underline the marked descents: $2\underline{63}7\underline{54}1$. Only the descents 63 and 54 contribute to the weight of $(w, \{2,5\})$, yielding the weight $(q^1 - 1)(q^2 - 1)$. Similarly, $W(w, \{4,5,6\}) = (q^2 - 1)(q^2 - 1)(q^0 - 1) = 0$ and $W(w, \emptyset) = 1$ because of the empty product. It is easily checked that the sum of the weights each of the $2^{\mathrm{des}(w)}$ marked versions of $w$ equals $q^{\sigma(w)} = q^5$.

From the cluster method, we partition the set of marked words into two classes:

1. Those marked words ending with a marked descent, i.e. those $(w,S)$ such that $\ell(w) - 1 \in S$.

2. Those marked words ending with a letter which is not part of a marked descent, i.e. those $(w,S)$ such that $\ell(w) - 1 \notin S$.

Taking the weights of these two classes shows us that

$$F_\sigma(q) = \sum_{\substack{(w,S):w\in\mathcal{S}_\mathbf{m}\\ S\subseteq Des(w)\\ \ell(w)-1\in S}} W(w,S) + \sum_{\substack{(w,S):w\in\mathcal{S}_\mathbf{m}\\ S\subseteq Des(w)\\ \ell(w)-1\notin S}} W(w,S) \tag{2.2}$$

We now relate these two classes. If a marked word $(w,S)$ ends with a marked descent, it must end in a marked descending run $w_i > w_{i+1} > \cdots > w_\ell$ such that $\{i, i+1, \ldots, \ell-1\} \subseteq S$. We call $w_i \cdots w_\ell$ a *terminal marked run* if $\{i, i+1, \ldots, \ell-1\} \subseteq S$, and such a run is *maximal* if $i-1 \notin S$. We will refer to a maximal terminal marked run as a *cluster*. When we remove the cluster, we are left with $(w_1 \cdots w_{i-1}, S \setminus \{i, i+1, \ldots, \ell-1\})$ which is a word in the second category above. To continue the example above, $(2637541, \{2,5,6\})$ has 541 as its cluster. To contrast, note that $(2637541, \{2,4,6\})$ has only 41 as its cluster, since the descent 54 is not marked. Weights are multiplicative, so $W(2637541, \{2,5,6\})$ is the product of the contribution from the cluster, $W(2637541, \{5,6\})$, and the contribution of marked descents outside the cluster,

$W(2637541,\{2\})$. Furthermore, rearranging the letters outside the cluster does not affect the weight of the cluster, e.g. $W(2637541,\{5,6\}) = W(7362541,\{5,6\})$. Our next steps are guided by fixing a cluster and letting the prefix vary.

Consider the set of marked words $(w,S)$ ending in the cluster $t_1\cdots t_k$ and write $w = w'\,t_1t_2\cdots t_k$. Note this implies $S \supseteq \{\ell-1,\ldots,\ell-k+1\}$. Let $S' = S\setminus\{\ell-1,\ldots,\ell-k+1\}$ be those marked descents appearing outside the cluster, and so we see that $W(w,S') = W(w',S')$ since the letters of $w$ which appear in the cluster do not contribute to the weight of $(w,S')$. Thus if $w$ ends with cluster $t_1\cdots t_k$, then

$$W(w,S) = W(w',S')W(w,\{\ell-1,\ldots,\ell-k+1\}).$$

Note that $\nu(w') = \nu(w) - \nu(t_1\cdots t_k)$. Next observe that $W(w,\{\ell-1,\ldots,\ell-k+1\})$ is constant over all marked words with cluster $t_1\cdots t_k$, and we will denote this weight $W(\mathbf{m},T)$ where we identify $T = \{t_1,\ldots,t_k\} \subseteq [n]$ with the descending run $t_1\cdots t_k$. This follows from the fact that our patterns have shape $\sigma_1{-}\sigma_2\sigma_3$, and the descent-weight $(\sigma)_d(w,i)$ does not care where that $\sigma_1$ appears so long as it appears to the left of $w_i$.

$$\begin{aligned}
\sum_{\substack{(w,S):w\in\mathcal{S}_{\mathbf{m}}\\ S\subseteq Des(w)\\ \{\ell-k,\cdots\ell-1\}\subseteq S\\ \ell-k-1\notin S}} W(w,S) &= \sum_{\substack{(w,S):w\in\mathcal{S}_{\mathbf{m}}\\ S\subseteq Des(w)\\ \{\ell-1,\ldots,\ell-k+1\}\subseteq S\\ \ell-k-1\notin S}} W(w,S')W(w,\{\ell-1,\ldots,\ell-k+1\})\\
&= \sum_{\substack{T\subseteq[n]\\ |T|=k}}\ \sum_{\substack{(w',S'):w\in\mathcal{S}_{\mathbf{m}'}\\ \mathbf{m}'=\mathbf{m}-\nu(T)\\ S'\subseteq Des(w')}} W(w',S')W(\mathbf{m},T) \qquad (2.3)\\
&= \sum_{\substack{T\subseteq[n]\\ |T|=k}} W(\mathbf{m},T)F_\sigma^{\mathcal{S}_{\mathbf{m}'}}(q)
\end{aligned}$$

Summing over $k = 2,\ldots,n$ gets us the weight of all words in $\mathcal{S}_{\mathbf{m}}$ ending with a cluster.

Now consider a marked word $(w,S)$ which does not end in a marked descent and let $w' = w_1\cdots w_{\ell-1}$. Then $S \subseteq Des(w')$, so $(w',S)$ is a marked word. Furthermore, the last letter of $w$ does not contribute toward the weight of $(w,S)$, and so $W(w,S) = W(w',S)$. Therefore we may write the weight of the set of marked words which do not end with a marked descent in terms of weight enumerators $F_\sigma^{\mathcal{S}_{\mathbf{m}'}}(q)$ where $\mathbf{m}' = \nu(w_1\cdots w_{\ell-1})$.

Specifically we may make the following algebraic manipulations:

$$\begin{aligned}\sum_{\substack{(w,S):w\in\mathcal{S}_{\mathbf{m}}\\ S\subseteq Des(w)\\ \ell(w)-1\notin S}} W(w,S) &= \sum_{\substack{(w,S):w\in\mathcal{S}_{\mathbf{m}}\\ S\subseteq Des(w)\\ \ell(w)-1\notin S}} W(w_1\cdots w_{\ell-1},S)\\ &= \sum_{w_\ell=1}^{n} \sum_{\substack{(w',S):w'\in\mathcal{S}_{\mathbf{m}'}\\ \mathbf{m}'=\mathrm{red}(\mathbf{m}-\nu(w_\ell))\\ S\subseteq Des(w')}} W(w',S) \qquad (2.4)\\ &= \sum_{w_\ell=1}^{n} F_\sigma^{\mathcal{S}_{\mathbf{m}'}}(q)\end{aligned}$$

Comparing Equations (2.3) and (2.4) suggests the simplification to define $W(\mathbf{m},T)=1$ for singleton sets $T$. Strictly speaking $t_1$ could not be considered a cluster on its own, but one cannot ignore the similarities between the forms $w = w't_1\cdots t_k$ for cluster $t_1\cdots t_k$ and the form $w=w'w_\ell$ for word $w$ which does not end in a cluster. This allows us to combine Equations (2.2), (2.3), and (2.4) in the following way, where $\mathbf{m}\backslash T := \mathrm{red}(\mathbf{m}-\nu(t_1\cdots t_k))$ for $T=\{t_1,\ldots,t_k\}$:

$$\begin{aligned}F_\sigma(q) &= \sum_{\substack{(w,S):w\in\mathcal{S}_{\mathbf{m}}\\ S\subseteq Des(w)\\ \ell(w)-1\in S}} W(w,S) + \sum_{\substack{(w,S):w\in\mathcal{S}_{\mathbf{m}}\\ S\subseteq Des(w)\\ \ell(w)-1\notin S}} W(w,S)\\ &= \sum_{k=2}^{n}\sum_{\substack{T\subseteq[n]\\ |T|=k}} W(\mathbf{m},T)F_\sigma^{\mathcal{S}_{\mathbf{m}\backslash T}}(q) + \sum_{t=1}^{n} F_\sigma^{\mathcal{S}_{\mathbf{m}\backslash\{t\}}}(q) \qquad (2.5)\\ &= \sum_{k=1}^{n}\sum_{\substack{T\subseteq[n]\\ |T|=k}} W(\mathbf{m},T)F_\sigma^{\mathcal{S}_{\mathbf{m}\backslash T}}(q)\end{aligned}$$

It now remains to consider the values for $W(\mathbf{m},T)$ for the various descent-based type-$(1,2)$ patterns: $1$–$21, 2$–$31, 3$–$32, 1$–$21, 2$–$21$. We also can consider 21 in this same context, although the distribution of 21 is a well-studied object. Suppose the marked word $(w,\{\ell-1,\ldots,\ell-k+1\})$ ends with cluster $t_1\cdots t_k$ so that $w=w_1\cdots w_{\ell-k}t_1\cdots t_k$. Consider the descent $t_it_{i+1}$ and the letters lying to the left of it. There are $\nu(1)+\cdots+\nu(t_{i+1}-1)$ letters which are less than $t_{i+1}$. This count includes $t_{i+2}\cdots t_k$ which lie to the right of $t_{i+1}$ so there are only $\nu(1)+\cdots+\nu(t_{i+1}-1)-(k-i-1)$ . Hence $t_it_{i+1}$ is involved in $\nu(1)+\cdots+\nu(t_{i+1}-1)-(k-i-1)$ copies of 1–32 using $t_it_{i+1}$. Similarly but more straight-forward, there are $\nu(t_{i+1}+1)+\cdots+\nu(t_i-1)$ copies of 2–31 using $t_it_{i+1}$ and there are $\nu(t_i+1)+\cdots+\nu(n)$ copies of 3–21 using $t_it_{i+1}$. Last, there $\nu(t_{i+1})$ copies of 1–21 using $t_it_{i+1}$ and there are $\nu(t_i)$ copies of 2–21 using $t_it_{i+1}$. Hence we have the following descent-weights for the descent $t_it_{i+1} = w_{\ell-k+i}w_{\ell-k+i+1}$, which we state as a lemma:

**Lemma 15.** *For $w \in \mathcal{S}_{\mathbf{m}}$ ending with a descending run $w_{\ell-k+1}\cdots w_\ell = t_1\cdots t_k$ we have the following descent weights:*

1. $(21)_d(w,\ell-k+i) = 1$
2. $(1\text{–}32)_d(w,\ell-k+i) = \nu(1)+\cdots+\nu(t_{i+1}-1)-(k-i-1)$
3. $(2\text{–}31)_d(w,\ell-k+i) = \nu(t_{i+1}+1)+\cdots+\nu(t_i-1)$
4. $(3\text{–}21)_d(w,\ell-k+i) = \nu(t_i+1)+\cdots+\nu(n)$
5. $(1\text{–}21)_d(w,\ell-k+i) = \nu(t_{i+1})$
6. $(2\text{–}21)_d(w,\ell-k+i) = \nu(t_i)$

Since $W(\mathbf{m},T) = \prod_{j=\ell-k+1}^{\ell-1}(q^{\sigma_d(w,j)}-1)$, we can fill in the appropriate values for $W(\mathbf{m},T)$ to complete the recurrences for $F_\sigma^{\mathcal{S}_{\mathbf{m}}}(q)$ in Equation (2.5).

As mentioned above, symmetries can be exploited to get distributions for other pattern functions. For example, $2\text{–}13(w) = 2\text{–}31^c(w^c)$, and so $F_{2\text{-}13}^{\mathcal{S}_{\mathbf{m}}}(q) = F_{2\text{-}31}^{\mathcal{S}_{\mathbf{m}'}}$ where $\mathbf{m}'_i = n+1-m_i$ for any nonzero $m_i$ and $m'_i = 0$ if $m_i = 0$.

This method's strength lies in its applicability to joint distributions. Let $\sigma$ and $\tau$ be two descent-based pattern functions and let $F_{(\sigma,\tau)}^{\mathcal{S}_{\mathbf{m}}}(q,t) := \sum_{w\in\mathcal{S}_{\mathbf{m}}} q^{\sigma(w)}t^{\tau(w)}$. Then just as before we may write $F_{\sigma,\tau}^{\mathcal{S}_{\mathbf{m}}}$ as

$$F_{\sigma,\tau}^{\mathcal{S}_{\mathbf{m}}}(q,t) = \sum_{w\in\mathcal{S}_{\mathbf{m}}}\sum_{S\subseteq Des(w)}\prod_{i\in S}(q^{\sigma_d(w,i)}t^{\tau_d(w,i)}-1)$$

Hence we weight the marked word $(w,S)$ by:

$$W(w,S) := \prod_{i\in S}(q^{\sigma_d(w,i)}t^{\tau_d(w,i)}-1).$$

This implies the cluster-weights:

$$W(\mathbf{m}, T) = \prod_{j=\ell-k+1}^{\ell-1} (q^{\sigma_d(w,j)} t^{\tau_d(w,j)} - 1).$$

Of course this generalizes in the same way to multistatistics with any number of descent-based patterns of type-$(1, 2)$. For the most extreme example, consider the holistic weight enumerator for $\mathcal{S}_{\mathbf{m}}$ where the weight of $w$ is given by

$$F_{all}^{\mathcal{S}_{\mathbf{m}}}(q_1, q_2, q_3, q_4, q_5, q_6) := {q_1}^{(21)(w)} {q_2}^{(1\text{–}32)(w)} {q_3}^{(2\text{–}31)(w)} {q_4}^{(3\text{–}21)(w)} {q_5}^{(1\text{–}21)(w)} {q_6}^{(2\text{–}21)(w)}.$$

This implies the following weights on marked words:

$$W(w, S) := \prod_{i \in S} \left( {q_1}^{(21)(w)} {q_2}^{(1\text{–}32)(w)} {q_3}^{(2\text{–}31)(w)} {q_4}^{(3\text{–}21)(w)} {q_5}^{(1\text{–}21)(w)} {q_6}^{(2\text{–}21)(w)} - 1 \right)$$

The remaining details are left to the reader.

We can also consider linear combinations of pattern functions by specialization. For example, the major index for words is given by

$$\begin{aligned} \text{MAJ}(w) &= \sum_{i \in Des(w)} i \\ &= (21)(w) + (1\text{–}32)(w) + (2\text{–}31)(w) + (3\text{–}21)(w) + (1\text{–}21)(w) + (2\text{–}21)(w). \end{aligned}$$

The specialization $F_{all}^{\mathcal{S}_{\mathbf{m}}}(q, q, q, q, q, q)$ gives the weight-enumerator for $\mathcal{S}_{\mathbf{m}}$ where the weight of a $w$ is given by $q^{\text{MAJ}(w)}$. The Euler-Mahonian distribution can be recovered similarly.

The greatest limitation is that one may only combine descent-based statistics, or combine ascent-based statistics, and must keep within type-(1,2) or type-(2,1). Problems combining descent-based and ascent-based patterns arise since both descents and ascents are marked, resulting in non-monotone clusters which are much more varied than monotone clusters. In particular, for each $T \subseteq [n]$ there is more than one cluster involving the letters in $T$. Furthermore clusters may involve repeated letters, so one must consider submultisets $T \subseteq [n]$ to compute distributions over multiset permutations.

## 2.4 Distributions over $\mathcal{S}_n$

We now move to the special case of distributions over the permutations, considering the weight-enumerator:

$$F_\sigma(n) := F_\sigma^{\mathcal{S}_n}(q) = \sum_{\pi\in\mathcal{S}_n} q^{\sigma(\pi)}. \tag{2.6}$$

We will denote permutations with $\pi$ rather than $w$. Also note that $\ell(\pi) = n$. Since there are no repeated letters in permutations, $(1{-}21)(\pi) = (2{-}21)(\pi) = 0$ for any permutation $\pi$.

When $\mathbf{m} = \langle 1, 1, \ldots, 1\rangle$ for $n$ 1's, let $W(n, T) = W(\mathbf{m}, T)$. Then the recurrence in Equation (2.5) specializes to

$$\begin{aligned} F_\sigma(n) &= \sum_{k=1}^{n} \sum_{\substack{T\subseteq[n]\\|T|=k}} W(n,T)\ F_\sigma(n-k) \\ &= \sum_{k=1}^{n} F_\sigma(n-k)\Bigg(\sum_{\substack{T\subseteq[n]\\|T|=k}} W(n,T)\Bigg) \end{aligned} \tag{2.7}$$

We wish to find a recurrence of the form $F_\sigma(n)\sum_{k=1}^{n} F_\sigma(n-k)\ a_\sigma(n,k)$, where

$$\begin{aligned} a_\sigma(n,k) &:= \sum_{\substack{T\subseteq[n]\\|T|=k}} W(n,T) \\ &= \sum_{n\geq t_1>\cdots>t_k\geq 1} \prod_{i=1}^{k-1} (q^{(\sigma)_d(\pi,n-k+j)} - 1) \end{aligned}$$

for any permutation $\pi \in \mathcal{S}_n$ such that $\pi_{n-k+1}\cdots\pi_n = t_1\cdots t_k$.

Lemma 15 specializes to permutations as follows

**Lemma 16.** *For $\pi \in \mathcal{S}_n$ ending in the descending run $\pi_{n-k+1}\cdots\pi_n = t_1\cdots t_k$, we have the following descent weights:*

1. $(21)_d(\pi, n-k+i) = 1$

2. $(1{-}32)_d(\pi, n-k+i) = t_{i+1} + i - k$

3. $(2{-}31)_d(\pi, n-k+i) = t_i - t_{i+1} - 1$

4. $(3{-}21)_d(\pi, n-k+i) = n - t_i$

In the next four subsections, we will go through each pattern and develop recurrences for $a_\sigma(n,k)$. In each case we will omit the $\sigma$ subscripts.

### 2.4.1 The Pattern (21)

When $\sigma = 21$, Lemma 16 tells us $(21)_d(w,i) = 1$. Therefore $W(n,T) = (q^1 - 1)^{k-1}$ for any $|T| = k$, and so $a(n,k) = \binom{n}{k}(q-1)(k-1)$. Therefore we have the recurrence

$$F(n) = \sum_{k=1}^{n} \binom{n}{k} (q-1)^{k-1} F(n-k)$$

Observe that $qF(n)$ are the well-known Eulerian polynomials.

### 2.4.2 The Pattern (1–32)

When $\sigma = 1\text{–}32$, Lemma 16 tells us $(1\text{–}32)_d(\pi, n-k+i) = t_{i+1} + i - k$. Therefore

$$a(n,k) = \sum_{n \geq t_1 > \cdots > t_k \geq 1} \prod_{j=1}^{k-1} (q^{t_{j+1}+j-k} - 1)$$

To get a faster recurrence for $a(n,k)$, define the secondary function $b(n,k)$:

$$b(n,k) = \sum_{n \geq t_1 > \cdots > t_k \geq 1} \prod_{j=1}^{k} (q^{t_j+j-k-1} - 1).$$

Conditioning on whether $t_1 = n$, we get that

$$\begin{aligned} a(n,k) &= \sum_{n-1 \geq t_1 > \cdots > t_k \geq 1} \prod_{j=1}^{k-1} (q^{t_{j+1}+j-k} - 1) + \sum_{n-1 \geq t_2 > \cdots > t_k \geq 1} \prod_{j=1}^{k-1} (q^{t_{j+1}+j-k} - 1) \\ &= a(n-1,k) + \sum_{n-1 \geq t_1 > \cdots > t_{k-1} \geq 1} \prod_{j=1}^{k-1} (q^{t_j+j-k} - 1) \\ &= a(n-1,k) + b(n-1,k-1) \end{aligned}$$

Similarly we can derive a recurrence for $b(n,k)$ by conditioning on $t_1$.

$$\begin{aligned} b(n,k) &= \sum_{n\geq t_1>\cdots>t_k\geq 1} \prod_{j=1}^{k} (q^{t_j+j-k-1}-1) \\ &= \sum_{n-1\geq t_1>\cdots>t_k\geq 1} \prod_{j=1}^{k} (q^{t_j+j-k-1}-1) \\ &\qquad + \sum_{n-1\geq t_2>\cdots>t_k\geq 1} (q^{n-k}-1) \prod_{j=2}^{k} (q^{t_j+j-k-1}-1) \\ &= b(n-1,k) + (q^{n-k}-1) \sum_{n-1\geq t_1>\cdots>t_{k-1}\geq 1} \prod_{j=1}^{k-1} (q^{t_j+j-k}-1) \\ &= b(n-1,k) + (q^{n-k}-1) b(n-1,k-1) \end{aligned}$$

These recurrences, along with the initial conditions below yield fast computation of $F(n)$:

$$\begin{array}{ll} a(n,1) = n & b(n,1) = \sum_{i=1}^{n}(q^{i-1}-1) \\ a(n,k) = 0 \text{ for } k \geq n & b(n,k) = 0 \text{ for } k \geq n \end{array}$$

This recurrence confirms Claesson and Mansour's values listed in Table 2 in [33].

### 2.4.3 The Pattern (2–31)

When $\sigma = 2\text{–}31$, Lemma 16 tells us $(2\text{–}31)_d(\pi, n-k+i) = t_i - t_{i+1} - 1$. Therefore

$$a(n,k) = \sum_{n\geq t_1>\cdots>t_k\geq 1} \prod_{j=1}^{k-1} (q^{t_j-t_{j+1}-1}-1).$$

We will also make use of a secondary function $b(n,k)$, defined as

$$b(n,k) = \sum_{n\geq t_1>\cdots>t_k\geq 1} q^{-t_1} \prod_{j=1}^{k-1} (q^{t_j-t_{j+1}-1}-1). \tag{2.8}$$

Again conditioning on whether $t_1 = n$, we obtain

$$
\begin{aligned}
a(n,k) &= a(n-1,k) + \sum_{n-1\geq t_2>\cdots>t_k\geq 1} (q^{n-t_2-1}-1)\prod_{j=1}^{k-1}(q^{t_j-t_{j+1}-1}-1) \\
&= a(n-1,k) + q^{n-1}\sum_{n-1\geq t_2>\cdots>t_k\geq 1} q^{-t_2}\prod_{j=2}^{k-1}(q^{t_j-t_{j+1}-1}-1) \\
&\qquad - \sum_{n-1\geq t_2>\cdots>t_k\geq 1}\prod_{j=1}^{k-1}(q^{t_j-t_{j+1}-1}-1) \\
&= a(n-1,k) + q^{n-1}b(n-1,k-1) - a(n-1,k-1) \\
b(n,k) &= b(n-1,k) + \sum_{n-1\geq t_2>\cdots>t_k\geq 1} q^{-n}(q^{n-t_2-1}-1)\prod_{j=2}^{k-1}(q^{t_j-t_{j+1}-1}-1) \\
&= b(n-1,k) + q^{-n}\left(\sum_{n-1\geq t_2>\cdots>t_k\geq 1}(q^{n-t_2-1}-1)\prod_{j=1}^{k-1}(q^{t_j-t_{j+1}-1}-1)\right) \\
&= b(n-1,k) + q^{-n}\left(q^{n-1}b(n-1,k-1) - a(n-1,k-1)\right)
\end{aligned}
$$

With the initial conditions below we can quickly compute terms of $F(n)$.

$$
\begin{aligned}
a(n,1) &= n & b(n,1) &= \textstyle\sum_{i=1}^{n}(q^{-i}-1) \\
a(n,k) &= 0 \text{ for } k \geq n & b(n,k) &= 0 \text{ for } k \geq n
\end{aligned}
$$

Note that Parviainen in [73] gives closed-form formulas for the coefficient of $q^k$ in $F(n)$ for $1 \leq k \leq 8$ and provides the method for higher $k$. This recurrence confirms Claesson and Mansour's values in Table 3 of [33].

### 2.4.4 The Pattern (3–21)

When $\sigma = 3\text{–}21$, Lemma 16 tells us $(3\text{–}21)_d(\pi, n-k+i) = n - t_i$. Therefore

$$
a(n,k) = \sum_{n\geq t_1>\cdots>t_k\geq 1}\prod_{j=1}^{k-1}(q^{n-t_j}-1).
$$

We will also need to use the secondary functions $b(n,k)$ and $c(n,k)$:

$$
\begin{aligned}
b(n,k) &= \sum_{n\geq t_1>\cdots>t_k\geq 1} t_k\prod_{j=1}^{k}(q^{n-t_j}-1) \\
c(n,k) &= \sum_{n\geq t_1>\cdots>t_k\geq 1}\prod_{j=1}^{k}(q^{n-t_j}-1).
\end{aligned}
\tag{2.9}
$$

First observe that $a(n,k) = b(n-1,k-1)$, since

$$\begin{aligned} a(n,k) &= \sum_{n\geq t_1>\cdots>t_{k-1}\geq 2} (t_k - 1)\prod_{j=1}^{k-1}(q^{n-t_j}-1) \\ &= \sum_{n-1\geq t_1>\cdots>t_{k-1}\geq 1} t_k \prod_{j=1}^{k-1}(q^{n-1-t_j}-1) \\ &= b(n-1,k-1) \end{aligned}$$

Now consider the sum in $b(n,k)$ and condition on whether $t_k = 1$. Then we get

$$\begin{aligned} b(n,k) &= \sum_{n\geq t_1>\cdots>t_k\geq 2} t_k \prod_{j=1}^{k}(q^{n-t_j}-1) + \sum_{n\geq t_1>\cdots>t_{k-1}\geq 2, t_k=1} \prod_{j=1}^{k}(q^{n-t_j}-1) \\ &= \sum_{n-1\geq t_1>\cdots>t_k\geq 1} (t_k+1) \prod_{j=1}^{k}(q^{n-1-t_j}-1) \\ &\quad + (q^{n-1}-1) \sum_{n-1\geq t_1>\cdots>t_{k-1}\geq 1, t_k=1} \prod_{j=1}^{k-1}(q^{n-1-t_j}-1) \\ &= b(n-1,k) + c(n-1,k) + (q^{n-1}-1)c(n-1,k-1) \end{aligned}$$

We also condition on whether $t_k = 1$ to get a recurrence for $c(n,k)$.

$$\begin{aligned} c(n,k) &= \sum_{n\geq t_1>\cdots>t_k\geq 2} \prod_{j=1}^{k}(q^{n-t_j}-1) + \sum_{n\geq t_1>\cdots>t_{k-1}\geq 2, t_k=1} (q^{n-1}-1) \prod_{j=1}^{k-1}(q^{n-t_j}-1) \\ &= \sum_{n-1\geq t_1>\cdots>t_k\geq 1} \prod_{j=1}^{k}(q^{n-1-t_j}-1) + (q^{n-1}-1) \sum_{n-1\geq t_1>\cdots>t_{k-1}\geq 1} \prod_{j=1}^{k-1}(q^{n-1-t_j}-1) \\ &= c(n-1,k) + (q^{n-1}-1)c(n-1,k-1) \end{aligned}$$

This recurrence confirms Claesson and Mansour's values listed in Table 1 of [33].

## 2.5 Maple Implementation

A corresponding package of Maple procedures illustrating the above methods, ClusterGPP, can be downloaded on the author's homepage. The primary procedures are listed in Table 2.1).

The reader is encouraged to experiment with the above package to see resulting distributions. In particular, we did not discuss the joint distributions over permutations, but ClusterGPP can certain compute them.

Table 2.1: Procedures in the Maple package ClusterGPP

| Procedure Name | Description |
|---|---|
| *PatternCount* | Counts the number of occurrences of a given pattern in a given word |
| *BFdist* | Computes the distribution of a given set of patterns over a given set of words. |
| *Rdist* | Computes the distribution $F_{\sigma}^{\mathcal{S}_{\mathbf{m}}}(q)$ for a single pattern $\sigma$ using Equation (2.5). |
| *RdistM* | Recursively computes the distribution of multiple patterns, e.g. $F_{\sigma,\tau}^{\mathcal{S}_{\mathbf{m}}}(q_1, q_2)$ using using Equation (2.5). This subsumes *Rdist*, as singletons may be entered. |

## 2.6 Conclusions and Future Directions

This chapter uses the cluster method to get recurrences for joint distributions $F_{\sigma}^{\mathcal{S}_{\mathbf{m}}}(q)$ of pattern functions of the form $\sigma_1 – \sigma_2\sigma_3$, $\sigma_2 > \sigma_3$. Restricting attention to $\mathcal{S}_n$ and single patterns allowed us to get some more specialized recurrences in section 2.4. This is a generalization of the basic avoidance question, since $F_{\sigma}^{\mathcal{S}_{\mathbf{m}}}(0)$ gives the number of words in $\mathcal{S}_{\mathbf{m}}$ with no occurrences of $q$.

The above work could apply to questions of *pattern packing*. In [23] Burstein et al. consider the highest number of occurrences of a given pattern or patterns which can occur in a word in $[n]^{\ell}$. This is simply the degree of the distributions we have calculated by summing over the appropriate $\mathcal{S}_{\mathbf{m}}$. For example, the first 20 terms of the sequence

$$a(n) := \max\{(1–32)(\pi) : \pi \in \mathcal{S}_n\}$$

can be computed in under a minute using procedures from ClusterGPP. The interested reader may adapt the above procedures to generate recurrences for the degrees of the polynomials.

The above approach could extend nicely to get distributions for other pattern functions of the shape $\sigma_1\sigma_2 \cdots \sigma_k – \sigma_{k+1}$, although in this case the clusters would be more complicated than monotone runs. The cluster-based approach of Nakamura in [69] to compute the number of permutations avoiding a consecutive pattern should blend well with the approach outlined in this chapter.

# Chapter 3

# Enumeration Schemes and Dashed Patterns

*This chapter represents joint work with Lara Pudwell.*

## 3.1 Introduction

Zeilberger introduced enumeration schemes, discussed in section 1.3.4 of Chapter 1, to compute the number of permutations avoiding a set of classical patterns. In this chapter we extend the tools of enumeration schemes to compute the number of permutations avoiding a set of *dashed* patterns.

Recall the definition for dashed patterns from section 1.1.2 of Chapter 1. A dashed pattern is a pair $(\sigma, X)$ for permutation $\sigma \in \mathcal{S}_k$ and a set of adjacencies $X \subseteq \{0\} \cup [k]$. Permutation $\pi \in \mathcal{S}_n$ contains dashed pattern $(\sigma, X)$ if there is a sequence $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that the following are satisfied:

1. $\text{red}(\pi_{i_1}\pi_{i_2}\cdots\pi_{i_k}) = \sigma$
2. $i_{x+1} = i_x + 1$ for each $x \in X \setminus \{0, k\}$
3. $i_1 = 1$ if $0 \in X$ and $i_k = n$ if $k \in X$

For an example, observe that $\pi = 12534$ contains $(1234, \emptyset)$ but avoids $(1234, \{2\})$.

We will restrict our attention to pairs $(\sigma, X) \in \mathcal{S}_k \times [k-1]$, so this third condition is moot. We write $(\sigma, X)$ as a permutation with a dash between $\pi_j$ and $\pi_{j+1}$ if $j \notin X$. For example, $(12435, \{1, 3\})$ is written 12–43–5. Hence we rephrase the example above by saying 12534 contains 1–2–3–4 but avoids 1–23–4.. Thus we will often refer to "the dashed pattern $\sigma$" without explcitly referring to $X$.

To avoid confusion, we will consider all patterns in this chapter to be dashed patterns, writing classical patterns with all dashes in place. Hence $123 = (123, \{1, 2\})$

represents a consecutive pattern $\pi_i < \pi_{i+1} < \pi_{i+2}$ while the classical pattern 123 is written 1–2–3.

In section 3.2 we will provide a brief summary of how enumeration schemes work, as well as construct a scheme for 23–1-avoiding permutations by hand. In section 3.3 we outline how the discovery of schemes can be done via a finite computer search. In section 3.4 we prove that any finite set $B$ containing only consecutive patterns has a finite enumeration scheme, as well as any singleton set $B = \{\sigma_1\sigma_2\cdots\sigma_t\text{–}\sigma_{t+1}\}$. In section 3.5 we analyze the rate of success as we use the computer to attempt automated discovery of schemes.

## 3.2 Summary of Enumeration Schemes

Recall that an enumeration scheme $E$ is the encoding of a divide-and-conquer recurrence. The set of $B$-avoiders $\mathcal{S}_n(B)$ is partitioned according to the prefix patterns $p = \text{red}(\pi_1\pi_2\cdots\pi_k)$ to form smaller sets $\mathcal{S}_n(B)[p]$. These subsets are further partitioned according to the actual prefix word $w = \pi_1\pi_2\cdots\pi_k$ to form $\mathcal{S}_n(B)[p; w]$. For each prefix $p$, (at least) one of the following conditions is satisfied:

(1) $\mathcal{S}_n(B)[p]$ is $\{p\}$ or $\emptyset$.

(2) For each prefix word $w$ such that $\text{red}(w) = p$, one of the following happens:

 (2a) $\mathcal{S}_n(B)[p; w]$ is empty.

 (2b) $\mathcal{S}_n(B)[p; w]$ is in bijection with some $\mathcal{S}_{\hat{n}}(B)[\hat{p}; \hat{w}]$ where $\hat{n} < n$.

(3) $\mathcal{S}_n(B)[p]$ must be partitioned further, so $s_n(B)[p] = \sum_{p'} s_n(B)[p';]$, where the sum runs over all children $p'$, which are permutations of length $k+1$ such that $\text{red}(p'_1 \dots p'_k) = p$.

Condition (1) only applies when $n = |p|$, and so serves as a base case for the recurrence. Condition (2a) is achieved according to the *gap vector criteria* described in section 1.3.4 of the introduction. Condition (2b) is achieved according to the deletion map $d_R(\pi)$, which deletes $\pi_r$ from $\pi$ for each $r \in R$, and then reduces to form a smaller permutation.

This is a bijection only when the set $R$ is *reversibly deletable* as described in section 1.3.4 of the introduction.

Also recall that formally an enumeration scheme $E$ is a set of triples $(p, G_p, R_p)$ such that:

1. $(\epsilon, \emptyset, \emptyset) \in E$.

2. If $(p, G_p, R_p) \in E$ and $R_p = \emptyset$, then $(p', G_{p'}, R_{p'}) \in E$ for every child $p'$ of $p$.

3. If $(p, G_p, R_p) \in E$ and $R_p \neq \emptyset$, then $(\hat{p}, G_{\hat{p}}, R_{\hat{p}}) \in E$ for $\hat{p} = d_{R_p}(p)$.

For an example, we present the scheme for 1–2–3-avoiding permutations as explained in equation (1.5) of the introduction:

$$\{(\epsilon, \emptyset, \emptyset), (1, \emptyset, \emptyset), (12, \{\langle 0, 0, 1\rangle\}, \{2\}), (21, \emptyset, \{1\})\}\,. \tag{3.1}$$

As a proof-of-concept for extending schemes to dashed patterns, let us construct by hand an enumeration scheme for 23–1-avoiding permutations. As for 1–2–3-avoiding permutations, we will need to consider prefixes of length 2. We will first consider the prefix pattern 12. Observe that $\mathcal{S}_n(23\text{–}1)[12; ab]$ is empty if $a > 1$, since if $\pi \in \mathcal{S}_n(23\text{–}1)[12; ab]$ then $\pi_i = 1$ for some $i \geq 3$. Thus $\mathrm{red}(ab\pi_i) = 231$ and so $\pi$ contains 23–1. Hence $\mathcal{S}_n(B)[12; ab]$ is non-empty only if $1 = a < b \leq n$, so we have the gap vector $\langle 1, 0, 0\rangle$. Next observe that the deletion map

$$d_1 : \mathcal{S}_n(23\text{–}1)[12; 1b] \to \mathcal{S}_{n-1}(23\text{–}1)[1; b-1]$$

is a bijection: deleting a letter at the beginning of a permutation cannot create a new copy of 23–1, and inserting a 1 at the start of a permutation cannot create a new 23–1.

We now consider permutations starting with prefix pattern 21. For $\pi \in \mathcal{S}_n(23\text{–}1)[21; ab]$, observe that $\pi_1$ cannot take part in a 23–1 pattern at all since this would require $\mathrm{red}(\pi_1\pi_2) = \mathrm{red}(23) = 12$ while it is known that $\pi_1 > \pi_2$. Hence the map $d_1$ restricts to a bijection $\mathcal{S}_n(23\text{–}1)[21; ab] \to \mathcal{S}_{n-1}(23\text{–}1)[1; b]$. There are no gap vectors for the prefix $p = 21$. Hence we have the following scheme for 23–1-avoiding permutations:

$$E = \{(\epsilon, \emptyset, \emptyset), (1, \emptyset, \emptyset), (12, \{\langle 1, 0, 0\rangle\}, \{1\}), (21, \emptyset, \{1\})\} \tag{3.2}$$

This scheme translates into the following system of recurrences:

$$
\begin{aligned}
s_n(23\text{–}1) &= s_n(23\text{–}1)[\epsilon] \\
&= s_n(23\text{–}1)[1] \\
&= \sum_{a=1}^{n} s_n(23\text{–}1)[1;a] \\
s_n(23\text{–}1)[1;a] &= \sum_{b=1}^{a-1} s_n(23\text{–}1)[21;ab] + \sum_{b=a+1}^{n} s_n(23\text{–}1)[12;ab] \\
s_n(23\text{–}1)[21;ab] &= s_{n-1}(23\text{–}1)[1;b] \\
s_n(23\text{–}1)[12;ab] &= \begin{cases} s_{n-1}(23\text{–}1)[1;b-1], & a=1 \\ 0, & a>1 \end{cases}
\end{aligned}
$$

This system simplifies to:

$$
s_n(23\text{–}1)[1;a] = \begin{cases} \sum_{b=1}^{n-1} s_{n-1}(23\text{–}1)[1;b], & a=1 \\ \sum_{b=1}^{a-1} s_{n-1}(23\text{–}1)[1;b], & 1<a\leq n \end{cases}
$$

which can be used to compute arbitrarily many terms of the sequence $s_n(23\text{–}1)$ in polynomial time. Alternately, one can use the methods from Chapter 5 to get the following functional equation for the weight-enumerator $F(z,x) = \sum_{n\geq 1}\sum_{n\in\mathcal{S}_n(23\text{-}1)} z^n x^{\pi_1}$:

$$
F(z,x) = zx + zxF(z,1) + \frac{zx^2}{x-1}F(zx,1) - \frac{zx}{x-1}F(z,x)
$$

## 3.3 Automated Discovery

We now turn to the process of automating the discovery of enumeration schemes for dashed patterns, since this automatation is the most outstanding feature of this method. The overall algorithm proceeds as follows:

**Algorithm 17.**

1. *Initialize* $E := \{(\epsilon, \emptyset, \emptyset)\}$

2. *Let* $P$ *be the set of all children of all prefixes in* $E$ *such that* $R_p = \emptyset$*. If there are no such prefixes, return* $E$*. Otherwise proceed to step 3.*

*3. For each $p \in P$, find a basis of gap vectors $G_p$.*

*4. For each $p \in P$, find a non-empty reversibly deletable set of indices $R_p$ given the gap vector criteria in $G_p$. If no such $R_p$ exists, let $R_p = \emptyset$.*

*5. Let $E = E \cup \{(p, G_p, R_p) : p \in P\}$.*

*6. Return to step 2.*

Steps 1, 2, 5, and 6 are routine computations for a computer algebra system. In the following subsections we present algorithms to automate steps 3 and 4.

### 3.3.1 Gap Vectors

We first look at automating step (3) of Algorithm 17. As mentioned previously, the set of gap vectors forms an order ideal in $\mathbb{N}^{k+1}$ and therefore it suffices to find a finite basis of minimal gap vectors. We attack this problem by dividing it into two sub-problems: (1) How can you test whether a given $\vec{v}$ is a gap vector and (2) what is the largest norm $M = |\vec{v}| = v_1 + v_2 + \cdots + v_{k+1}$ which needs to be checked? We answer the first question in Theorem 18 below, but leave the second question to be answered by a search parameter when one uses the computer to construct schemes.

This approach mimics that of [102], rather than [91], since its implementation allows the user more control over the algorithm via a parameter that lets us set the maximum allowed gap norm artificially low. This speeds computation time since it reduces the candidate pool for putative gap vectors, but this is at the cost of missing gap vectors which could make the enumeration scheme finite. For example, it is shown in [91] that there is no finite enumeration scheme for the forbidden set $B = \{1423, 1432\}$ using only gap vectors of norm 1.

Vatter actually presents a stronger notion of gap vector in [91], which allows him to provide an *a priori* bound on the norm of basis gap vectors. In particular, to construct a scheme for $B$-avoiding permutations one does not need to check for basis gap vectors of norm greater than $\max\{|\sigma| : \sigma \in B\} - 1$. In a search for a scheme for 1–2–3–4–5-avoiding permutations, however, one must search for a depth 7 scheme. A search for a depth 7 scheme with maximum gap vector norm 4 is impractical with the current implementation. A search for a depth 7 scheme with maximum gap vector norm 1, however, completes in under thirty minutes, returning a finite scheme.

We first present a test for whether a specific vector $\vec{v}$ is a gap vector by checking finitely many cases. It is important to note that Theorem 18 does not present a necessary condition, only a sufficient one. Hence using this test may "miss" some gap vectors during a search.

Given a set of forbidden patterns $B$, prefix $p \in \mathcal{S}_k$, and vector $\vec{v} = \langle v_1, v_2, \ldots, v_{k+1} \rangle \in \mathbb{N}^{k+1}$, define the set of permutations with prefix $p$ and spacing vector $\vec{v}$:

$$A(p, \vec{v}) = \{\pi \in \mathcal{S}_{|p|+|\vec{v}|} : \pi_1 \cdots \pi_k \sim p, \vec{g}(\pi_1 \cdots \pi_k) = \vec{v}\}.$$

Define the *head* of a dashed pattern $(\sigma, X)$ to be the subpattern $(\mathrm{red}(\sigma_1 \cdots \sigma_{\ell+1}), X)$ where $\ell = \max X$. For example, the head of $(241652, \{2\})$ = 2–41–6–5–3 is $(\mathrm{red}(241), \{2\})$ = 2–31. The part of $\sigma$ following the head is a classical pattern, with dashes between every letter.

**Theorem 18.** *Consider prefix $p \in \mathcal{S}_k$ and spacing vector $\vec{v} \in \mathbb{N}^{k+1}$. If every permutation $\pi \in A(p, \vec{v})$ contains a copy of some $\sigma \in B$ such that $\pi_1 \cdots \pi_k$ contains the head of the copy, then $\vec{v}$ is a gap vector, i.e., $A(p, \vec{u})$ has only $B$-containing permutations for any $\vec{u} \geq \vec{v}$.*

*Proof.* Any permutation in $\pi \in A(p, \vec{u})$ for $\vec{u} \geq \vec{v}$ may be constructed by inserting letters into some permutation $\pi' \in A(p, \vec{v})$. We will construct $\pi'$ from $\pi$, and this construction will have an obvious inverse.

Let $\pi \in A(p, \vec{u})$ where $|p| = k$. Let $c_i$ be the $i^{th}$ smallest letter in $\pi_1 \cdots \pi_k$ and let $c_0 = 0$ and $c_{k+1} = n + 1$. Define $C_i := \{\pi_j : j > k, c_{i-1} < \pi_j < c_i\}$ for $i \in [k + 1]$, and observe that $u_i = \#C_i$. For each $i$, choose $u_i - v_i$ letters of $C_i$, delete these letters from $\pi$, and reduce. Note that the deleted letters all lie outside of the prefix $\pi_1 \cdots \pi_k$, so this process forms $\pi' \in A(p, \vec{v})$. Reversing this process by re-inserting the letters provides the necessary construction of $\pi$ from $\pi'$. By our hypothesis, $\pi'$ contains $\sigma \in B$ such that the head of $\sigma$ lies in the prefix ${\pi'}_1 \cdots {\pi'}_k$. Inserting letters after the prefix will not destroy this copy of $\sigma$ since the portion of $\sigma$ lying outside the head has no adjacency restrictions. Hence $\pi \in A(p, \vec{u})$ also contains $\sigma$, and our result is proven. □

Note that $A(p, \vec{v})$ contains $|\vec{v}|!$ permutations, and each of these must be checked for $B$-containment. Hence keeping $|\vec{v}|$ small is a significant computational advantage.

Note that the criterion that every permutation in $A(p, \vec{v})$ contains a copy of $\sigma \in B$ *such that* $\pi_1 \cdots \pi_k$ *contains the head of the copy* is required. For example, consider $B = \{124\text{–}3, 123\text{–}5\text{–}4\}$. Here $A(123, \langle 0, 0, 0, 2\rangle) = \{12345, 12354\}$, both of which contain a forbidden pattern although the copy of 124–3 contained in 12354 does not have its head entirely in $p$. Now observe that $234165 \in A(123, \langle 1, 0, 0, 2\rangle)$ avoids $B$ even though $\langle 1, 0, 0, 2\rangle \geq \langle 0, 0, 0, 2\rangle$: the inserted 1 severs the occurence of 124–3 without creating any other forbidden pattern. It is not clear whether Theorem 18 is exhaustive, that is, whether there exist gap vectors $\vec{v}$ which do not satisfy the given criterion. We have observed no such vectors in practice.

In the computer implementation of this test, one must construct $A(p, \vec{v})$ explicitly but this is a simple matter. The definition of $\mathrm{red}(w)$ extends naturally to words with letters which are non-integer rational numbers. The smallest letter in the prefix $w$ can be assumed to be 1, and the $v_1$ letters in the tail which are smaller than this 1 can be written:

$$\frac{1}{v_1 + 1}, \frac{2}{v_1 + 1}, \ldots, \frac{v_1}{v_1 + 1}.$$

More generally, the $i^{th}$ smallest letter in $w$ can be assumed to be $i$, and the $v_i$ letters between the $i - 1$ and $i$ can be written:

$$(i-1) + \frac{1}{v_i + 1}, (i-1) + \frac{2}{v_i + 1}, \ldots, (i-1) + \frac{v_i}{v_i + 1}.$$

Finally, the $v_{k+1}$ letters which exceed all letters in the prefix can be written

$$k + \frac{1}{v_{k+1} + 1}, k + \frac{2}{v_{k+1} + 1}, \ldots, k + \frac{v_{k+1}}{v_{k+1} + 1}.$$

Thus it follows that every permutation in $A(p, \vec{v})$ can be written uniquely as the reduction of $p$ followed by some permutation of these $v_1 + v_2 + \cdots + v_{k+1}$ fractional elements. For example, for $\vec{v} = \langle 0, 0, 2\rangle$ and $p = 12$,

$$A(12, \langle 0, 0, 2\rangle) = \{\mathrm{red}(12(2 + \tfrac{1}{3})(2 + \tfrac{2}{3})), \mathrm{red}(12(2 + \tfrac{2}{3})(2 + \tfrac{1}{3}))\} = \{1234, 1243\}$$

As an example, consider the $B = \{23\text{–}1\}$ avoiding permutations, with prefix $p = 12$ and suppose we search over all vectors with norm at most 2. Table 3.3.1 gives the relevant information for each of the ten candidates.

| $\vec{v}$ | $A(12, \vec{v})$ | Gap vector? |
|---|---|---|
| $\langle 0,0,0\rangle$ | $\{12\}$ | No |
| $\langle 1,0,0\rangle$ | $\{231\}$ | Yes |
| $\langle 0,1,0\rangle$ | $\{132\}$ | No |
| $\langle 0,0,1\rangle$ | $\{123\}$ | No |
| $\langle 1,1,0\rangle$ | $\{2413, 2431\}$ | Yes |
| $\langle 1,0,1\rangle$ | $\{2314, 2341\}$ | Yes |
| $\langle 0,1,1\rangle$ | $\{1324, 1342\}$ | No |
| $\langle 2,0,0\rangle$ | $\{3412, 3421\}$ | Yes |
| $\langle 0,2,0\rangle$ | $\{1423, 1432\}$ | No |
| $\langle 0,0,2\rangle$ | $\{1234, 1243\}$ | No |

Looking at the set of gap vectors determined $\{\langle 1,0,0\rangle, \langle 1,1,0\rangle, \langle 1,0,1\rangle, \langle 2,0,0\rangle\}$, we see the order ideal generated by these vectors has minimal basis $\{\langle 1,0,0\rangle\}$.[1] Hence in the enumeration scheme we see $G_{12} = \{\langle 1,0,0\rangle\}$.

### 3.3.2 Reversibly Deletable Sets

We now turn our attention to automating step (4) of Algorithm 17: discovering reversibly deletable sets of indices for a given prefix $p$. The approach parallels [102].

Recall that for set of indices $R$, the map $d_R$ deletes $\pi_r$ for each $r \in R$. This forms a bijection $d_R : \mathcal{S}_n(\emptyset)[p; w] \to \mathcal{S}_{n-|R|}(\emptyset)[d_R(p); d_R(w)]$, and when this map restricts to a bijection $\mathcal{S}_n(B)[p; w] \leftrightarrow \mathcal{S}_{n-|R|}(B)[d_R(p); d_R(w)]$ we say that $R$ is reversibly deletable for prefix $p$. In the classical case, the deletion of a letter or letters could not create an occurrence of a forbidden pattern. For dashed patterns, however, deleting a letter may create the adjacency required to form an occurrence of a dashed pattern. For example, 3142 avoids 23–1 but $d_2(3142) = 231$ does not since the 3 and 4 have moved adjacent to each other. This does not preclude the existence of bijective maps $d_R$, it merely requires additional checks for the automated discovery. In the end, a finite search for a reversibly deletable set suffices as in the classical case: it is only the manner in which

[1]This example is perhaps misleading regarding the actual implementation. In the package gVatter, once the computer discovers gap vector $\langle 1,0,0\rangle$ it will not bother testing any other $\vec{v} \geq \langle 1,0,0\rangle$.

we check each candidate which differs. The need to check both directions of the map first appears in [75] when extending schemes for barred pattern avoidance. The added twist needed for dashed pattern avoidance is the introduction of the "null" symbol $\bullet$.

Note $R$ is reversibly deletable when every $\pi \in \mathcal{S}_n(\emptyset)[p;w]$ avoids $B$ if and only $d_R(\pi)$ also avoids $B$. Inversely, we could check whether every $\pi$ which contains some $\sigma \in B$ has image $d_R(\pi)$ which also contains some $\sigma' \in B$. This approach was introduced by Zeilberger in [102] and used by Pudwell in [74] and [75] when extending enumeration schemes to other contexts.

Let us illustrate the approach via an example before moving to the general case. Consider $B = \{124\text{–}3\}$ and prefix $p = 132$. We ask "which letters of the prefix can participate in a $\sigma = 124\text{–}3$ pattern?". Suppose $\pi$ is a permutation with prefix pattern 132, and that at least one letter of the prefix is part of a copy of $\sigma$. If $\pi$ has minimal length, then $\pi$ must have the form $\text{red}(132abc)$ where $a, b, c \in \mathbb{Q}$ such that $2abc \sim \sigma$: $\sigma$ starts with two rises and the descent 32 in the prefix prevents a $\sigma$ from starting earlier. There are four such permutations: $13\underline{2465}$, $14\underline{2365}$, $15\underline{2364}$, $16\underline{2354}$ (the occurence of $\sigma$ is underlined in each). It will be necessary to keep track of where when dash(es) in the contained copy of $\sigma$ appear outside of the prefix, which we denote with the "null" symbol $\bullet$. This special character denotes the possibility for intervening letters but cannot participate in patterns itself. Thus we write these four permutations as $13\underline{246}\bullet\underline{5}$,$14\underline{236}\bullet\underline{5}$, $15\underline{236}\bullet\underline{4}$, $16\underline{235}\bullet\underline{4}$. Denote this set of *containment scenarios* for $p = 132$ by $A_{132}$. We now apply $d_R$ for each $R \subseteq [3]$ and check whether the images under $d_R$ each contain $\sigma$. This is done in Table 3.1. If $d_R(\pi)$ contains $\sigma$ for each $\pi \in A_{132}$, then $R$ passes the first test[2] for reverse deletability: the insertion $d_R^{-1}$ does not create any forbidden patterns in $B$ when applied to a permutation which already avoids $B$. Looking down the columns of Table 3.1, we see that $\{1\}$, $\{2\}$, and $\{1, 2\}$ pass this test since every permutation in those columns contains $\sigma$.

Since a deletion map creates new adjacencies and potentially a copy of $\sigma$, there is a second test that $R$ must pass to be reversibly deletable. Consider $R = \{2\}$, which passed

[2] In the classical case, this was the *only* test.

the previous test. Applying $d_{\{2\}}$ to a permutation with prefix pattern 132 will create a permutation with prefix patttern 12, so we must consider the containment scenarios for the prefix $p = 12$: $A_{12} = \{124\bullet3, 1235\bullet4\}$. We then consider all ways each of these containment scenarios could have arisen by applying $d_{\{2\}}$ to a permutation with prefix patern 132; i.e., every permutation of the form red($1a24\bullet3$) for $a \in \{2+\frac{1}{2}, 3+\frac{1}{2}, 4+\frac{1}{2}\}$ or red($1b235\bullet4$) for $b \in \{2+\frac{1}{2}, 3+\frac{1}{2}, 4+\frac{1}{2}, 5+\frac{1}{2}\}$. In particular, this list includes $1325\bullet4$, which avoids $\sigma$ while $d_{\{2\}}(1325\bullet4) = 124\bullet3$ contains $\sigma$. Since one can use $d_{\{2\}}$ to create a $\sigma$-containing permutation from a $\sigma$-avoiding permutation, $R = \{2\}$ cannot be reversibly deletable. On the other hand, for $R = \{1\}$ one can check that the containment scenarios for $d_{\{1\}}(132) = 21$ are $A_{21} = \{2135\bullet4, 3125\bullet4, 4125\bullet3, 5124\bullet3\}$ and that the permutations starting with 132 which map to some $\pi \in A_{21}$ are precisely $\{13246\bullet5, 14236\bullet5, 15236\bullet4, 16235\bullet4\}$. Since each of these pre-image permutations contains $\sigma$, $R = \{1\}$ passes the second test for reversible deletability. Hence $\{1\}$ is reversibly deletable. Similarly, for $R = \{1, 2\}$ we get only the containment scenario $A_1 = \{124\bullet3\}$ and the same set of pre-images with prefix 132:

$$d_{\{1,2\}}\left(\{13246\bullet5, 14236\bullet5, 15236\bullet4, 16235\bullet4\}\right) = A_1.$$

Again, each of the permutations on the lefthand side contains $\sigma$, so $R = \{1, 2\}$ is reversibly deletable. Hence we have two non-empty reversibly deletable sets for prefix 132. While either set will lead to a valid enumeration scheme, we follow a convention to choose the largest one and break ties lexicographically by the smallest elements.

To demonstrate a subtlety of containment scenarios, consider the forbidden set $B = \{3–21, 32–1\}$ and prefix $p = 21$. Here we see that we have the basis gap vector $\langle 1, 0, 0\rangle$, and so any permutation starting with prefix word $ab$ for $a > b > 1$ necessarily contains a forbidden pattern. Hence to prove $R$ is reversibly deletable, we only need to

| $\pi \in A_{132}$ | $d_{\{1\}}(\pi)$ | $d_{\{2\}}(\pi)$ | $d_{\{3\}}(\pi)$ | $d_{\{1,2\}}(\pi)$ | $d_{\{1,3\}}(\pi)$ | $d_{\{2,3\}}(\pi)$ | $d_{\{1,2,3\}}(\pi)$ |
|---|---|---|---|---|---|---|---|
| 13246•5 | 2135•4 | 1235•4 | 1235•4 | 124•3 | 124•3 | 124•3 | 13•2 |
| 14236•5 | 3125•4 | 1235•4 | 1325•4 | 124•3 | 214•3 | 124•3 | 13•2 |
| 15236•4 | 4125•3 | 1235•4 | 1425•3 | 124•3 | 314•2 | 134•2 | 13•2 |
| 16235•4 | 5124•3 | 1235•4 | 1524•3 | 124•3 | 413•2 | 143•2 | 13•2 |

Table 3.1: $d_R(\pi)$ for each $R$, $\pi$.

show $d_R$ is bijective starting from sets of the form $\mathcal{S}_n(B)[21; a1]$. Therefore even though 42•31 contains a forbidden pattern and begins with 21, we know that $\mathcal{S}_n(B)[21; 42] = \emptyset$ and so we do not need to check whether $d_R(42 \bullet 31)$ contains a forbidden pattern. In fact the only containment scenario worth checking for $p = 21$ is $41 \bullet 32$. Hence $R = \{2\}$ passes the first test for reversible deletability. We then move on to consider the containment scenarios for prefix pattern $d_2(21) = 1$. These are $A_1 = \{3\bullet 21, 32\bullet 1\}$. The pre-images under $d_2$ starting with 21 include 413-2, however, which does not contain either forbidden pattern. Hence $R = \{2\}$ fails the second for reversible deletability. If we had not kept track of dashes with the null character •, however, the preimage 4132 would have contained a forbidden pattern and $\{2\}$ would have appeared to be reversibly deletable.

We now outline in general the scenarios method to test whether a set $R$ is reversibly deletable for prefix $p$ with respect to forbidden pattern $B$. We begin with a formal definition for a containment scenario.

**Definition 19.** *Let $(\sigma, X) \in \mathcal{S}_\ell \times [\ell - 1]$ be a dashed pattern and $p \in \mathcal{S}_k$ be a prefix pattern with known set of gap vectors $G$. Let $w \in ([n] \cup \{\bullet\})^{n+a}$ be a word with $a$ copies of • and no other letters repeated. Then $w$ is a* containment scenario *for $p$ if the following criteria are satisfied:*

1. *$w_1 \cdots w_k \sim p$. Note this implies • does not appear in the first $k$ letters.*
2. *There is some subsequence $1 \leq i_1 < \cdots < i_\ell \leq n + a$ such that $w(i_1) \cdots w(i_\ell) \sim \sigma$ and $i_{x+1} = \bullet$ for each $x \notin X$ such that $i_x \geq k$.*
3. *No subsequence of $w$ is a containment scenario (i.e., $w$ has minimal length).*
4. *$w_1 \cdots w_k$ fails all gap vector criteria in $G$.*

The set of containment scenarios for a forbidden set $B$ is simply the union of the sets of containment scenarios for each $\sigma \in B$. We will denote the set of containment scenarios for forbidden set $B$, prefix $p$, and set of gap vectors $G$ by $A_p$.

One can compute $A_p$ via brute force over all $2^{|p|} - 1$ nonempty subsequences of $p$. A set of indices $1 \leq i_1 < \cdots i_t \leq |p|$ is a *partial match* for $(\sigma, X) \in B$ if $i_x + 1 = i_{x+1}$

for each $x \in X$ and $p(i_1)p(i_2)\cdots p(i_t) \sim \sigma_1\sigma_2\cdots\sigma_t$. Note that a set of indices may be a partial match for more than one pattern in $B$. For each partial match of $(\sigma, X)$, insert the $|\sigma| - t$ letters and necessary number of $\bullet$ on the right end of $p$ in such a way to complete the occurrence of $\sigma$ using the letters in the partial match. Repeating this process for each $(\sigma, X) \in B$ gives us the complete set of containment scenarios. We may then throw out any containment scenarios whose first $|p|$ letters satisfy a gap vector criterion for some basis gap vector.

We now present the algorithm to check whether a given $R \subseteq [k]$ is reversibly deletable for prefix $p \in \mathcal{S}_k$ with respect to forbidden set $B$.

**Algorithm 20.**

1. *Compute the set of containment scenarios* $A_p$.
2. *For each* $\pi \in A_p$*, check if* $d_R(\pi)$ *contains a forbidden pattern in* $B$*. If any* $\pi$ *avoids* $B$*, then* $R$ *is not reversibly deletable. Otherwise, proceed to step 3.*
3. *Compute the set of containment scenarios* $A_{d_R(p)}$.
4. *Find the set of all permutations* $\pi$ *with prefix* $p$ *such that* $d_R(\pi) \in A_{d_R(p)}$*. If any of these* $\pi$ *avoids* $B$*, then* $R$ *is not reversibly deletable. If each of these contains some forbidden pattern, then* $R$ *is reversibly deletable.*

Therefore we can compute non-empty reversibly deletable sets automatically by a finite computer search. This concludes our discussion on automated discovery of enumeration schemes. These procedures have been implemented in the Maple package gVatter, available on the author's homepage.

## 3.4 Special Cases of Guaranteed Success

Knowing *a priori* whether a set of patterns $B$ has a finite enumeration scheme remains an open question. As a partial result, we show here that any forbidden set consisting only of consecutive patterns must have a finite enumeration scheme. As a corollary we show that any forbidden singleton set of the form $B = \{\sigma_1\sigma_2\cdots\sigma_t\text{–}\sigma_{t+1}\}$ also has a finite enumeration scheme.

**Theorem 21.** *If $B$ contains only consecutive patterns of length $t$, then $B$ admits a finite enumeration scheme of depth $t$.*

*Proof.* For $k \leq t$ let $B_k$ be the set of $k$-prefixes of patterns in $B$, that is, $B_k = \{\text{red}(\sigma_1 \cdots \sigma_k) : \sigma \in B\}$. Note that $B_t = B$. Suppose $p \in \mathcal{S}_k$ for $k < t$ and suppose $\pi$ is a permutation starting with pattern $p$. Notice that $\pi_1 \cdots \pi_k \pi_{k+1} \cdots \pi_t$ forms a pattern in $B$ only if $\text{red}(\pi_1 \cdots \pi_k) = p \in B_k$. From this observation, we can construct the requisite triples $(p, G_p, R_p)$ for our scheme. We follow Algorithm 17, and for each prefix $p$ we determine $R_p$ and $G_p$ as follows.

First suppose $p \in \mathcal{S}_k$ such that $k < t$ and $p \in B_k$. Then a permutation starting with $p$ has the potential to contain a forbidden pattern. Hence we let $G_p = \emptyset$ and $R_p = \emptyset$.

Next suppose $p \in \mathcal{S}_k$ such that $k \leq t$ and $p \notin B_k$. Then a permutation $\pi$ starting with $p$ could not possibly start with a forbidden pattern $\text{red}(\pi_1 \cdots \pi_t) \in B$. From this it follows that the first index is reversibly deletable. There may be a stronger statement, however, since the first *few* letters may all be reversibly deletable. Choose $a$ minimally such that $\text{red}(p_a p_{a+1} \cdots p_k) \notin B_{k-a+1}$ but $\text{red}(p_{a+1} \cdots p_k) \in B_{k-a}$. Note that $a \leq k-1$. We will show that $R = \{1, 2 \ldots, a\}$ is reversibly deletable. First note $d_R$ creates no new adjacencies, and so it cannot create a new copy of $\tau$. For the inverse, suppose $\pi \in \mathcal{S}_{n-a}(B)[d_R(p); d_R(w)]$ such that $\pi' = d_R^{-1}(\pi)$ has a copy of some forbidden $\sigma \in B$ starting at index $i$. In other words $\pi'_i \pi'_{i+1} \cdots \pi'_{i+t-1} \sim \sigma$. We know $\pi$ avoids $\sigma$, so $i < a+1 \leq k$. Hence $\pi'_i \pi'_{i+1} \cdots \pi'_k \in B_{k-i+1}$. Note that $\pi'_i \pi'_{i+1} \cdots \pi'_k \sim p_i p_{i+1} \cdots p_k$, however so $p_i p_{i+1} \cdots p_k \in B_{k-i+1}$. Now we know that $\text{red}(p_a p_{a+1} \cdots p_k) \notin B_{k-a+1}$ by construction, and so $i \neq a$. Hence $i < a$, contradicting minimality for our choice of $a$. Hence $d_R^{-1}$ cannot create any $\sigma \in B$, and so we see that $R$ is reversibly deletable. Thus we let $G_p = \emptyset$ and $R_p = \{1, 2, \ldots, a\}$. For example, consider $B = \{13254\}$ and the prefix $p = 1423$. Observe $1423 \notin B_4 = \{1324\}$, $\text{red}(423) = 312 \notin B_3 = \{132\}$, but $\text{red}(23) = 12 \in B_2 = \{12\}$ so $a = 2$.

Last, consider $p \in \mathcal{S}_t$ such that $p \in B$. Any permutation starting with $p$ most certainly contains a forbidden pattern. Hence we may set $G_p = \{\langle 0, 0, \ldots, 0 \rangle\}$, letting any prefix $w$ to satisfy the gap vector criteria and thus imply $\mathcal{S}_n(B)[p; w] = \emptyset$. We arbitrarily set $R_p = \{1\}$ since any set $R \subseteq [t]$ would be vacuously reversibly deletable in this case.

Since every prefix of length $t$ has non-empty $R_p$, we have constructed an enumeration scheme for permutations avoiding $B$.

□

We can extend Theorem 21 to consider dashed patterns of the form $\sigma_1\sigma_2\cdots\sigma_{t-1}\text{–}\sigma_t$.

**Corollary 22.** *The dashed pattern $\sigma = \sigma_1\sigma_2\cdots\sigma_t\text{–}\sigma_{t+1}$ has an enumeration scheme of depth $t$.*

*Proof.* Let $E_\tau$ be the enumeration scheme of depth $t$ for the consecutive portion $\tau = \text{red}(\sigma_1\cdots\sigma_t)$, and we will construct a scheme $E_\sigma$ for $\sigma$ based on $E_\tau$.

First let $(p, G_p, R_p) \in E_\tau$ for $p \neq \tau$. The construction of $E_\tau$ from Theorem 21 implies $G_p = \emptyset$. By the same reasoning as shown in the proof of Theorem 21, $R_p = \{1, 2, \ldots, a\}$ is reversibly deletable for $p$ with respect to $\sigma$, where $a$ is chosen minimally such that $p_ap_{a+1}\cdots p_k \not\sim \tau_1\cdots\tau_{k-a+1}$ and $p_{a+1}\cdots p_k \sim \tau_1\cdots\tau_{k-a}$. This follows from the same proof since containment of $\sigma$ implies containment of $\tau$, and so avoidance of $\tau$ implies avoidance of $\sigma$. Thus we let $(p, \emptyset, R_p) \in E_\sigma$.

Now consider $p = \tau$. If a permutation $\pi$ starts with $p$ then the letters must be spaced so as to leave no room for a latter $\sigma_t$. Hence we have gap vector $\vec{v} = \langle 0, \ldots, 0, 1, 0, \ldots, 0\rangle$, where $\vec{v}_{\sigma_{t+1}} = 1$ and $\vec{v}_i = 0$ for other $i$. It can be seen $\vec{v}$ forms a basis for the gap vectors, so we let $G_p = \{\vec{v}\}$. Next choose $a$ as above, that is, minimally such that $p_ap_{a+1}\cdots p_k \not\sim \tau_1\ldots\tau_{k-a+1}$ but $p_{a+1}\cdots p_k \sim \tau_1\ldots\tau_{k-a}$). Then $R_p = \{1, \ldots, a\}$ is reversibly deletable for the same reasons as discussed above. Add $(p, G_p, R_p)$ to the set $E_\sigma$.

Looking at the resulting $E_\sigma$, we see it is a depth $t$ enumeration scheme for $\sigma$. □

The proof above can be generalized to obtain a finite scheme for any set of patterns of the form $B = \{\sigma_1\sigma_2\cdots\sigma_t\text{–}\sigma_{t+1}\text{–}\sigma_{t+2}, \sigma_1\sigma_2\cdots\sigma_t\text{–}\sigma_{t+2}\text{–}\sigma_{t+1}\}$ based on the enumeration scheme for their common head $\tau = \text{red}(\sigma_1\sigma_2\cdots\sigma_t)$. In this case the prefix $p = \tau$ has a gap vector basis $G_p = \{\vec{v}\}$, where $v_{\sigma_{t+1}} = 1$, $v_{\sigma_{t+2}-1} = 1$ and $v_i = 0$ for other $i$ for

$\sigma_{t+1} < \sigma_{t+2} - 1$. If $\sigma_{t+1} = \sigma_{t+2} - 1$, then $v_{\sigma_{t+1}} = v_{\sigma_{t+2}-1} = 2$ instead. The triples for other prefixes $p \neq \tau$ appearing in $E_\tau$ transfer unchanged to $E_\sigma$.

One may similarly construct a scheme for the set of $k!$ patterns formed by appending to consecutive pattern $\tau$ a dashed tail of $k$ letters $\sigma_{t+1}, \sigma_{t+1}+1, \ldots, \sigma_{t+k}$ in all possible orderings, e.g. $B = \{12\text{–}3\text{–}4\text{–}5, 12\text{–}3\text{–}5\text{–}4, 12\text{–}4\text{–}3\text{–}5, 12\text{–}4\text{–}5\text{–}3, 12\text{–}5\text{–}3\text{–}4, 12\text{–}5\text{–}4\text{–}3\}$. If we suppose $\sigma_{t+j} < \sigma_{t+j+1}$, then let $\vec{u}^{(j)}$ be the $0-1$ vector with a 1 in position $\sigma_{t+j} - (j-1)$. Then for prefix $p = \tau$ we have gap vector $\vec{v} = \sum_j \vec{u}^{(j)}$. The remaining triples $(p, G_p, R_p) \in E_\tau$ transfer unchanged to $E_\sigma$. These pattern sets are similar to Kitaev's notion of partially-ordered generalized patterns in [61], where some letters of the pattern are incomparable (or rather, do not need to be compared). Thus the example $B$ would be written as a single such pattern 12–3–3′–3″ where the letters acting as 3, 3′, 3″ are incomparable.

One might hope we can continue this trend of adding dashed portions to patterns with known schemes to get new schemes, but of course this does not work in general[3]. Still, there may be some interesting relationships between two dashed patterns with the same underlying permutation. For example, every pattern $(1234, X)$ for $X \subseteq \{1, 2, 3\}$ has a finite scheme, whose depths (based on the implementation in gVATTER) are summarized in Table 3.2. There do not appear to be any clear patterns dictating scheme depth for dashed pattern $(1234, X)$ given the depths of other patterns $(1234, X')$, based on subset relations between $X$ and $X'$.

| $\sigma$ | $X$ | Scheme Depth |
|---|---|---|
| 1234 | $\{1,2,3\}$ | 4 |
| 123–4 | $\{1,2\}$ | 3 |
| 12–34 | $\{1,3\}$ | 4 |
| 1–234 | $\{2,3\}$ | 4 |
| 12–3–4 | $\{1\}$ | 4 |
| 1–23–4 | $\{2\}$ | 3 |
| 1–2–34 | $\{3\}$ | 5 |
| 1–2–3–4 | $\emptyset$ | 4 |

Table 3.2: Scheme depth for dashed pattern $(1234, X)$

[3]If it did then we would get enumeration schemes for all classical patterns, which certainly is not the case.

## 3.5 Analysis of Success Rates

Aside from the results of the previous section, there is no known classification of which pattern-sets admit finite enumeration schemes. In this section we present empirical results obtained from the implementation of the above algorithms in the Maple package gVATTER. We will say that a set of forbidden patterns $B$ is $(d, M)$*-scheme countable*, or $(d, M)$-SC, if either $B$, $B^r$, or $B^{-1}$ (when well-defined) admits a finite enumeration scheme of depth at most $d$ with basis gap vectors with norm at most $M$. As discussed in the introduction, $B$ is $(d, M)$-SC if and only if its set of complement patterns $B^c$ is $(d, M)$-SC.

The following data were assembled by checking whether each dashed pattern $(\sigma, X)$ is $(5, 2)$-SC, where we chose 5 and 2 as a practical computational considerations. While there are $k! \cdot 2^{k-1}$ dashed patterns of length $k$, we took advantage of symmetry when able to reduce the number of patterns to check. To refine analysis, we separated the patterns of length $k$ by the locations of their dashes. These are represented in Table 3.3 by "block type," which is a vector describing the number of letters between each dash. For example, the block type of the pattern 12–35–467 is $(2, 2, 3)$.

| Block type | Number of trivial symmetry classes | Number of (5, 2)-SC classes | Percentage |
|---:|---:|---:|---:|
| (2) | 1 | 1 | 100% |
| (1,1) | 1 | 1 | 100% |
| (3) | 2 | 2 | 100% |
| (2,1) | 3 | 3 | 100% |
| (1,1,1) | 2 | 2 | 100% |
| (4) | 8 | 8 | 100% |
| (3,1) | 12 | 12 | 100% |
| (2,2) | 8 | 3 | 37.5% |
| (2,1,1) | 12 | 4 | 25% |
| (1,2,1) | 8 | 6 | 75% |
| (1,1,1,1) | 7 | 2 | 28.6% |

Table 3.3: Success rate by block type

It would appear that the success rate is not solely dependent on the number of dashes. For example, of the 20 pattern classes with a single dash, the five which are not $(5, 2)$-SC are all of block type $(2, 2)$. Of the classes with two or more dashes, the

| Set type | Number of trivial symmetry classes | Number of (5, 2)-SC classes | Percentage |
|---|---|---|---|
| {2} | 2 | 2 | 100.0% |
| {2, 2} | 3 | 3 | 100.0% |
| {2, 3} | 11 | 11 | 100.0% |
| {3} | 7 | 7 | 100.0% |
| {3, 3} | 70 | 68 | 97.1% |
| {3, 3, 3} | 358 | 354 | 98.9% |
| {3, 4} | 914 | 639 | 69.9% |
| {4} | 55 | 35 | 63.6% |
| {5} | 479 | 144 | 30.1% |

Table 3.4: Success rate for sets of patterns, $B$.

most successful block type is $(1, 2, 1)$ where the dashes do not follow one another.

In the classical case, schemes were most successful when avoiding multiple patterns simultaneously. Table 3.4 lists the success rates for finding sets $B$ which are $(5, 2)$-SC, for various $B \subseteq \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5$. In the leftmost column, the "set type" of a set $B$ refers to the multiset $\{|\sigma| : \sigma \in B\}$.

## 3.6 Wilf-classification of Dashed Patterns

We now present some preliminary Wilf-classification results based on the data generated by the schemes. Recall from section 1.1.3 of the introduction that two patterns $\sigma, \tau$ are said to be *Wilf-equivalent* if $s_n(\alpha) = s_n(\beta)$ for all $n$. Claesson enumerates permutations avoiding a length 3 pattern with one dash in [32], and Elizalde and Noy enumerate permutations avoiding length 3 patterns with no dashes (i.e. consecutive) in [43]. Thus we turn our attention to length 4 patterns. All patterns of length 4 with finite schemes of depth at most 5 are listed in Table 3.5. Solid black lines separate classes whose sequences are observed to diverge before the 31st term.

The first steps towards general results were taken in [42, 61], which we summarize below in Proposition 23:

**Proposition 23** (Elizalde [42], Kitaev [61])**.** *Suppose $\sigma, \tau$ are Wilf equivalent consecutive patterns of length $k$. Then the following are also Wilf equivalent:*

- $\sigma\text{–}(k+1) \equiv \tau\text{–}(k+1)$

| $\sigma$ | $\{\mathcal{S}_n(\sigma)\}_n$ | OEIS [53] | Comments |
|---|---|---|---|
| 123-4 | $1, 2, 6, 23, 108, 598, 3815, 27532, 221708, 197025, \ldots$ | A071076 | |
| 321-4 | | | $\equiv$ 123-4 by Proposition 23 |
| 132-4 | $1, 2, 6, 23, 107, 585, 3671, 25986, 204738, 1776327, \ldots$ | A071075 | |
| 231-4 | | | $\equiv$ 132-4 by Proposition 23 |
| 312-4 | | | $\equiv$ 132-4 by Proposition 23 |
| 213-4 | | | $\equiv$ 132-4 by Proposition 23 |
| 142-3 | | | Conjecture 24 |
| 241-3 | | | Conjecture 24 |
| 124-3 | $1, 2, 6, 23, 107, 584, 3660, 25910, 204564, 1782520, \ldots$ | New | |
| 421-3 | | | Conjecture 24 |
| 143-2 | $1, 2, 6, 23, 107, 582, 3622, 25369, 197523, 1692535, \ldots$ | New | |
| 214-3 | $1, 2, 6, 23, 107, 583, 3637, 25548, 199506, 1714383, \ldots$ | New | |
| 12-34 | $1, 2, 6, 23, 107, 585, 3669, 25932, 203768, 1761109, \ldots$ | A113226 | |
| 12-43 | | | $\equiv$ 12-34 by Proposition 23 |
| 21-43 | | | $\equiv$ 21-43 by Proposition 23 |
| 1-24-3 | $1, 2, 6, 23, 105, 549, 3207, 20577, 143239, 1071704, \ldots$ | A137538 | Wilf equivalent to $25\bar{1}34$? |
| 1-42-3 | | | Conjecture 24 |
| 1-23-4 | $1, 2, 6, 23, 105, 549, 3207, 20577, 143239, 1071704, \ldots$ | A113227 | |
| 1-32-4 | | | $\equiv$ 1-23-4 by Proposition 23 |
| 1-34-2 | | | Conjecture 24 |
| 1-43-2 | | | Conjecture 24 |
| 12-3-4 | $1, 2, 6, 23, 105, 550, 3228, 20878, 146994, 1116000, \ldots$ | New | |
| 12-4-3 | | | Conjecture 24 |
| 21-3-4 | | | Conjecture 24 |
| 21-4-3 | | | Conjecture 24 |

Table 3.5: Dashed patterns of length 4 admitting schemes of depth at most 5

- $\sigma$–$(k+2)(k+1) \equiv \tau$–$(k+2)(k+1)$
- $\sigma$–$(k+2)(k+1) \equiv \sigma$–$(k+1)(k+2)$
- 1–$(\sigma_1+1)(\sigma_2+1)\cdots(\sigma_k+1)$–$(k+1) \equiv$ 1–$(\tau_1+1)(\tau_2+1)\cdots(\tau_k+1)$–$(k+1)$

It is clear from symmetries of the square that the following consecutive patterns are Wilf equivalent:

- $12 \equiv 21$
- $123 \equiv 321$
- $132 \equiv 213 \equiv 231 \equiv 312$

Thus we see that Proposition 23 gets us many of the equivalences which appear in Table 3.5. There remain many conjectured pairs, which we summarize below in Conjecture 24. Note that in each case, the conjectured equivalence is confirmed computationally for permutations of length $n \leq 30$.

**Conjecture 24.** *The following pairs are Wilf equivalent:*

- 132–4 $\equiv$ 142–3
- 132–4 $\equiv$ 241–3
- 124–3 $\equiv$ 421–3
- 1–24–3 $\equiv$ 1–42–3
- 1–34–2 $\equiv$ 1–43–2
- 1–23–4 $\equiv$ 1–34–2
- 1–23–4 $\equiv$ 1–43–2
- 12–3–4 $\equiv$ 12–4–3
- 21–3–4 $\equiv$ 21–4–3
- 12–3–4 $\equiv$ 21–3–4

The first three conjectured pairs suggest a common generalization:

**Conjecture 25.** *Let* $\sigma, \tau \in \mathcal{S}_{k+1}$ *such that* $\mathrm{red}(\sigma_1 \cdots \sigma_k) = \mathrm{red}(\tau_1 \cdots \tau_k)$. *Then the dashed patterns* $\sigma_1\sigma_2\cdots\sigma_k\text{—}\sigma_{k+1} \equiv \tau_1\tau_2\cdots\tau_k\text{—}\tau_{k+1}$.

Corollary 22 constructs schemes for $\sigma_1\sigma_2\cdots\sigma_k\text{—}\sigma_{k+1}$ and $\tau_1\tau_2\cdots\tau_k\text{—}\tau_{k+1}$ which differ only in the gap vectors for the prefix $p = \mathrm{red}(\sigma_1\sigma_2\cdots\sigma_k) = \mathrm{red}(\tau_1\tau_2\cdots\tau_k)$. This symmetry suggests the existence of a nice bijection between $\mathcal{S}_n(\sigma_1\sigma_2\cdots\sigma_k\text{—}\sigma_{k+1})$ and $\mathcal{S}_n(\tau_1\tau_2\cdots\tau_k\text{—}\tau_{k+1})$.

## 3.7 Conclusions and Future Directions

This chapter developed automatable methods to compute $s_n(B)$ for many sets of dashed patterns $B$. This was accomplished by extending the enumeration schemes developed by Vatter and Zeilberger in [100, 91, 102]. The restrictions on adjacencies which dashed patterns present introduced complications when discovering gap vectors and reversibly deletable sets. Theorem 18 demonstrates that gap vectors can only be discovered when prefixes are long enough to contain a large portion of a dashed pattern. Section 3.3.2 explains how the discovery of reversibly deletable sets requires two tests rather than one as well as the introduction of a "null" character. With the Maple implementation in gVatter[4], someone interested in a particular set of patterns $B$ has a ready tool to compute $s_n(B)$ if $B$ admits a finite scheme.

Despite the added complications, Theorem 21 proves that any consecutive pattern admits a finite scheme. Hence enumeration schemes may be added to the list of methods to analyze problems in consecutive pattern avoidance. Corollary 22 extends this result to another large class of dashed patterns which are guaranteed to have a finite scheme. Classical patterns admitting a finite scheme have not been classified, and there have been few results about infinite classes of patterns which admit finite schemes.

In [91], Vatter presents two questions which apply equally well to the schemes contained in this chapter. First, is every sequence produced by a finite enumeration scheme

[4]Available for download from the author's homepage.

holonomic? An answer in the affirmative would imply that computing finitely many terms would be sufficient to prove Wilf-equivalence for two sequences generated by finite schemes. In this same vein, Vatter also asks whether it is decidable whether two finite enumeration schemes produce the same sequence. An answer in the affirmative would provide an alternative method to use enumeration schemes to prove Wilf-equivalence results among sets of patterns with finite enumeration schemes.

Enumeration schemes provide powerful tools for generating the sequences $s_n(B)$. Thus far they have been developed for dashed patterns and barred patterns. This project originated as an attempt to develop enumeration schemes for the bivincular patterns discussed in section 1.1.2 of Chapter 1, but it quickly became apparent that the maps $d_R$ wreak havoc on vertical adjacencies among letters following the prefix and would be unsuitable. A different recursive structure for $\mathcal{S}_n$ would need to be exploited to make enumeration schemes work for bivincular patterns with both horizontal and vertical adjacency requirements.

In [74] Pudwell extends enumeration schemes to pattern avoidance by words, as discussed in section 1.1.2 of the introduction. The work above should extend nicely to this context with little difficulty. Similarly, schemes could be developed to handle permutations (or even words) avoiding barred dashed patterns by combining the techniques of this chapter and [75].

The most fruitful applications of the schemes in this chapter could be from using them in conjunction with Chapters 4 and 5. Chapter 4 develops methods to use an enumeration scheme, such as the ones developed in this chapter, to weight-count $\mathcal{S}_n(B)$ according to inversion number and according to consecutive pattern functions. Chapter 5 develops methods to convert enumeration schemes into a system of functional equations involving the generating function $F(z) = \sum_{n\geq 0} s_n(B)z^n$. These chapters are written in for schemes for permutations avoiding classical patterns, but all results in those chapters extend unchanged to enumeration schemes for permutations avoiding dashed patterns.

# Chapter 4

# Enumeration Schemes and Permutation Statistics

## 4.1 Introduction

Enumeration schemes, introduced in section 1.3.4 of the Introduction, were developed with the intent of computing $s_n(B)$ for a given pattern set $B$. This chapter shows how to use schemes to compute weight enumerators for $\mathcal{S}_n(B)$ other than $s_n(B) = \sum_{\pi \in \mathcal{S}_n(B)} 1$. In particular, we weight according to the inversion number $\text{INV}(\pi) = \#\{(i,j) : i < j, \pi_i > \pi_j\}$, as well as the number of copies of any consecutive pattern.

In [13] Barcucci, Del Lungo, Pergola, and Pinzani adapted generating trees to compute the weight enumerator with respect to inversion number for $B = \{321\}$, $B = \{321, 3\bar{1}42\}$, and $B = \{4231, 4132\}$, which provide $q$-analogues of the Catalan, Motzkin, and Schröder numbers, respectively. Based on these results, Cheng, Eu, and Fu in [29] constructed a correspondence between $\mathcal{S}_n(321)$ and Catalan paths of length $2n$, where a permutation $\pi$ with $\text{INV}(\pi) = k$ maps to a Catalan path which contains a certain region with area $k$. In the same vein, Bandlow and Killpatrick in [12] constructed a similar INV-to-area bijection between $\pi \in \mathcal{S}_n(312)$ and Dyck paths, which was later extended with Egge in [11] to an inv-to-area bijection between $\pi \in \mathcal{S}_n(4231, 4132)$ and Schröder paths. Chen, Deng, and Yang in [28] also constructed an INV-to-area bijection between $\mathcal{S}_n(321, 3\bar{1}42)$ and Motzkin paths. The tools described in this paper can aid in similar endeavors to construct such bijections.

In section 4.3 we describe how to adapt an enumeration scheme for $\mathcal{S}_n(B)$ to compute the distribution of inversion number over $\mathcal{S}_n(B)$. In sections 4.4 and 4.5 we summarize Pudwell's extensions to pattern-avoiding words and barred pattern-avoiding permutations, as well the relevant adaptations which compute the refinement according to inversion number. Section 4.6 details the implementation each adaptations in the Maple packages qVatter, qmVatter, and qbVatter. Section 4.7 lists some of the results obtained from these packages. Section 4.8 outlines how to use schemes to compute the distribution of permutation statistics based on the number of occurrences of a consecutive pattern.

## 4.2 Summary of Enumeration Schemes

Recall that an enumeration scheme $E$ is the encoding of a divide-and-conquer recurrence. The set of $B$-avoiders $\mathcal{S}_n(B)$ is partitioned according to the prefix pattern $p = \text{red}(\pi_1\pi_2\cdots\pi_k)$ to form smaller sets $\mathcal{S}_n(B)[p]$. These subsets are further partitioned according to the actual prefix word $w = \pi_1\pi_2\cdots\pi_k$ to form $\mathcal{S}_n(B)[p;w]$. For each prefix $p$, (at least) one of the following conditions is filled:

(1) $\mathcal{S}_n(B)[p]$ is $\{p\}$ or $\emptyset$.

(2) For each prefix word $w$ such that $\text{red}(w) = p$, one of the following happens:

  (2a) $\mathcal{S}_n(B)[p;w]$ is empty

  (2b) $\mathcal{S}_n(B)[p;w]$ is in bijection with some $\mathcal{S}_{\hat{n}}(B)[\hat{p};\hat{w}]$ for $\hat{n} < n$ and $\hat{p}$ contained in $p$.

(3) $\mathcal{S}_n(B)[p]$ must be partitioned further, so $s_n(B)[p] = \sum_{p'} s_n(B)[p';]$, where the sum runs over all *children* $p'$: permutations of length $k+1$ such that $\text{red}(p'_1\ldots p'_k) = p$.

Condition (1) only applies when $n = |p|$, and so serves as a base case for the recurrence. Condition (2a) is achieved according to the *gap vector criteria* described in section 1.3.4 of Chapter 1. Condition (2b) is achieved according to the deletion map $d_R(\pi)$, which deletes $\pi_r$ from $\pi$ for each $r \in R$, and then reduces to form a smaller permutation. This is a bijection when the set $R$ is *reversibly deletable* as described in section 1.3.4 of Chapter 1.

A set $E$ of triples $(p, G_p, R_p) \in \bigcup_k(\mathcal{S}_k \times \mathbb{N}^k \times 2^{[k]})$ is a valid enumeration scheme if the following criteria are met:

1. $(\epsilon, \emptyset, \emptyset) \in E$.

2. If $R_p = \emptyset$, then $(p', G_{p'}, R_{p'}) \in E$ for every child $p'$ of $p$.

3. If $R_p \neq \emptyset$, then $(\hat{p}, G_{\hat{p}}, R_{\hat{p}}) \in E$ for $\hat{p} = d_{R_p}(p)$

For an example, we present the scheme for 1–2–3-avoiding permutations as explained in equation (1.5) of the introduction.

$$\{(\epsilon, \emptyset, \emptyset), (1, \emptyset, \emptyset), (12, \{\langle 0, 0, 1\rangle\}, \{2\}), (21, \emptyset, \{1\})\} \tag{4.1}$$

## 4.3 Refining According to Inversion Number

In this section we will focus on refining according to the inversion number INV. For pattern set $B$, let $F(n, B, q)$ be the weight enumerator of $\mathcal{S}_n(B)$ according to $q^{\text{INV}(\pi)}$, i.e.,

$$F(n, B, q) := \sum_{\pi \in \mathcal{S}_n(B)} q^{\text{INV}(\pi)}.$$

Since schemes operate on the divide-and-conquer approach, we make the following refinements:

$$F(n, B, q)[p] := \sum_{\pi \in \mathcal{S}_n(B)[p]} q^{\text{INV}(\pi)}$$
$$F(n, B, q)[p; w] := \sum_{\pi \in \mathcal{S}_n(B)[p;w]} q^{\text{INV}(\pi)}.$$

Given an enumeration scheme $E$ for $\mathcal{S}_n(B)$ we now show how to re-interpret it to form a recurrence for $F(n, B, q)[p; w] = \sum_{\pi \in \mathcal{S}_n(B)[p;w]} q^{\text{INV}(\pi)}$. We begin with the initial conditions. When $n$ is small, $\mathcal{S}_n(B)[p]$ is either $\{p\}$, and hence $F(n, B, q)[p] = q^{\text{INV}(p)}$, or $\mathcal{S}_n(B)[p] = \emptyset$, in which case $F(n, B, q)[p] = 0$. The other initial conditions derive from the gap vector criteria. When $w$ satisfies a gap vector criterion, $\mathcal{S}_n(B)[p; w] = \emptyset$ and so $F(n, B, q)[p; w] = 0$.

Next, partitioning the set $\mathcal{S}_n(B)[p]$ into $\mathcal{S}_n(B)[p']$ for children $p'$ has the same effect on $F(n, B, q)[p]$ as for $s_n(B)[p]$, that is, $F(n, B, q)[p] = \sum_{p'} F(n, B, q)[p']$ as the sum runs over all children $p'$.

It remains to consider the bijections $d_r$ and $d_R$ formed by reversibly deletable indices. Deleting the $r^{th}$ letter from $\pi$ causes the loss of any inversion involving $\pi_r$. Let $\delta_r(\pi) :=$

$\text{INV}(\pi) - \text{INV}(d_r(\pi))$ denote the number of inversions lost when applying $d_r$. Then Lemma 26 gives a formula for $\delta_r(\pi)$ below is given in terms of the prefix $\pi_1\pi_2\dots\pi_r$:

**Lemma 26.** *Let $\pi \in \mathcal{S}_n$ and fix index $1 \le r \le n$. Then*

$$\delta_r(\pi) = (\pi_r - 1) + \sum_{i<r} \text{sgn}(\pi_i - \pi_r) \tag{4.2}$$

*where* $\text{sgn}(x)$ *is the signum function*

$$\text{sgn}(x) := \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases} \cdot$$

*Proof.* From the definition of INV we see that $\delta_r(\pi) = \#\{i : i < r, \pi_i > \pi_r\} + \#\{i : r < i, \pi_r > \pi_i\}$. This second set may be rewritten as $\{i : \pi_i < \pi_r\} \setminus \{i : i < r\}$. Hence we get that

$$\begin{aligned} \delta_r(\pi) =& \#\{i : i < r, \pi_i > \pi_r\} + \#\{i : r < i, \pi_r > \pi_i\} \\ =& \#\{i : i < r, \pi_i > \pi_r\} + \#\{i : \pi_i < \pi_r\} - \#\{i : i < r, \pi_i < \pi_r\} \end{aligned}$$

The formula follows from combining the first and third summands to get $\sum_{i<r} \text{sgn}(\pi_i - \pi_r)$ and the fact that $\#\{i : \pi_i < \pi_r\} = \pi_r - 1$. □

When deleting multiple letters simultaneously, we make use of the following corollary.

**Corollary 27.** *Let $\pi \in \mathcal{S}_n$ and $R \subseteq [n]$ so that $R = \{r_1, \dots, r_t\}$ for $r_i < r_{i+1}$. Then*

$$\delta_R(\pi) := \text{INV}(\pi) - \text{INV}(d_R(\pi)) = \sum_{r \in R} \delta_r(\pi) - \text{INV}(\pi_{r_1}\pi_{r_2}\dots\pi_{r_t}) \tag{4.3}$$

*Proof.* The deletion of each $\pi_{r_i}$ causes the loss of $\delta_{r_i}(\pi)$ of the inversions in $\pi$. Inversions among the deleted letters (i.e., those of the form $\pi_{r_i} > \pi_{r_j}$ for $i < j$) are double-counted and thus we must subtract $\text{INV}(\pi_{r_1}\pi_{r_2}\dots\pi_{r_t})$. □

Now consider a reversibly deletable set $R \subseteq \{1, \dots k\}$ for prefix $p \in S_k$ and prefix word $w$. In this case we have the bijection $d_R : \mathcal{S}_n(B)[p; w] \to \mathcal{S}_{n-|R|}(B)[d_R(p); d_R(w)]$. First observe that for $r \le k$, $\pi_r = w_r$ and that $\text{sgn}(\pi_i - \pi_r) = \text{sgn}(p_i - p_r)$ for $i < r$.

Thus by Lemma 26 $\delta_r(\pi)$ is dependent only on $r$, $p$ and $w$ and hence is constant over $\mathcal{S}_n(B)[p;w]$. Further, Corollary 27 implies that $\delta_R(\pi)$ is constant over all $\pi \in \mathcal{S}_n(B)[p;w]$. Hence it follows that $F(n,B,q)[p;w] = q^{\delta_R(p)} F(n-|R|,B,q)[d_R(p);d_R(w)]$.

To illustrate, we again consider $\mathcal{S}_n(1\text{–}2\text{–}3)$ whose scheme is

$$E = \big\{(\epsilon,\emptyset,\emptyset),(1,\emptyset,\emptyset),(12,\{\langle 0,0,1\rangle\},\{2\}),(21,\emptyset,\{1\})\big\}$$

Equation (1.6) in Chapter 1 demonstrates $E$ leads to the following recurrence:

$$s_n(1\text{–}2\text{–}3)[1;i] = \sum_{j=1}^{i} s_{n-1}(123)[1;j] \tag{4.4}$$

As in the classical case, $F(n,1\text{–}2\text{–}3,q) = \sum_{i=1}^{n} F(n,1\text{–}2\text{–}3,q)[1,\{i\}]$. For any $\pi \in \mathcal{S}_((1)[-;2]\text{–}3)21i\,j$ we get $\delta_{\{1\}}(\pi) = i-1$, and for any $\pi \in \mathcal{S}_n(1\text{–}2\text{–}3)[12;i\,n]$ we get $\delta_{\{2\}}(\pi) = n-2$. The enumeration scheme leads directly to the following refinement of equation 4.4:

$$\begin{aligned} F(n,1\text{–}2\text{–}3,q)[1;i] = &\sum_{j=1}^{i-1} q^{i-1}\cdot F(n-1,1\text{–}2\text{–}3,q)[1;j] + \\ &\qquad q^{n-2}\cdot F(n-1,1\text{–}2\text{–}3,q)[1;i] \end{aligned} \tag{4.5}$$

The generating functions $F(n,3\text{–}2\text{–}1,q) = q^{\binom{n}{2}} F(n,1\text{–}2\text{–}3,q^{-1})$ were studied via generating trees in Barcucci et al. in [13], as well as Cheng et al. in [29].

## 4.4 Pattern-Avoiding Words

We now turn our attention toward Pudwell's extensions of enumeration schemes,. We first consider the extension to words avoiding permutations as developed in [76]. Pattern-avoiding words are described in section 1.1.2 of Chapter 1.

For word $\pi \in [n]^\ell$, let $\nu_i(\pi)$ represent the number of copies of $i$ in $\pi$, and $\nu(\pi) := \langle \nu_1(\pi), \nu_2(\pi), \ldots\rangle \in \mathbb{N}^{\mathbb{N}}$. For alphabet vector $\mathbf{m} \in \mathbb{N}^{\mathbb{N}}$ with finitely many nonzero entries, we may define $\mathcal{S}_{\mathbf{m}}$ to be all words $\pi$ so that $\nu(\pi) = \mathbf{m}$. Let $n(\mathbf{m})$ be the index of the largest nonzero entry in $\mathbf{m}$, i.e., the largest letter appearing in $\pi \in \mathcal{S}_{\mathbf{m}}$. We may define the reduction of an alphabet vector red($\mathbf{m}$) to be the vector formed by removing internal zeroes from $\mathbf{m}$. For example $\text{red}(\langle 2,0,0,1,0,3,2,0,0,\ldots\rangle) = \langle 2,1,3,2,0,0,\ldots\rangle$. This is natural since if $\pi \in \mathcal{S}_{\mathbf{m}}$, then $\text{red}(\pi) \in \mathcal{S}_{\text{red}(\mathbf{m})}$.

Define $\mathcal{S}_{\mathbf{m}}(B)$ to be the subset of $S_{\mathbf{m}}$ avoiding all patterns in $B$. We will restrict our attention to sets $B$ containing only classical patterns with no repeated letters, since this was the context in [76]. Let $s_{\mathbf{m}}(B) = \#\mathcal{S}_{\mathbf{m}}(B)$ and $F(\mathbf{m}, B, q) := \sum_{\pi \in \mathcal{S}_{\mathbf{m}}(B)} q^{\text{INV}(\pi)}$.

As in the case for permutations, we will partition $\mathcal{S}_{\mathbf{m}}(B)$ according to prefixes. For a prefix $p$ of length $k$, we define $\mathcal{S}_{\mathbf{m}}(B)[p]$ to be those words $\pi \in \mathcal{S}_{\mathbf{m}}$ such that $\text{red}(\pi_1\pi_2\ldots\pi_k) = p$. We refine this further for a prefix word $w$ where $\text{red}(w) = p$, letting $\mathcal{S}_{\mathbf{m}}(B)[p;w]$ be the set of words $\pi \in S_{\mathbf{m}}(B)[p]$ such that $\pi_1\pi_2\ldots\pi_k = w$. Observe that these prefix patterns may have repeated letters, so there are now three prefixes of length 2: 12, 21, and 11. As in the classical case, Pudwell's schemes compute $s_{\mathbf{m}}(B)[p;w] := \#\mathcal{S}_{\mathbf{m}}(B)[p;w]$, while our refinement will compute $F(\mathbf{m}, B, q)[p;w] := \sum_{\pi \in \mathcal{S}_{\mathbf{m}}(B)[p;w]} q^{\text{INV}(\pi)}$. Again, each prefix $p$ leads to one of three cases:

(1) $\mathcal{S}_{\mathbf{m}}(B)[p]$ is $\{p\}$ or $\emptyset$.

(2) For each prefix word $w$ such that $\text{red}(w) = p$, one of the following happens:

  (2a) $\mathcal{S}_{\mathbf{m}}(B)[p;w]$ is empty

  (2b) $\mathcal{S}_{\mathbf{m}}(B)[p;w]$ is in bijection with some $\mathcal{S}_{\hat{\mathbf{m}}}(B)[\hat{p};\hat{w}]$ via a deletion map $d_R$ for nonempty $R \subseteq [k]$.

(3) $\mathcal{S}_{\mathbf{m}}(B)[p]$ must be partitioned further, so $s_{\mathbf{m}}(B)[p] = \sum_{p'} s_{\mathbf{m}}(B)[p';]$, where the sum runs over all *children* $p'$: words of length $k+1$ such that $\text{red}(p'_1 \ldots p'_k) = p$.

As before, case (1) provides base cases for the recurrences, while case (2a) is handled by an analogue of gap vectors, and the bijections in (2b) are deletion maps $d_r$.

**Gap Vectors**

Again, gap vectors are restrictions on the amount of vertical space which may appear between letters in the prefix of a permutation. Consider prefix $p$ with $k$ letters and prefix word $w$. Define the multiset $C = \{c_1, c_2, \ldots, c_k\} = \{w_1, w_2, \ldots, w_k\}$ such that

$c_i \leq c_{i+1}$. Let $c_0 = 1$ and $c_{k+1} = n(\mathbf{m})$, and form the $(k+1)$-vector $\vec{h} = \vec{h}(\mathbf{m}, w)$ so that $h_i = c_i - c_{i-1}$. Observe the change in interpretation of the $h_i$ compared to the $g_i$ in the case of pattern-avoiding permutations. We call $\vec{h}$ the *spacing vector* for $w$. Here $h_i$ for $2 \leq i \leq k$ represents *one more than* the number of letters between $c_{i-1}$ and $c_i$, or $h_i = 0$ when $c_{i-1} = c_i$, while $h_1$ and $h_{k+1}$ are still the number of letters lying below $c_1$ and above $c_k$, respectively. While this convention may seem odd, it allows for neater computation elsewhere. In particular if $v = \langle v_1, v_2, \ldots, v_{k+1}\rangle$ is a spacing vector for $w$, then $w_1 \ldots w_{t-1} w_{t+1} \ldots w_k$ has the spacing vector $v = \langle v_1, v_2, \ldots, v_t + v_{t+1}, \ldots v_{k+1}\rangle$. This is of great use when computing the schemes, although once the scheme is computed either convention would suffice. This alternate convention is presented here for the sake of clarifying the output from the Maple package mVatter and its refinement qmVatter.

We may again define an analogue of gap vectors for words by finding those $w$ such that $\mathcal{S}_{\mathbf{m}}(B)[p; w]$ is necessarily empty.

**Definition 28.** *Given a set of forbidden patterns $B$ and prefix $p$, then $v$ is a* word gap vector *for prefix $p$ if, for all $\mathbf{m}$, $\mathcal{S}_{\mathbf{m}}(B)[p; w] = \emptyset$ for any $w$ such that $h(\mathbf{m}, w) \geq v$. When this happens, we say that $w$ satisfies the* word gap vector criterion *for $v$.*

As before, the word gap vectors for $p$ form an upper ideal in $\mathbb{N}^{k+1}$ and it suffices to find a finite basis. The details of their automated discovery are given in [76] and will not be repeated here.

### Reversible Deletability

The deletion operator $d_r$ is defined similarly to the permutation case, but without reduction. For word $\pi$ let $d_r(\pi) = \pi_1 \cdots \pi_{r-1}\pi_{r+1} \cdots \pi_n$. For alphabet vector $\mathbf{m}$ and word $\pi$ such that $\nu(\pi) \leq \mathbf{m}$, define $d_\pi(\mathbf{m}) = \mathbf{m} - \nu(\pi)$, which is the alphabet vector for the word formed by deleting the letters of $\pi$ from a word with alphabet vector $\mathbf{m}$.

Again some maps $d_r : \mathcal{S}_{\mathbf{m}}(B)[p; w] \to \mathcal{S}_{d_{w_r}(\mathbf{m})}(B)[d_r(p); d_r(w)]$ are bijections for all $w$ which fail all word gap vector criteria for $p$ with respect to $B$. In this case $r$ is called *reversibly deletable*, as before. Unlike before, however, if $r$ and $s$ are both reversibly deletable, the set $\{r, s\}$ may not be. For example any $\pi \in \mathcal{S}_{\mathbf{m}}(1\text{–}2\text{–}3)[11;11]$ has the form $11\pi_3\pi_4\dots\pi_n$, where each $\pi_i \geq \pi_{i+1}$ for $i \geq 3$. Here both $d_1$ and $d_2$ are bijections to $\mathcal{S}_{\mathbf{m}-\nu(1)}(1\text{–}2\text{–}3)[1;1]$, however $d_1 \circ d_2$ is not surjective since $\mathcal{S}_{\mathbf{m}-\nu(11)}(1\text{–}2\text{–}3)[\epsilon;\epsilon]$ contains words other than the nonincreasing word. It is not necessary to perform deletions $d_R$ for $|R| > 1$, however, so long as the scheme contains $(p'', G_{p''}, R_{p''})$ for all patterns $p''$ contained in $p$. Since schemes may always be constructed as to contain all subpatterns, we need not consider simultaneous deletions. When there are several reversibly deletable indices, we use the convention to delete only the first.

The maps $d_r$ affect the number of inversions appearing analogously to Lemma 26.

**Lemma 29.** *Let* $\pi \in \mathcal{S}_{\mathbf{m}}$ *and fix some index* $1 \leq r \leq n$. *Then*

$$\delta_r(\pi) := \textsc{inv}(\pi) - \textsc{inv}\left(d_r(\pi)\right) = (\nu_1(\pi) + \cdots + \nu_{\pi_r - 1}(\pi)) + \sum_{i<r} \operatorname{sgn}(\pi_i - \pi_r) \tag{4.6}$$

*where* $\operatorname{sgn}(x)$ *is the signum function.*

The proof for Lemma 26 applies for this lemma as well, except now $\#\{i : \pi_i < \pi_r\} = \nu_1(\pi) + \cdots + \nu_{\pi_r - 1}(\pi)$ instead of simply $\pi_r - 1$.

Observe that once again $\delta_r$ remains constant over all $\pi \in \mathcal{S}_{\mathbf{m}}(B)[p;w]$.

There is a second batch of bijections which are vital to Pudwell's schemes. The maps $d_r$ result in words $\pi$ such that $\nu(\pi)$ has internal zeroes. Hence to achieve step (2a), whenever we apply a deletion map $d_r$ we must also apply the reduction map to the image get to a set $\mathcal{S}_{\hat{\mathbf{m}}}(B)[\hat{p};\hat{w}]$ which has been previously counted. Since $\textsc{inv}(\pi)$ depends only on the relative heights of the letters of $\pi$ and not the values of the letters themselves, we see $\textsc{inv}(\pi) = \textsc{inv}(\operatorname{red}(\pi))$ so reduction has no effect on inversion number.

Hence we have refined Pudwell's enumeration schemes for permutation-avoiding words to compute

$$F(\mathbf{m}, B, q)[p;w] := \sum_{\pi \in \mathcal{S}_{\mathbf{m}}(B)[p;w]} q^{\textsc{inv}(\pi)} \tag{4.7}$$

via a system of recurrences. For an example, consider $F(\mathbf{m}, \{1\text{–}2\text{–}3\}, q)$, which has the enumeration scheme

$$\{(\epsilon, \emptyset, \emptyset), (1, \emptyset, \emptyset), (21, \emptyset, \{1\}), (11, \emptyset, \{1,2\}), (12, \{\langle 0,1,1\rangle\}, \{2\})\} \tag{4.8}$$

This translates into the following set of recurrences.

$$
\begin{aligned}
F(\mathbf{m}, B, q) &= F(\mathbf{m}, B, q)[\epsilon] = F(\mathbf{m}, B, q)[1] \\
&= \sum_{i=1}^{m(\mathbf{m})} F(\mathbf{m}, B, q)[1; i] \\
F(\mathbf{m}, B, q)[1; i] &= \sum_{j=1}^{i-1} F(\mathbf{m}, B, q)[21; ij] + F(\mathbf{m}, B, q)[11; ii] + \sum_{j=i+1}^{m(\mathbf{m})} F(\mathbf{m}, B, q)[12; ij] \\
F(\mathbf{m}, B, q)[21; ij] &= q^{\mathbf{m}_1 + \cdots + \mathbf{m}_{i-1}} F(\mathbf{m} - \nu(i), B, q)[1; j] \\
F(\mathbf{m}, B, q)[11; ii] &= q^{\mathbf{m}_1 + \cdots + \mathbf{m}_{i-1}} F(\mathbf{m} - \nu(i), B, q)[1; i] \\
F(\mathbf{m}, B, q)[12; ij] &= \begin{cases} 0 & j - i \geq 1 \text{ and } n - j \geq 1 \\ q^{\mathbf{m}_1 + \cdots + \mathbf{m}_{j-1} - 1} F(\mathbf{m} - \nu(j), B, q)[1; i] & \text{otherwise} \end{cases}
\end{aligned}
\tag{4.9}
$$

Using this recurrence and implementation discussed below, it takes less than a minute on a personal computer for Maple to compute the first 12 terms of the sequence given by $F(\mathbf{m}, \{1\text{–}2\text{–}3\}, q)$ where $\mathbf{m} = \nu(1122 \cdots nn) = \langle 2, 2, \ldots, 2, 0, 0, \ldots \rangle$. The first 4 terms are given in Table 4.1, and more may be viewed at this paper's website on the author's homepage (see Section 4.6).

| $n$ | $F(\nu(1122 \cdots nn), \{123\}, q)$ |
|---|---|
| 1 | $1$ |
| 2 | $1 + q + 2\,q^2 + q^3 + q^4$ |
| 3 | $3\,q^4 + 3\,q^5 + 6\,q^6 + 7\,q^7 + 9\,q^8 + 7\,q^9 + 5\,q^{10} + 2\,q^{11} + q^{12}$ |
| 4 | $q^8 + q^9 + 4\,q^{10} + 6\,q^{11} + 15\,q^{12} + 18\,q^{13} + 28\,q^{14} + 35\,q^{15} +$ |
| | $+44\,q^{16} + 47\,q^{17} + 49\,q^{18} + 42\,q^{19} + 31\,q^{20} + 18\,q^{21} + 9\,q^{22} + 3\,q^{23} + q^{24}$ |

Table 4.1: Distributions of inversion number over the set of permutations of the multiset $\{1, 1, 2, 2, \ldots, n, n\}$

## 4.5 Permutations Avoiding Barred Patterns

Pudwell also extended enumeration schemes to count permutations avoiding barred patterns in [75]. Barred patterns were introduced in section 1.1.2.

These schemes again partition according to prefixes, so define $\mathcal{S}_n(B)[p]$ and $\mathcal{S}_n(B)[p; w]$ as before. We again meet three possibilities for a given prefix $p \in \mathcal{S}_k$:

(1) $\mathcal{S}_n(B)[p]$ is $\{p\}$ or $\emptyset$.

(2) For each prefix word $w$ such that $\text{red}(w) = p$, one of the following happens:

(2a) $\mathcal{S}_n(B)[p; w]$ is empty

(2b) $\mathcal{S}_n(B)[p; w]$ is in bijection with some $\mathcal{S}_{\hat{n}}(B)[\hat{p}; \hat{w}]$ via a deletion map $d_R$ for nonempty $R \subseteq [k]$.

(3) $\mathcal{S}_n(B)[p]$ must be partitioned further, so $s_n(B)[p] = \sum_{p'} s_n(B)[p';]$, where the sum runs over all children $p'$, which are words of length $k+1$ such that $\text{red}(p'_1 \dots p'_k) = p$.

As before, case (1) applies only if $p$ has length $n$ and depends on whether $p$ avoids $B$. There are additional base cases, however, which are hidden in the statement for case (2). In the case of classical pattern avoidance, if we knew that $\mathcal{S}_n(B)[p] = \emptyset$ then it followed that $\mathcal{S}_{n+1}(B)[p] = \emptyset$ since the addition of an extra letter could not cause the avoidance of a pattern. For barred patterns, however, that added letter may be exactly the barred letter we require to avoid a barred pattern. Hence for certain small $n$, it is possible that $\mathcal{S}_n(B)[p] = \emptyset$ while $\mathcal{S}_{n+s}(B)[p] \neq \emptyset$ for some $s > 0$. These thresholds $s$ are called *stop points*, maximal values $s_p$ such that $\mathcal{S}_n(B)[p] = \emptyset$ for any $n \leq s_p$. Stop points are built into the schemes for barred patterns, which have a structure as a set of 4-tuples $(p, G_p, R_p, s_p)$ instead of triples $(p, G_p, R_p)$. These additional base cases are trivial to incorporate into our refinement, since they imply additional cases when $F(n, B, q)[p; w] = 0$.

When case (2) applies and stop points are not to blame, the subcase (2a) is indicated by gap vector criteria — their application remains unchanged from the description in section 1.3.4 of the introduction. Similarly, the bijections in subcase (2b) are performed via the deletion map $d_R$, so Lemma 26 applies as written. The discovery of these gap vectors and reversibly deletable indices meet additional wrinkles as explained in [75], but again the discovery is not our present concern. Hence we have refined enumeration schemes to compute $F(n, B, q)$ for pattern sets $B$ containing barred patterns.

## 4.6 Implementation in Maple

To accompany their papers, Zeilberger wrote WILFfor [100], Vatter wrote WILFPLUS for [91], and Zeilberger wrote VATTER for [102]. Pudwell continued this tradition, writing MVATTER to accompany the multiset permutation results in [76] and BVATTER to accompany the barred pattern results in [75]. Here we continue that tradition, presenting three suites of additional Maple procedures: QVATTER, QMVATTER, and QBVATTER to implement the refinements above for each of the schemes produced by VATTER, MVATTER, and BVATTER, respectively. Tables 4.2, 4.3, and 4.4 summarize the primary procedures in each package. All files, along with dozens of pre-computed examples of $F(n, B, q)$ and $F(\mathbf{m}, B, q)$, may be downloaded from the author's homepage. Further comments on syntax are presented upon loading the packages successfully.

| | |
|---|---|
| qWilf(n,B,q) | A brute-force computation of $F(n, B, q)$. This is based on the procedure Wilf(n,B) in VATTER, which recursively constructs the set $\mathcal{S}_n(B)$. |
| SchemeF(B, Depth, GapLimit) | Attempts to compute an enumeration scheme for pattern set B, limiting the search to prefixes of length Depth and gap vectors $g$ such that $\sum g_i \leq$ GapLimit. If no such scheme exists, it returns FAIL. This procedure is contained in VATTER. |
| qMiklos(E, Prefix, GapVector, q) | Computes $F(n, B, q)[\text{Prefix}, C]$ using the scheme $E$, where $C$ is the prefix set such that $g(n, C)$=GapVector. To compute $F(n, B, q)$, let Prefix=[] and GapVector=[n]. |
| qSeqS(E, N, q) | Computes $F(n, B, q)$ for $n = 1, 2, \ldots, N$, where $E$ is the scheme for $B$. |

Table 4.2: Primary procedures in QVATTER

It should be noted the scheme-construction algorithms implemented follow the algorithms described by Zeilberger in [102] and as a result contain the parameter "GapLimit." To find the minimal gap vectors $G_p$ for $p \in S_k$, the algorithm considers all vectors $g \in \mathbb{Z}^{k+1}$ such that $\sum_i g_i \leq$ GapLimit. Vatter showed this parameter is not strictly necessary by providing a subtle variant of gap vectors. For this variant, the maximum norm of basis gap vectors is bounded above by one less than the longest

| | |
|---|---|
| qmWilf(**m**,B,q) | Computes $F(\mathbf{m}, B, q)$ via brute force, based on mWilf(**m**,B) from mVatter which recursively constructs the set $S_{\mathbf{m}}(B)$. |
| SchemeF(B, Depth, GapLimit) | Attempts to compute an enumeration scheme for pattern set B, limiting the search to prefixes of length Depth and gap vectors $g$ such that $\sum g_i \leq$ GapLimit. If no such scheme exists, it returns FAIL. This procedure is contained in mVatter. |
| qMiklosA(E, Prefix, Remaining,q) | Computes $F(\mathbf{m}, B, q)[p, C]$ where $E$ is the enumeration scheme for $B$, Prefix is $p$, and Remaining is the frequency vector for the letters lying outside of the prefix. |
| qSeqS11(E,r,N,q) | Computes the sequence $F(\mathbf{m}, B, q)$ where $\mathbf{m}$ is the alphabet vector for $r$ copies of each letter $1, \ldots, n$ as $n$ ranges from 1 to $N$. |
| qMiklosTot(E,k,n,q) | Computes $\sum_{w \in [k]^n(B)} q^{\text{INV}(w)}$, the distribution over all words avoiding $B$ of length $n$ with letters at most $k$, where $E$ is the enumeration scheme for $B$. |
| qSeqSkn(E,k,n,q) | Computes the sequence given by qMiklosTot(E,k,N,q) for $n = 1, 2, \ldots, N$. |

Table 4.3: Primary procedures in qmVatter

| | |
|---|---|
| qWilf(n,B,q) | A brute-force computation of $F(n, B, q)$. This is based on the procedure Wilf(n,B) in bVatter, which recursively constructs the set $\mathcal{S}_n(B)$. |
| SchemeFast(B, Depth, GapLimit) | Attempts to compute an enumeration scheme for pattern set B, limiting the search to prefixes of length Depth and gap vectors $g$ such that $\sum g_i \leq$ GapLimit. If no such scheme exists, it returns FAIL. This procedure is contained in bVatter. |
| qMiklos(E, Prefix, GapVector, q) | Computes $F(n, B, q)[\text{Prefix}, C]$ using the scheme $E$, where $C$ is the prefix set such that $g(n, C)$=GapVector. To compute $F(n, B, q)$, let Prefix=[ ] and GapVector=[n]. |
| qSeqS(E, N, q) | Computes $F(n, B, q)$ for $n = 1, 2, \ldots, N$, where $E$ is the scheme for $B$. |

Table 4.4: Primary procedures in qbVatter

pattern appearing in $B$.

## 4.7 Applications

### 4.7.1 Exploring $\mathcal{S}_n(B)$

#### An Infinite Family of Patterns

As an example of how schemes can help us explore $\mathcal{S}_n(B)$ and form conjectures, we consider the family of patterns $B_k = \{3\text{–}1\text{–}2, k\text{–}(k-1)\text{–}\cdots\text{–}2\text{–}1\}$. Chow and West showed these have generating functions related to Chebyshev polynomials in [30], and Vatter showed in [91] that each of these $B_k$'s has an enumeration scheme of depth 2 and exhibits the scheme in tree form. In our notation, this is the scheme

$$\{(\epsilon, \emptyset, \emptyset), (1, \emptyset, \emptyset), (12, \emptyset, \{1\}), (21, \{\langle 0, 1, 0\rangle, \langle k-2, 0, 0\rangle\}, \{1\})\}$$

This yields the following family of recurrences for $F(n, B_k, q)[p; w]$:

$$\begin{aligned}
F(n, B_k, q) &= \sum_{i=1}^{n} F(n, B_k, q)[1; i] \\
F(n, B_k, q)[1; i] &= \sum_{j=1}^{i-1} F(n, B_k, q)[21; ij] + \sum_{j=i+1}^{n} F(n, B_k, q)[12; ij] \\
F(n, B_k, q)[12; ij] &= q^{i-1} F(n-1, B_k, q)[1, j-1] \\
F(n, B_k, q)[21; ij] &= \begin{cases} 0, & \text{if } i-j > 1 \text{ or } j > k-2 \\ q^j F(n-1, B_k, q)[1; i-1], & \text{otherwise} \end{cases}
\end{aligned} \tag{4.10}$$

It can be seen from [30] that the generating functions $f_k(x) = \sum_{x \geq 0} s_n(B_k) x^n$ satisfy the relation

$$f_k(x) = \frac{1}{1 - x\, f_{k-1}(x)}.$$

A similar relationship holds for $f_k(x, q) = \sum_{x \geq 0} F(n, B_k, q) x^n$. With a combination of the above recurrences and the Maple package GFUN, a suite of tools to work with generating functions, we may quickly conjecture the generating functions $f_k(x, q)$ for small $k$. After a bit of tinkering and luck we may conjecture the following proposition, which when combined with the obvious initial condition $f_1(x,q) = 1$ allows us to compute $f_k(x,q)$ for any $k$.

**Proposition 30.** *If* $f_k(x,q) = \sum_{n\geq 0} F(n, B_k, q)x^n$*, then*

$$f_k(x,q) = \frac{1}{1 - x\, f_{k-1}(qx,q)}.$$

*Proof.* We will follow the method of weight enumerators, as used in [79]. Define the weight function $W_{(x,q)}(\pi) = x^{|\pi|}q^{\textsc{inv}(\pi)}$, where $|\pi|$ represents the length of $\pi$. Note $W_{(x,q)}(\epsilon) = 1$. We will weight-count $\mathcal{S}(B_k) := \bigcup_{n\geq 0} \mathcal{S}_n(B_k)$, where

$$W_{(x,q)}(\mathcal{S}(B_k)) := \sum_{\pi\in\mathcal{S}(B_k)} W_{(x,q)}(\pi).$$

Observe $f_k(x,q) = W_{(x,q)}(\mathcal{S}(B_k))$. The proposition follows from a proof of the following:

$$W_{(x,q)}(\mathcal{S}(B_k)) = 1 + x\, W_{(qx,q)}(\mathcal{S}(B_{k-1})) \cdot W_{(x,q)}(\mathcal{S}(B_k)).$$

Any $\pi \in \mathcal{S}(B_k)$ of positive length must contain a 1, say $\pi_i = 1$, and so we may decompose $\pi$ into $\pi^{(1)}\, 1\, \pi^{(2)}$, where $\mathrm{red}(\pi^{(1)}) \in \mathcal{S}(B_{k-1})$ and $\mathrm{red}(\pi^{(2)}) \in \mathcal{S}(B_k)$. Clearly $|\pi| = |\pi^{(1)}| + |\pi^{(2)}| + 1$. Since $\pi$ avoids 3–1–2, every letter in $\pi^{(1)}$ lies below every letter in $\pi^{(2)}$ otherwise the $\pi_i = 1$ would play the role of "1" in a 3–1–2 pattern. From this we determine that $\textsc{inv}(\pi) = \textsc{inv}(\pi^{(1)}) + \textsc{inv}(\pi^{(2)}) + |\pi^{(1)}|$, where the last term is the number of inversions involving the 1 of $\pi$. Combining these observations we get:

$$\begin{aligned} W_{(x,q)}(\pi) &= W_{(x,q)}(\pi^{(1)}\, 1\, \pi^{(2)}) \\ &= x^{|\pi^{(1)}\, 1\, \pi^{(2)}|} q^{\textsc{inv}(\pi^{(1)}\, 1\, \pi^{(2)})} \\ &= x\, x^{|\pi^{(1)}|}\, x^{|\pi^{(2)}|}\, q^{\textsc{inv}(\pi^{(1)})}\, q^{\textsc{inv}(\pi^{(2)})}\, q^{|\pi^{(1)}|} \\ &= x\, W_{(qx,x)}(\pi^{(1)})\, W_{(q,x)}(\pi^{(2)}) \end{aligned}$$

Summing over all non-empty $\pi \in \mathcal{S}(B_k)$, we get the relation $W_{(x,q)}(\mathcal{S}(B_k)) - 1 = xW_{(qx,q)}(\mathcal{S}(B_{k-1})) \cdot W_{(x,q)}(\mathcal{S}(B_k))$, from which our proposition follows. □

**Inversions Over $\mathcal{S}_n(1\text{–}32)$**

In Proposition 3 of [32] Claesson shows that the Bell numbers enumerate 1–32-avoiding permutations via a bijection, where the number of descents in the permutation is one

| | k = 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| n = 0 | 1 | | | | | | | | | | |
| 1 | 1 | | | | | | | | | | |
| 2 | 1 | 1 | | | | | | | | | |
| 3 | 1 | 1 | 2 | 1 | | | | | | | |
| 4 | 1 | 1 | 2 | 4 | 3 | 3 | 1 | | | | |
| 5 | 1 | 1 | 2 | 4 | 7 | 8 | 9 | 9 | 6 | 4 | 1 |

Table 4.5: Number of permutations avoiding 1–32 of length $n$ with $k$ inversions

more than the number of blocks in the corresponding set partition. We now find the number of permutations with $k$ inversions avoiding 1–32, leading to a new refinement of the Bell numbers which is shown in Table 4.5.

It is interesting to note that if one continues this chart, the columns are each eventually constant. Specifically, if $f(n,k)$ is the number of permutations avoiding 1–32 of length $n$ with $k$ inversions, then $f(n,k) = f(n+1,k)$ for all $n \geq k+2$. This contrasts with the unrestricted case, where the number of permutations with a fixed number of inversions continues to increase with $n$. The stagnation can be seen as follows. For $k = 1$, we see the only such permutation of length $n$ is given by $2134\cdots n$. Otherwise suppose $\pi$ is a permutation of length $n \geq k+2$ with $k > 1$ inversions which avoids 1–32. This pattern continues so that a permutation avoiding 1–32 with $k$ inversions has the form $\sigma \oplus 12..(n-k-1)$ where $\sigma \in \mathcal{S}_{k+1}(1\text{–}32)$ with $k$ inversions. The sequence of $\left\{\lim_{n\to\infty}(f,k)\right\}_{k\geq 0}$ is given by $1, 1, 2, 4, 7, 13, 22, 38, 63, 105, \ldots$, which has been added to the OEIS, [53], as A188920. These numbers should describe set partitions where a certain statistic is $k$, but it is unclear whether such a statistic should be. It should be noted that when one considers the analogous question of permutations avoiding 1–3–2 of length $n$ with $k$ inversions, one see that as $n \to \infty$ the number is eventually constant at the number of integer partitions of $k$. See [49] for details of this proof.

### 4.7.2 Symmetry Applications

We may also consider symmetry questions which were studied by Simion and Schmidt in [81]. Define $\mathcal{E}_n(B)$ to be the set of even permutations avoiding $B$ and $\mathcal{O}_n(B)$ to be the set of odd permutations avoiding $B$. Let $E_n(B) = \#\mathcal{E}_n(B)$ and $O_n(B) = \#\mathcal{O}_n(B)$.

Then since permutation $\pi$ is even if and only if $\text{INV}(\pi)$ is even, we see that $F(n, B, -1) = E_n(B) - O_n(B)$. Of course if we have already gone through the trouble to find the scheme, we can also compute $E_n(B) = \frac{F(n,B,-1)+s_n(B)}{2}$ and $E_n(B) = \frac{F(n,B,-1)-s_n(B)}{2}$.

We call a set of patterns $B$ *evenly-split* if $E_n(B) = O_n(B)$ for all $n \geq 2$, i.e., $F(n, B, -1) = 0$. An obvious necessary condition is that $B$ must contain the same number of even and odd patterns of length $k$ for each $k$, since otherwise $E_k(B) \neq O_k(B)$. This is not sufficient, however, as shown by $B = \{1\text{–}2\text{–}3\text{–}4, 1\text{–}3\text{–}2\text{–}4\}$ where $E_7(B) = O_7(B) - 1 = 918$. Interestingly, data for $n \leq 10$ do suggest that $E_{2n}(B) = O_{2n}(B)$. From exploring various pairs $\{\sigma, \tau\} \subseteq S_4$, we were able to observe and then prove the following theorem.

**Theorem 31.** *For any* $\sigma \in \mathcal{S}_k$, $B = \{\sigma_1\text{–}\sigma_2\text{–}\cdots\text{–}\sigma_k, \sigma_2\text{–}\sigma_1\text{–}\cdots\text{–}\sigma_k\}$ *is evenly-split.*

*Proof.* Let $\sigma$ denote $\sigma_1\text{–}\sigma_2\text{–}\cdots\text{–}\sigma_k$ and $\hat{\sigma}$ denote $\sigma_2\text{–}\sigma_1\text{–}\cdots\text{–}\sigma_k$.

Consider the map transposing the first two letters of a permutation, $\tau : \pi_1\pi_2 \ldots \pi_n \mapsto \pi_2\pi_1 \ldots \pi_n$. This is an involution $E_n \to O_n$ which we will see preserves the $B$-avoiding property. Suppose otherwise so that for $\pi \in \mathcal{S}_n(B)$, $\hat{\pi} = \tau(\pi)$ contains $\sigma$ or $\hat{\sigma}$. If $\hat{\pi}(i_1\hat{\pi}(i_2) \ldots \hat{\pi}(i_k)$ is a copy of $\sigma$, then $i_1 = 1$ and $i_2 = 2$ or else $\pi$ would contain $\sigma$ as well. If both are involved in a copy of $\sigma$, however, then $\pi(i_1)\pi(i_2) \ldots \pi(i_k)$ is a copy of $\hat{\sigma}$. Hence $\tau$ preserves $B$-avoidance and reverses sign. $\square$

We of course have similar symmetric criteria by transposing the *last* two letters, the *smallest* two letters, and the *largest* two letters. This theorem is not a full characterization, as there are $B$ of size 4 which are evenly-split (for example, $\{1\text{–}2\text{–}3, 1\text{–}3\text{–}2, 2\text{–}1\text{–}3, 2\text{–}3\text{–}1\}$). Even among pairs, $E_n(1\text{–}2\text{–}3\text{–}4, 1\text{–}4\text{–}3\text{–}2) = O_n(1\text{–}2\text{–}3\text{–}4, 1\text{–}4\text{–}3\text{–}2)$ for $2 \leq n \leq 20$ and so it appears $\{1\text{–}2\text{–}3\text{–}4, 1\text{–}4\text{–}3\text{–}2\}$ is evenly-split without satisfying this criterion (nor any of its symmetries). It should also be said that transposing two adjacent letters in the middle of a permutation does not lead to an evenly-split pair, as demonstrated above by $\{1\text{–}2\text{–}3\text{–}4, 1\text{–}3\text{–}2\text{–}4\}$.

We close this section with a remark regarding multiset permutations avoiding $B = \{3\text{–}2\text{–}1, 3\text{–}1\text{–}2\}$. These have the enumeration scheme

$$\{(\epsilon, \emptyset, \emptyset), (1, \{\langle 2, 0\rangle\}, \{1\})\}.$$

This implies

$$F(\mathbf{m}, B, q) = F(\mathbf{m} - \nu(1), B, q) + q^{\mathbf{m}_1} F(\mathbf{m} - \nu(2), B, q). \tag{4.11}$$

However, for $w \in S_{\langle a_1, a_2, a_3 \dots \rangle}$, let $\phi(w)$ be the word in $S_{\langle a_2, a_1, a_3 \dots \rangle}$ obtained by changing each 1 in $w$ to a 2 and each 2 to a 1. When we restrict the domain to $B$-avoiding words, observe that $\phi$ is a (bijective) involution $S_{\langle a_1, a_2, a_3 \dots \rangle}(B) \to S_{\langle a_2, a_1, a_3 \dots \rangle}(B)$. If 3–2–1 did appear in $\phi(w)$, then if "2–1" were played by a literal 21 in $\phi(w)$ then $w$ must have contained a corresponding 3–1–2. If only the "1" of a 3–2–1 were played by a 1, then the "2" must be played by some letter of $\phi(w)$ which is at least 3, so a 2 may also play the "1". If the "1" were played by a 2, then obviously it may be played by a 1 instead. The analogous arguments apply should 3–1–2 appear in $\phi(w)$ as well. Hence $w$ avoids $B$ if and only if $\phi(w)$ avoids $B$.

Next consider the effect of $\phi$ on $\textsc{inv}(w)$. Since only the 1's and 2's change places, it can easily be seen that $\textsc{inv}(\phi(w)) - \textsc{inv}(w) = a_1 a_2 - 2\textsc{inv}_{12}(w)$ where $\textsc{inv}_{12}(w)$ denotes the number of pairs $i < j$ such that $w_i = 2$ and $w_j = 1$. It follows that

$$F(\langle a_1, a_2, a_3, \dots \rangle, B, -1) = (-1)^{a_1 a_2} F(\langle a_2, a_1, a_3, \dots \rangle, B, -1)$$

We return to the recurrence in (4.11) and specialize to $q = -1$, $\mathbf{m}_1 = \mathbf{m}_2 = a$ to see:

$$\begin{aligned} F(\langle a, a, a_3, \dots \rangle, B, -1) &= F(\langle a-1, a, a_3, \dots \rangle, B, -1) \\ &\qquad + (-1)^a F(\langle a, a-1, a_3, \dots \rangle, B, -1) \\ &= (-1)^{(a-1)a} F(\langle a, a-1, a_3, \dots \rangle, B, -1) \\ &\qquad + (-1)^a F(\langle a, a-1, a_3, \dots \rangle, B, -1) \\ &= (1 + (-1)^a) F(\langle a, a-1, a_3, \dots \rangle, B, -1) \end{aligned}$$

Hence we get the following proposition:

**Proposition 32.** *If* $\mathbf{m}_1 = \mathbf{m}_2$ *are odd then* $F(\mathbf{m}, \{3\text{–}2\text{–}1, 3\text{–}1\text{–}2\}, -1) = 0$.

## 4.8 Consecutive Pattern Functions

In this section we will show how to refine schemes to compute weight-enumerators for $\mathcal{S}_n(B)$ for other permutation statistics given by pattern functions. For any $\sigma \in \mathcal{S}_k$, this section outlines how to refine enumeration schemes according to the number of occurrences of the consecutive pattern $\sigma_1\sigma_2\cdots\sigma_k$.

We will restrict our exposition to schemes for classical and for dashed patterns, ignoring the extensions discussed above in sections 4.4 and 4.5. The same arguments apply, but we will not repeat them here.

For pattern $\sigma$, recall we can construct the pattern function $\sigma(\pi)$ to be the number of copies of $\sigma$ appearing in $\pi$. We use this to construct the weight-enumerator[1]

$$F^{\sigma}(n,B,q) := \sum_{\pi\in\mathcal{S}_n(B)} q^{\sigma(\pi)}. \tag{4.12}$$

We will also use $F^{\sigma}(n,B,q)[p]$ to denote the weight-enumerator for $\mathcal{S}_n(B)[p]$, and $F^{\sigma}(n,B,q)[p;w]$ for the weight-enumerator for $\mathcal{S}_n(B)[p;w]$.

We again have the initial condition $\mathcal{S}_n(B)[p] = \{p\}$, which implies $F^{\sigma}(n,B,q)[p] = q^{\sigma(p)}$. Similarly the gap criteria present conditions for when $\mathcal{S}_n(B)[p;w] = \emptyset$, which implies that $F^{\sigma}(n,B,q)[p;w] = 0$. When there are no reversibly deletable indices, we must re-partition the set $\mathcal{S}_n(B)[p]$ into $\mathcal{S}_n(B)[p']$ for children $p'$. This as the same effect on $F^{\sigma}(n,B,q)[p]$ as when we weight according to inversion number, that is, $F^{\sigma}(n,B,q)[p] = \sum_{p'} F^{\sigma}(n,B,q)[p']$ as the sum runs over all children $p'$.

Hence we see that as in the case for INV, the only obstacle in using schemes to compute $F^{\sigma}(n,B,q)[p;w]$ lies in the bijections $d_R$ and their effect on the number of copies of $\sigma$. The general case requires a modification, called "deepening," of the given original scheme, however, which we discuss below in section 4.8.1. Once this modification is completed, then we may compute the refinement as discussed in section 4.8.2.

### 4.8.1 Deepening Schemes

Suppose that for some pattern set $B$ and prefix $p \in \mathcal{S}_k$ the only non-empty reversibly deletable index is $k$. Then for $\pi \in \mathcal{S}_n(B)[p;w]$, the value of $\mathrm{des}(\pi) - \mathrm{des}(d_k(\pi))$ depends on the pattern formed by $\pi_{k-1}\pi_k\pi_{k+1}$ and hence is not constant over $\mathcal{S}_n(B)[p;w]$. The solution is to partition $\mathcal{S}_n(B)[p;w]$ even further, in a process we call *deepening* the

[1] In the notation of Chapter 2, this is the weight-enumerator $F_{\sigma}^{\mathcal{S}_n(B)}(q)$.

scheme. In this section we show this process can always be accomplished for a given scheme. The main result of this section is Theorem 34.

**Definition 33.** *Given enumeration scheme $E$, the* margin *of a triple $(p, G_p, R_p)$ is the number of letters in the prefix after the last reversibly deletable index given by $R_p$, i.e. $|p| - \max R_p$. The scheme $E$ has* margin $t$ *if every triple in $E$ has margin at least $t$.*

For example the triple $(1342, \emptyset, \{1, 3\})$ has margin 1, while the scheme for 1–2–3-avoiding permutations given in (4.1) has margin 0 because of the triple $(12, \{\langle 0, 0, 1\rangle\}, \{2\})$.

**Theorem 34.** *Suppose the pattern set $B$ has an enumeration scheme $E$. Then given a threshold $t \in \mathbb{N}$ there is an enumeration scheme $E'$ with margin $t$. Further, $E'$ is finite if $E$ is finite.*

To prove Theorem 34, we will use two lemmas: one for reversible deletable indices and one for gap vectors.

**Lemma 35.** *Suppose that $R$ is a reversibly deletable set of indices with respect to prefix $p \in \mathcal{S}_k$ for forbidden pattern-set $B$. Then $R$ is also reversibly deletable for any child $p'$, that is, any prefix $p' \in \mathcal{S}_{k+1}$ such that* $\mathrm{red}(p'_1 p'_2 \cdots p'_k) = p$.

*Proof.* Consider the restriction of $d_R : \mathcal{S}_n(B)[p; w] \to \mathcal{S}_{n-|R|}(B)[d_R(p); d_R(w)]$ to the domain $\mathcal{S}_n(B)[p'; w'] \subseteq \mathcal{S}_n(B)[p; w]$. We need to show $d_R$ maps $\mathcal{S}_n(B)[p'; w']$ bijectively onto $\mathcal{S}_{n-|R|}(B)[d_R(p'); d_R(w')]$. If $\hat{\pi} = d_R(\pi)$ for $\pi \in \mathcal{S}_n(B)[p; w]$, then for the last reversibly deletable index $r = \max R$ we see $\mathrm{red}(\pi_{r+1}\pi_{r+2}\cdots\pi_n) = \mathrm{red}(\hat{\pi}_{r+1}\hat{\pi}_{r+2}\cdots\hat{\pi}_n)$. It follows that $\mathcal{S}_n(\emptyset)[p'; w']$ maps bijectively onto $\mathcal{S}_{n-|R|}(\emptyset)[d_R(p'); d_R(w')]$, and the bijectivity of $d_R$ on the original domain of $B$-avoiders shows that $\mathcal{S}_n(\emptyset)[p'; w']$ maps bijectively onto $\mathcal{S}_{n-|R|}(\emptyset)[d_R(p'); d_R(w')]$. $\square$

**Lemma 36.** *Suppose that $v \in \mathbb{N}^k$ is a gap vector with respect to prefix $p \in \mathcal{S}_k$ for forbidden pattern-set $B$, and consider the child $p' \in \mathcal{S}_{k+1}$ such that* $\mathrm{red}(p'_1 \cdots p'_k) = p$ *and $p'_{k+1} = i$. Then any vector of the form*

$$\langle v_1, \ldots, v_{i-1}, a, b, v_{i+1}, \cdots, v_{k+1}\rangle$$

*for $a + b = \max(v_i - 1, 0)$ is gap vector for $p'$.*

*Proof.* Consider $\mathcal{S}_n(B)[p';w']$, where

$$g(w') = \langle g_1, \cdots g_{k+2}\rangle \geq \langle v_1, \ldots, v_{i-1}, a, b, v_{i+1}, \cdots, v_{k+1}\rangle$$

for appropriate $a, b$. Let $w = w_1' w_2' \cdots w_k'$, so $\text{red}(w) = p$ and

$$g(w) = \langle g_1, \ldots, g_{i-1}, g_i + g_{i+1} + 1, g_{i+2}, \cdots, g_{k+2}\rangle$$
$$\geq \langle v_1, \ldots, v_{i-1}, a + b + 1, v_{i+1}, \cdots, v_{k+1}\rangle = v.$$

Therefore $w$ satisfies the gap criterion for $v$, so $\mathcal{S}_n(B)[p;w] = \emptyset$. It follows that $\mathcal{S}_n(B)[p';w'] \subseteq \mathcal{S}_n(B)[p;w]$ is also empty. Hence $\langle v_1, \ldots, v_{i-1}, a, b, v_{i+1}, \cdots, v_{k+1}\rangle$ is a gap vector. □

The transformation in Lemma 36 allows us to use the set $G_p$ of gap vectors for prefix $p$ to form the set of gap vectors for the child $p'$:

$$G_{p'} = \{\langle v_1, \ldots, v_{i-1}, a, b, v_{i+1}, \cdots, v_{k+1}\rangle : v \in G_p, a + b = \max(v_i - 1, 0)\} \qquad (4.13)$$

Note that if we begin with just the basis vectors for $G_p$ and apply this transformation to them we will get a basis for $G_{p'}$ (although not necessarily a minimal one).

We may now assemble the tricks from Lemmas 35 and 36 to prove Theorem 34.

*Proof of Theorem 34.* We will first prove the case for $t = 1$ before moving on to the more general case. We will construct a new scheme $E'$ from a given $E$. For each $(p, G_p, R_p) \in E$ such that $R_p = \emptyset$ or $\max R_p < |p|$, place a copy of $(p, G_p, R_p)$ in $E'$. Next consider $(p, G_p, R_p) \in E$ such that $\max R_p = |p|$. First place $(p, G_p, \emptyset)$ in $E'$. Then for each child $p'$ of $p$, let $G_{p'}$ be the set of gap vectors described in Equation (4.13). Lemma 35 shows that $R_p$ is also a reversibly deletable set for $p'$, so we place the triple $(p', G_{p'}, R_p)$ in $E'$. Now $E'$ contains only triples $(p, G_p, R_p)$ such that $|R_p| \leq |p| - 1$.

It remains to show that $E'$ is a valid scheme. Clearly $(\epsilon, \emptyset, \emptyset) \in E'$, and our replacement above ensures that any triple in $E'$ with $R_p = \emptyset$ has triples for each child of $p$. It remains to show that for any triple $(p, G_p, R_p) \in E'$, the triple corresponding to the prefix $d_{R_p}(p)$ also lies in $E'$. This was true for the original scheme $E$, so we need only check the triples added to form $E'$. Suppose $(p', G_{p'}, R_{p'}) \in E'$ is one of these triples, added as the child of $(p, G_p, R_p) \in E$. By construction, $R_{p'} = R_p$, so $d_{R_{p'}}(p')$ is a child of $d_R(p)$. Since $E$ is a valid scheme, the triple corresponding to $d_R(p)$ lies in $E$ and hence in $E'$. If the triple for $d_{R_{p'}}(p')$ does not already lie in $E'$, then we may use the triple corresponding to $d_R(p)$ to construct a triple for its child $d_{R_{p'}}(p')$ and add that to $E'$. We repeat this process as often as necessary until $E'$ is a valid scheme: this process terminates since it will never require us to add a triple for a prefix of length longer than the original $|p'|$.

We now may move on to the case for general $t$, which requires iterating the preceding argument. Given scheme $E$, we construct scheme $E^{(t)}$ which has has margin $t$, i.e. any triple $(p, G_p, R_p) \in E^{(t)}$, $|p| - \max R_p \geq t$. The preceding paragraph outlines the construction for $E^{(1)}$ from $E = E^{(0)}$. To construct $E^{(t)}$ from $E^{(t-1)}$, one must add the triples for each child $p'$ of a prefix $p$ such that $(p, G_p, R_p) \in E^{(t-1)}$ and $\max R_p > |p| - t$. Since $E^{(t-1)}$ has margin $t-1$ and $(p, G_p, R_p)$ witnesses this margin, the resulting triple $(p', G_{p'}, R_{p'} = R_p)$ has margin $|p'| - \max R_{p'} = |p| + 1 - \max R_p = t$. □

As an example, consider the following deepening of the scheme $E$ for 1–2–3-avoiding permutations as given in equation (4.1) above.

$$E^{(1)} = \Big\{(\epsilon, \emptyset, \emptyset), (1, \emptyset, \emptyset), (12, \{\langle 0,0,1\rangle\}, \emptyset), (21, \emptyset, \{1\}), (123, \{\langle 0,0,0,0\rangle\}, \{2\}), (132, \{\langle 0,0,0,1\rangle\}, \{2\}), (231, \{\langle 0,0,0,1\rangle\}, \{2\})\Big\} \tag{4.14}$$

Observe that the schemes constructed are not guaranteed to be efficient. For example, continuing the process above for $E^{(2)}$ implies $(213, \emptyset, \{1\}) \in E^{(2)}$ . Because of the rise which was added as we went from 21 to its child 213, the vector $\langle 0,0,0,1\rangle$ is a gap vector which the above algorithm does not provide. Missing gap vectors does not make the scheme inaccurate, just inefficient since $s_n(1\text{–}2\text{–}3)[213; w_1w_2w_3]$ is computed via a bijection with $s_{n-1}(1\text{–}2\text{–}3)[12; w_2(w_3-1)]$ rather than using gap criteria to first check if it is empty. If efficiency is desired, one can compute gap vectors for each child $p'$ "from scratch" by using the tools in Vatter.

### 4.8.2 Using Deepened Schemes

Now that we can construct schemes with margin $t$, we will use them to weight-count $\mathcal{S}_n(B)$ according to consecutive patterns with length $t$. We begin with an analogue of Corollary 27, adapted for consecutive patterns.

**Theorem 37.** *Let $\sigma \in \mathcal{S}_{t+1}$ be a consecutive pattern and consider the triple $(p, G, R)$ with margin at least $t$. Then for any $\pi \in \mathcal{S}_n(\emptyset)[p]$, the deletion map $d_R$ has the following effect on the number of copies of $\sigma$:*

$$\begin{aligned} \delta_R^\sigma(\pi) &:= \sigma(\pi) - \sigma(d_R(\pi)) \\ &= \sigma(p) - \sigma(d_R(p)). \end{aligned} \tag{4.15}$$

*In particular, $\delta_R^\sigma$ is constant over all permutations with prefix $p$. Note that unlike the case of the change in inversion number, $\delta_R^\sigma$ may be positive or negative since copies of $\sigma$ may be either created or destroyed by $d_R$.*

*Proof.* Let $r = \max R$. Then since $\sigma$ is a consecutive pattern of length $t$,

$$\sigma(\pi) = \sigma(\pi_1 \cdots \pi_{|p|}) + \sigma(\pi_{|p|-t+1} \cdots \pi_n)$$

This follows from the observation that an occurrence of $\sigma$ is either contained entirely in the prefix $p$, in which case the occurrence is counted among $\sigma(\pi_1 \cdots \pi_{|p|})$, or $\sigma$ contains at least one $\pi_i$ for $i > |p|$, in which case it is counted among $\sigma(\pi_{|p|-t+1} \cdots \pi_n)$. The rightmost copy of $\sigma$ which could disappear upon deleting $\pi_r$ for $r = \max R$ comes from the subword $\pi_r \pi_{r+1} \cdots \pi_{r+t}$ and since $(p, G, R)$ has margin at least $t$, $r + t \leq |p|$. Hence applying $d_R$ does not affect the number of occurrences of $\sigma$ which are not entirely contained within the prefix. Therefore we see that

$$\begin{aligned} \sigma(\pi) - \sigma(d_R(\pi)) &= \sigma(\pi_1 \cdots \pi_{|p|}) - \sigma(d_R(\pi_1 \cdots \pi_{|p|})). \\ &= \sigma(p) - \sigma(d_R(p)) \end{aligned}$$

□

It follows from 37 that if $E$ is a scheme of margin $t$, then for any triple $(p, G, R) \in E$,

$$F^\sigma(n, B, q)[p; w] = q^{\delta_R^\sigma(p)} F^\sigma(n - |R|, B, q)[d_R(p); d_R(w)]. \tag{4.16}$$

Since Theorem 34 tells us that we may always construct a finite margin $t$ scheme given any finite enumeration scheme, so all previously-known schemes may be re-read to instead give weighted counts for $\mathcal{S}_n(B)$.

As an example, consider weight-counting 1–2–3-avoiding permutations according to the number of descents, i.e. $\sigma = 21$. We use the scheme $E^{(1)}$ with margin 1, given in equation (4.14) to get the following recurrences for $F_n[p; w] := F^{21}(n, 1\text{–}2\text{–}3, q)[p; w]$.

$$\sum_{\pi \in \mathcal{S}_n(1\text{–}2\text{–}3)} q^{des(\pi)} = F_n[\epsilon; \epsilon]$$

$$\begin{aligned}
F_n[\epsilon; \epsilon] &= \sum_{a=1}^{n} F_n[1; a] \\
F_n[1; a] &= \sum_{b=1}^{a-1} F_n[12; ab] + \sum_{j=i+1}^{n} F_n[12; ab] \\
F_n[12; ab] &= \sum_{c=1}^{a-1} F_n[231; abc] + \sum_{c=a+1}^{b-1} F_n[132; abc] + \sum_{c=b+1}^{n} F_n[123; abc] \\
F_n[21; ab] &= q\, F_{n-1}[1; b] \text{ (via } d_1) \\
F_n[123; abc] &= 0 \text{ (via gap criteria)} \\
F_n[132; abc] &= \begin{cases} q\, F_{n-1}[12; ac] & b = n \text{ (via } d_2) \\ 0 & \text{otherwise (via gap criteria)} \end{cases} \\
F_n[231; abc] &= \begin{cases} q\, F_{n-1}[21; ac] & b = n \text{ (via } d_2) \\ 0 & \text{otherwise (via gap criteria)} \end{cases}
\end{aligned} \tag{4.17}$$

The first six terms are given in Table 4.6. Barnabei et al. study the distribution of descent number over $\mathcal{S}_n(1\text{–}2\text{–}3)$ via a bijection with Dyck paths in [14]. See OEIS sequence A166073, [54], for more details such as a generating function.

| $n$ | $\sum_{\pi \in \mathcal{S}_n(1\text{–}2\text{–}3)} q^{des(\pi)}$ |
|---|---|
| 1 | $1$ |
| 2 | $1 + q$ |
| 3 | $4\,q + q^2$ |
| 4 | $2\,q + 11\,q^2 + q^3$ |
| 5 | $15\,q^2 + 26\,q^3 + q^4$ |
| 6 | $5\,q^2 + 69\,q^3 + 57\,q^4 + q^5$ |

Table 4.6: Weight-enumerators for $\mathcal{S}_n(1\text{–}2\text{–}3)$ with respect to descent number

**Multiple Statistics**

Several statistics exist as linear combinations of pattern functions. Most well-known is the peak number, the number of indices $i$ such that $\pi_i \geq \pi_{i-1}$ and $\pi_i \geq \pi_{i+1}$. Denote this number $\textsc{pic}(\pi)$.[2] It is quickly seen that any occurrence of a consecutive 132 or 231 is a peak, and so we see that

$$\textsc{pic}(\pi) = (132)(\pi) + (231)(\pi)$$

Since a margin 2 scheme can weight according to either 132 or 231, one hopes it could weight according to both. This is in fact the case. For patterns $\sigma$ and $\tau$, we may construct the weight enumerator[3]

$$F^{(\sigma,\tau)}(n, B, q_1, q_2) = \sum_{\pi \in \mathcal{S}_n(B)} {q_1}^{\sigma(\pi)}\, {q_2}^{\tau(\pi)}.$$

Theorem 37 implies that for consecutive patterns $\sigma$ and $\tau$ of length at most $t+1$, if $(p, G, R)$ has margin at least $t$, then $d_R$ has the following effect on the joint weight enumerator:

$$F^{(\sigma,\tau)}(n, B, q_1, q_2) = {q_1}^{\delta^{\sigma}_R(p)}\, {q_2}^{\delta^{\tau}_R(p)}\, F^{(\sigma,\tau)}(n - |R|, B, q_1, q_2). \tag{4.18}$$

Hence schemes may be used to compute such two-statistic weight enumerators as well. Then the weight enumerator according to peak number is

$$F^{(132,231)}(n, B, q, q) = \sum_{\pi \in \mathcal{S}_n(B)} q^{\textsc{pic}(\pi)}$$

Also note that the inversion number may be computed jointly with a consecutive pattern function by letting $\delta^{\textsc{inv}}_R = \delta_R$ from Corollary 27. Further, we can combine any number of consecutive pattern functions $\sigma^{(1)}, \ldots, \sigma^{(k)}$ to compute their joint distribution over $\mathcal{S}_n(B)$:

$$\begin{aligned} F^{(\sigma^{(1)},\ldots,\sigma^{(k)})}(n, B, q_1, \ldots q_k) &:= \sum_{\pi \in \mathcal{S}_n(B)} {q_1}^{\sigma^{(1)}(\pi)} \cdots {q_k}^{\sigma^{(k)}(\pi)} \\ &= {q_1}^{\delta^{\sigma^{(1)}}_R(p)} \cdots {q_k}^{\delta^{\sigma^{(k)}}_R(p)}\, F^{(\sigma^{(1)},\ldots,\sigma^{(k)})}(n - |R|, B, q_1, \ldots, q_k). \end{aligned}$$

---

[2] Tradition dictates that permutation statistics be abbreviated to exactly three letters. Our notation here works best in French, or at least a French accent.

[3] Of course this definition may be extended for any number of statistics, but we will restrict ourselves to pairs for now.

## 4.9 Conclusions and Future Directions

In this chapter we have developed a method to compute the weight-enumerator for $\mathcal{S}_n(B)$ according to weights given by either inversion number or the number of copies of a given consecutive pattern. These are completely automated, providing recurrences which answer a broad class of enumeration questions about the number of permutations avoiding $B$ with $k$ copies of the pattern (2–1) (inversions) and/or $\ell$ copies of a given consecutive pattern $\sigma$.

Since $\text{INV}(\pi) = (2\text{–}1)(\pi)$, it is reasonable to believe there is analogue of the preceding results to use margin $(t-1)$ schemes to get weighted counts according to statistics of the form $\sigma_1\sigma_2\cdots\sigma_t\text{–}\sigma_{t+1}$. Such a a result would subsume the work in Chapter 2, which only computes distributions over the entire set $\mathcal{S}_{\mathbf{m}}$.

In [74] Pudwell used a similar notion of enumeration schemes to count words avoiding patterns with repeated letters. These schemes, however, partition the $S_{\mathbf{m}}(B)$ according to the locations of [copies of] the letters 1 through $k$ rather than the first $k$ letters. Further, the schemes hinge on a bijection other than $d_R$, instead deleting all copies of a certain letter. Hence Lemma 26 is inapplicable. The similarity in process, however, indicates a refinement according to INV or consecutive pattern functions is still possible.

One can also use the above recurrences to aid exploration of asymptotic distributions of INV over pattern-avoiding sets. It is well known, see for example [44], that INV is asymptotically normal over all of $S_n$. This argument is presented in Chapter 7. With the tools above, one may now explore the asymptotic distributions of INV or other statistics over a set of restricted permutations $\mathcal{S}_n(B)$.

Last, the techniques of Chapter 5 outline a method of deriving functional equations for the generating function $\sum_{n\geq 0} s_n(B)z^n$. The refinements in this chapter are adaptable to compute the generating function $\sum_{n\geq 0} F^{\sigma}(n,B,q)z^n$ for $\sigma = 2\text{–}1$ or a consecutive $\sigma$. A proof-of-concept is provided in section 5.6 of Chapter 5.

# Chapter 5

# The Umbral Transfer Matrix Method and Enumeration Schemes

## 5.1 Introduction

The goal of this chapter is to develop a method to convert a given enumeration scheme for $\mathcal{S}_n(B)$ into a functional equation for the weight enumerator of $\mathcal{S}(B) = \bigcup_{n\geq 1} \mathcal{S}_n(B)$ with weight $W(\pi) = z^{|\pi|}$. In particular, we will develop systems of functional equations for the weight-enumerators of $\mathcal{S}(B)[p] = \bigcup_n \mathcal{S}_n(B)[p]$ for certain prefix patterns $p \in \mathcal{S}_k$ with weight $z^{|\pi|}{x_1}^{\pi_1} \cdots {x_k}^{\pi_k}$, that is,

$$F_p(z, x_1, \ldots, x_k) := \sum_{\pi \in \mathcal{S}_n(B)[p]} z^{|\pi|}{x_1}^{\pi_1} \cdots {x_k}^{\pi_k}$$

The primary tool will be operators acting on the $F_p(z, \bar{x})$ in ways dictated by the scheme. These operators, and in fact the whole approach in general, is inspired and informed by the Umbral Transfer Matrix Method as introduced in [101]. Indeed, if one uses the methods outlined below for the enumeration scheme for permutations avoiding the consecutive pattern 321 as developed in Chapter 3, the steps taken toward a system of functional equation closely parallel those taken in Example 3 of [101] regarding compositions with no double-descents. Further, [15] develops the application of the Umbral Transfer Matrix Method to consecutive pattern problems. They exhibit a special case combining section 5.2 below with section 4.8 of Chapter 4 to find the number of permutations with $k_1, k_2, \ldots, k_t$ occurrences of given consecutive patterns $\sigma^{(1)}, \sigma^{(2)}, \ldots, \sigma^{(t)} \in \mathcal{S}_\ell$.

This chapter considers only classical patterns, and we will omit dashes to simplify notation. Hence the dashed pattern 1–3–2 is written simply 132. The results discussed here will indeed transfer to the schemes for dashed patterns as developed in Chapter 3, but all examples are chosen from the classical cases.

Enumeration schemes are discussed in section 1.3.4 of the Introduction, and discussed further in Chapters 3 and 4. Here we summarize the pertinent information. An enumeration scheme $E$ for a forbidden set of patterns $B$ is a set of triples $(p, G, R)$ where $p \in \mathcal{S}_k$ is a prefix pattern, $G \subseteq \mathbb{N}^{k+1}$ is a basis for a set of gap vectors, and $R$ is a reversibly deletable set for $p$. When $R$ is non-empty, we have a nontrivial map $d_R : \mathcal{S}_n(B)[p; w] \to \mathcal{S}_{n-|R|}(B)[d_R(p); d_R(w)]$ which is a bijection whenever $w$ fails all gap vector criteria set forth by $G$. We will invert these deletion maps to form insertions, the analysis of which forms the crux of our method.

This chapter consists primarily of examples which illustrate the method. In section 5.2 we present the most basic example by considering unrestricted permutations. We move into 123-avoiding permutations in section 5.3. We address other forbidden sets $B \subseteq \mathcal{S}_3$ in section 5.4. As a more complicated example, we consider 1234-avoiding permutations in section 5.5. Section 5.5 also provides a sketch of the general method. Section 5.6 gives an example of how the refinements discussed in Chapter 4 can also be incorporated into these functional equations.

## 5.2 Unrestricted Permutations

As a motivating example, let us consider the set of all permutations and refine our count according to the first letter, $\pi_1$. Specifically we are interested in the following generating function:

$$F(z, x) = \sum_{n=1}^{\infty} \sum_{\pi \in \mathcal{S}_n} z^n x^{\pi_1}$$

This is the weight-enumerator of $\mathcal{S} := \bigcup_{n \geq 1} \mathcal{S}_n$ where the weight of a given permutation is $W(\pi) = z^{|\pi|} x^{\pi_1}$ and the weight of a set $S$ is $W(S) = \sum_{\pi \in S} W(\pi)$. In this case, we can quickly see that

$$F(z, x) = \sum_{n \geq 1} \sum_{k=1}^{n} (n-1)! z^n x^k$$

but for the sake of illustration we will assume we do not know this.

For unrestricted permutations, we have the following depth-2 scheme:

$$E = \big\{ (\epsilon, \emptyset, \emptyset), (1, \emptyset, \emptyset), (12, \emptyset, \{1\}), (21, \emptyset, \{1\}) \big\}$$

While this is not the scheme of minimal depth[1], it serves to illustrate the method. The term $(12, \emptyset, \{1\})$ indicates that given a permutation beginning with a rise, we may delete the first letter $\pi_1 = a$ to form a bijection between $\{\pi \in \mathcal{S}_{n+1} : \pi_1 < \pi_2, \pi_1 = a\}$ and $\mathcal{S}_n$. Similarly the term $(21, \emptyset, \{1\})$ indicates we may delete the first letter to form a bijection between $\{\pi \in \mathcal{S}_{n+1} : \pi_1 > \pi_2, \pi_1 = a\}$ and $\mathcal{S}_n$. Reversing these maps we see that given a permutation $\pi$, we may insert any letter $b \in \{1, 2 \ldots, n+1\}$ at the front of the permutation, and incrementing each $\pi_i \geq b$ by one. Call this map $i_b$. For example $i_3(4231) = 35241$.

Now consider the effect $i_b$ has on the weight of a permutation $\pi$. If $W(\pi) = z^n x^a$, i.e. $|\pi| = n$ and $\pi_1 = a$, then $W(i_b(\pi)) = z^{n+1}x^b$. If we take the union over all $b \in \{1, \ldots, a\}$, the set of images is the set of all permutations of length $n+1$ which start with $b(a+1)$. Note that $b(a+1) \sim 12$. This set of images has weight

$$\begin{aligned}
W\Big(\bigcup_{b=1}^{a}\{i_b(\pi)\}\Big) &= \sum_{b=1}^{a} W(i_b(\pi)) \\
&= \sum_{b=1}^{a} z^{n+1}x^b \\
&= z^{n+1}\frac{x^{a+1}-x}{x-1} \\
&= \frac{zx}{x-1}z^n x^a - \frac{zx}{x-1}z^n
\end{aligned}$$

Hence the maps $i_b$ imply the following operator on the monomial $z^n x^a$:

$$\mathcal{P}_{12} : z^n x^a \mapsto \sum_{b=1}^{a} z^{n+1}x^b = \frac{zx}{x-1}z^n x^a - \frac{zx}{x-1}z^n$$

Extending linearly, the operator $\mathcal{P}_{12}$ acts on a formal power series $G(z,x)$ as follows:

$$\mathcal{P}_{12} \circ G(z,x) = \frac{zx}{x-1}G(z,x) - \frac{zx}{x-1}G(z,1) \tag{5.1}$$

In particular, if $F(z,x)$ is the weight enumerator of $\mathcal{S}$, then $\mathcal{P}_{12} \circ F(z,x)$ is the weight enumerator of the set of permutations which begin with a rise.[2]

[1] The minimal-depth scheme is $\{(\epsilon, \emptyset, \emptyset), (1, \emptyset, \{1\})\}$. We have deepened the scheme via Theorem 34 from Chapter 4.

[2] Any permutation $\pi_1 \cdots \pi_n$ arises uniquely from $i_{\pi_1}(\mathrm{red}(\pi_2 \cdots \pi_n))$, so there is no over- or under-counting.

Similarly, we may consider images of $\pi$ with $W(\pi) = z^n x^a$ under the maps $i_b$ for $b \in \{a+1, \ldots, n+1\}$. Each resulting permutation has $n+1$ and begins with a descent. As we did for $\mathcal{P}_{12}$, the set of images of a single permutation with weight $z^n x^a$ implies the following action on monomials:

$$\mathcal{P}_{21} : z^n x^a \mapsto \sum_{b=a+1}^{n+1} z^{n+1} x^b = z^{n+1} \frac{x^{n+2} - x^{a+1}}{x-1}.$$

This extends linearly to act on a formal power series $G(z, x)$ as follows:

$$\mathcal{P}_{21} \circ G(z, x) = \frac{zx^2}{x-1} G(zx, 1) - \frac{zx}{x-1} G(z, x). \tag{5.2}$$

Again, if $F(z, x)$ is the weight enumerator of $\mathcal{S}$, then $\mathcal{P}_{21} \circ F(z, x)$ is the weight enumerator of the set of permutations which begin with a descent.

Next, observe that we have the following decomposition of $\mathcal{S}$ into disjoint subsets.

$$\mathcal{S} = \{1\} \cup \{\pi \in \mathcal{S} : |\pi| \geq 2, \pi_1\pi_2 \sim 12\} \cup \{\pi \in \mathcal{S} : |\pi| \geq 2, \pi_1\pi_2 \sim 21\}$$

By taking the weights of each side we see that

$$W(\mathcal{S}) = W(\{1\}) + W(\{\pi \in \mathcal{S} : |\pi| \geq 2, \pi_1\pi_2 \sim 12\}) + W(\{\pi \in \mathcal{S} : |\pi| \geq 2, \pi_1\pi_2 \sim 21\})$$

$$F(z, x) = zx + \mathcal{P}_{12} \circ F(z, x) + \mathcal{P}_{21} \circ F(z, x)$$

Applying the operators as determined in (5.1) and (5.2) gives us the following functional equation after a bit of simplification

$$F(z, x) = zx + \frac{zx^2}{x-1} F(zx, 1) - \frac{zx}{x-1} F(z, 1) \tag{5.3}$$

Thus we have converted the scheme into a functional equation for the generating function $F(z, x)$. Usually we would be interested in $F(z, 1)$, treating the $x$ as a catalytic variable. As will be seen in other examples, it is difficult in practice to get a functional equation purely in terms of $z$, however.

Note that all terms on the right hand side of Equation (5.3) involve an extra $z$ term, which means we can use (5.3) to get finite approximations of $F(z, x)$ accurate up to degree $z^N$. If $F(z, x) = \sum_{n=1}^{\infty} \sum_{a=1}^{n} f(n, a) z^n x^a$ and $F_N(z, x) = \sum_{n=1}^{N} \sum_{a=1}^{n} f(n, a) z^n x^a$ is a truncation of $F(z, x)$, then it is clear that $F_1(z, x) = zx$ and

$$F_{N+1}(z, x) = zx + \frac{zx^2}{x-1} F_N(zx, 1) - \frac{zx}{x-1} F_N(z, 1).$$

In this way we may compute approximations of $F(z,x)$ in polynomial time to a degree bounded only by computation power and patience.

## 5.3 Avoiding 123

We now adapt the arguments of the previous section to count restricted permutations. In this section we will focus on permutations avoiding the classical pattern 123, leaving other restrictions to later sections.

We are considering the weight-enumerator of $\mathcal{S}(123) := \bigcup_{n\geq 1} \mathcal{S}_n(123)$ where the weight of a given permutation is $W(\pi) = z^{|\pi|}x^{\pi_1}$.

$$F(z,x) = W(\mathcal{S}(123)) = \sum_{n\geq 1} \sum_{\pi\in\mathcal{S}_n(123)} z^n x^{\pi_1}$$

We also use the same decomposition of $\mathcal{S}(123)$ according to the pattern formed by the first two letters:

$$\mathcal{S}(123) = \{1\} \cup \{\pi \in \mathcal{S}(123) : |\pi| \geq 2, \pi_1\pi_2 \sim 12\} \cup \{\pi \in \mathcal{S}(123) : |\pi| \geq 2, \pi_1\pi_2 \sim 21\}.$$

Taking the weights suggests a functional equation

$$\begin{aligned} W(\mathcal{S}(123)) = zx &+ W(\{\pi \in \mathcal{S}(123) : |\pi| \geq 2, \pi_1\pi_2 \sim 12\}) \\ &+ W(\{\pi \in \mathcal{S}(123) : |\pi| \geq 2, \pi_1\pi_2 \sim 21\}) \end{aligned}$$

We will proceed by determining operators $\mathcal{P}_{12}$ and $\mathcal{P}_{21}$ so that

$$\mathcal{P}_{12} \circ F(z,x) = W(\{\pi \in \mathcal{S}(123) : |\pi| \geq 2, \pi_1\pi_2 \sim 12\})$$
$$\mathcal{P}_{21} \circ F(z,x) = W(\{\pi \in \mathcal{S}(123) : |\pi| \geq 2, \pi_1\pi_2 \sim 21\})$$

From equation 1.5 in the Introduction we have the following enumeration scheme for 123-avoiding permutations, $E_{123}$:

$$E_{123} = \{(\epsilon, \emptyset, \emptyset), (1, \emptyset, \emptyset), (12, \{\langle 0,0,1\rangle\}, \{2\}), (21, \emptyset, \{1\}).$$

By comparison with the scheme $E$ for unrestricted permutations, we see a similar structure where permutations starting 12 or 21 reduce to a permutation with prefix 1 via the deletion of a single letter. Further, the term $(21, \emptyset, \{1\}) \in E_{123}$ also appears in

$E$. It follows that we may insert the letter $b$ at the first index of a permutation with weight $z^n x^a$ for any $a+1 \leq b \leq n+1$, just as in the unrestricted case. Hence we again have the operator generated by the following action on monomials:

$$\mathcal{P}_{21} : z^n x^a \mapsto \sum_{b=a+1}^{n+1} z^{n+1} x^b = z^{n+1} \frac{x^{n+2} - x^{a+1}}{x-1},$$

which extends linearly as

$$\mathcal{P}_{21} \circ G(z,x) = \frac{zx^2}{x-1} G(zx,1) - \frac{zx}{x-1} G(z,x).$$

We now consider the term $T = (12, \{\langle 0,0,1 \rangle\}, \{2\}) \in E_{123}$. The presence of non-trivial gap criteria, as well as the different deletion map, means that the corresponding operator $\mathcal{P}_{12}$ will differ from its analogue in the unrestricted case. Consider for the moment the triple $T' = (12, \emptyset, \{2\})$, which is $T$ with trivial gap vector criteria. Given $\pi \in \mathcal{S}_{n+1}(123)[12]$, $T'$ tells us the deletion map $d_2 : \pi_1\pi_2\pi_3 \cdots \pi_{n+1} \mapsto \text{red}(\pi_1\pi_3 \cdots \pi_{n+1})$ is a bijection when restricting to 123-avoiding permutations. This map has inverse $i_b$ for $b = \pi_2 > \pi_1$, and applying $i_b$ to a permutation with weight $z^n x^a$ yields a permutation with weight $z^{n+1} x^a$ since the first letter is preserved. We sum over all possible letters which may be inserted to form a 12 pattern, $b \in \{a+1, \ldots, n+1\}$, to arrive at the following operator on monomials:

$$\mathcal{P}'_{12} : z^n x^a \mapsto \sum_{b=a+1}^{n+1} z^{n+1} x^a.$$

This would be the operator $\mathcal{P}_{12}$ if not for the gap criterion $\langle 0,0,1 \rangle$. We now restore the gap criteria and return to original triple $T = (12, \{\langle 0,0,1 \rangle\}, \{2\}) \in E_{123}$. The gap criteria summarize tell us that we may only insert $b = n+1$ without forming a 123 (since otherwise $\pi_1\pi_2(n+1)$ would form a 123 pattern). Hence we must modify $\mathcal{P}'_{12}$ to get the true operator:

$$\mathcal{P}_{12} : z^n x^a \mapsto \sum_{b=n+1}^{n+1} z^{n+1} x^a = z^{n+1} x^a$$

which extends linearly to

$$\mathcal{P}_{12} \circ G(z,x) = zG(z,x)$$

Now that we have $\mathcal{P}_{12}$ and $\mathcal{P}_{21}$ we can write the following functional equation for $F(z,x)$:

$$\begin{aligned} F(z,x) &= xz + \mathcal{P}_{12} \circ F(z,x) + \mathcal{P}_{21} \circ F(z,x) \\ &= xz + zF(z,x) + \frac{zx^2}{x-1} F(zx,1) - \frac{zx}{x-1} F(z,x). \end{aligned}$$

Typically one does not know $F(z,1)$ *a priori*. In this particular case, however, we know $F(z,1) = \frac{1-\sqrt{1-4z}}{2z} - 1$ since we know that $s_n(123) = \frac{1}{n+1}\binom{2n}{n}$, the Catalan numbers. Hence we can get the closed form

$$F(z,x) = \frac{x - 2zx - x\sqrt{1-4zx}}{2(z+x-1)}.$$

## 5.4 Other Schemes of Depth 2

In this section we focus on several special cases where there is a depth-2 enumeration scheme for permutations avoiding a set of forbidden patterns $B$. In each case we wish to compute the weight-enumerator of $\mathcal{S}(B) := \bigcup_{n\geq 1} \mathcal{S}_n(B)$ by first partitioning according to the prefix pattern of the first two letters, i.e.,

$$\mathcal{S}(B) = \{1\} \cup \{\pi \in \mathcal{S}(B) : |\pi| \geq 2, \pi_1\pi_2 \sim 12\} \cup \{\pi \in \mathcal{S}(B) : |\pi| \geq 2, \pi_1\pi_2 \sim 21\}.$$

We then consider the weight enumerator $F(z,x) = W(\bigcup_{n\geq 1} \mathcal{S}_n(B))$, which will satisfy a functional equation of the form

$$F(z,x) = xz + \mathcal{P}_{12} \circ F(z,x) + \mathcal{P}_{21} \circ F(z,x)$$

for some operators $\mathcal{P}_{12}$ and $\mathcal{P}_{21}$. Each subsection below outlines the construct of these operators for the specific pattern. The examples were chosen to increase in complexity, each example introducing a new "twist."

### 5.4.1 Avoiding 132

To avoid 132, we begin with the enumeration scheme

$$E_{132} = \{(\epsilon, \emptyset, \emptyset), (1, \emptyset, \emptyset), (12, \{\langle 0,1,0\rangle\}, \{1\}), (21, \emptyset, \{1\})\}$$

As in the unrestricted case, the triple $(21, \emptyset, \{1\}) \in E_{132}$ again translates to the operator

$$\mathcal{P}_{21} \circ G(z,x) = \frac{zx^2}{x-1} G(zx,1) - \frac{zx}{x-1} G(z,x).$$

The triple $(12, \{\langle 0,1,0 \rangle\}, \{1\})$ now translates to the operator $\mathcal{P}_{12}$ which acts on monomials as

$$\mathcal{P}_{12} : z^n x^a \mapsto \sum_{b=a}^{a} z^{n+1} x^b = z^{n+1} x^a$$

and extends linearly as $\mathcal{P}_{12} \circ G(z,x) = zG(z,x)$.

Observe these operators are identical to those from the 123-avoiding permutation, leading to the same generating function. This provides another proof that the number of permutations starting with $a$ which avoid 123 is equal to the number of permutations starting with $a$ which avoid 132. This is proven bijectively by Simion and Schmidt in [81].

### 5.4.2 Avoiding 123 and 132

We start with a depth-2 enumeration scheme[3] for permutations avoiding 123 and 132 simultaneously.

$$E_{\{123,132\}} = \{(\epsilon, \emptyset, \emptyset), (1, \emptyset, \emptyset), (12, \{\langle 0,1,0 \rangle, \langle 0,0,1 \rangle\}, \{1\}), (21, \emptyset, \{1\})$$

and the implied functional equation for $F(z,x) = W(\mathcal{S}(123, 132))$

$$F(z,x) = zx + \mathcal{P}_{12} \circ F(z,x) + \mathcal{P}_{21} \circ F(z,x)$$

for some operators $\mathcal{P}_{12}$ and $\mathcal{P}_{21}$.

Once again we have the term $(21, \emptyset, \{1\}) \in E_{\{123,132\}}$, which gives us the operator

$$\mathcal{P}_{21} \circ G(z,x) = \frac{zx^2}{x-1} G(zx,1) - \frac{zx}{x-1} G(z,x).$$

We now focus on the triple $T = (12, \{\langle 0,1,0 \rangle, \langle 0,0,1 \rangle\}, \{1\})$ to get $\mathcal{P}_{12}$. For the first time we are seeing two basis gap vectors for the same prefix. Since we insert the first

[3]There is a depth-1 scheme, $\{(\epsilon, \emptyset, \emptyset), (1, \{\langle 0,2 \rangle\}, \{1\})\}$. If one uses the weight $W_\epsilon(\pi) = z^{|\pi|}$ and $F(z) = W_\epsilon(\mathcal{S}(123, 132))$, this scheme implies that $F(z)$ satisfies the functional equation $F(z) = W_\epsilon(1) + \mathcal{P}_1 F(z)$ where $\mathcal{P}_1$ is an operator corresponding to the map of inserting a letter $b \in \{n, n+1\}$ at the front of a permutation of length $n > 1$, i.e., $\mathcal{P}_1 : z^n \mapsto 2z^{n+1}$. Hence it can be seen that $F(z) = z + 2zF(z)$, and so $F(z) = \frac{z}{1-2z}$.

letter, $b$, the weight of a single permutation changes from $z^n x^a$ to $z^{n+1}x^b$. Since we may only choose $b$ which form a 12 prefix and create permutations failing all gap vector criteria, we can expect $\mathcal{P}_{12}$ to act on monomials in the form $z^n x^a \mapsto \sum_{b=b_0}^{b_1} z^{n+1}x^b$ for some $1 \le b_0 \le b_1 \le a$. Let $\pi'$ be the permutation obtained by inserting $b$ at the start of $\pi$ with weight $z^n x^a$. Then ${\pi'}_1 = b < {\pi'}_2 = a+1$. The gap vector $\langle 0,1,0\rangle$ tells us that ${\pi'}_2 - {\pi'}_1 - 1 < 1$, or else $\pi'$ will contain a forbidden pattern (in particular, 132); thus we need $b = a$ and so $b_0 = b_1 = a$. Furthermore, the gap vector $\langle 0,0,1\rangle$ tells us that ${\pi'}_2 = n+1$ or else $\pi'$ will contain a forbidden pattern (in particular, 123). Combining this information tells us that $\mathcal{P}_{12}$ acts on monomials in the following way:

$$\mathcal{P}_{12} : z^n x^a \mapsto \begin{cases} z^{n+1}x^a & a = n \\ 0 & a \neq n. \end{cases}$$

This extends linearly to any formal power series $G(z,x)$, although the piecewise nature of $\mathcal{P}_{12}$ makes its action awkward to write in this context. Luckily we may simulate the action of $\mathcal{P}_{12}$ as follows. We know that our $F(z,x)$ has the special property that the coefficient of $z^n x^a$ is nonzero only if $1 \le a \le n$. Suppose $G(z,x)$ is an arbitrary formal power series such that the only nontrivial terms of $G(z,x)$ have the form $z^n x^a$ for $n \ge a$. Then the nontrivial terms of $G(z\,u, x\,u^{-1})$ have the form $u^{n-a}z^n x^a$, and so specializing to $u = 0$ leaves only those terms where $n = a$. Hence we see that the action of $\mathcal{P}_{12}$ on such a formal power series $G(z,x)$ is realized by the following change of variables and evaluation.

$$\mathcal{P}_{12} \circ G(z,x) = z\,G(z\,u, x\,u^{-1})\big|_{u=0} \tag{5.4}$$

Now that we have our operators, we get the functional equation

$$F(z,x) = zx + z\,F(z\,u, x\,u^{-1})\big|_{u=0} + \frac{zx^2}{x-1}F(zx,1) - \frac{zx}{x-1}F(z,x). \tag{5.5}$$

This functional equation provides both the data to conjecture (via successively accurate approximations) *and the proof of correctness* (via confirming the conjectured solution indeed satisfies the functional equation and initial conditions) for the closed form

$$F(z,x) = \frac{zx + z^2x - z^2\,x^2}{1-2zx}.$$

This example may be generalized to consider permutations avoiding $\{132, 12\cdots k\}$ for a fixed $k \geq 3$. This has the enumeration scheme[4]

$$E_k = \{(\epsilon, \emptyset, \emptyset), (1, \emptyset, \emptyset), (12, \{\langle 0,1,0\rangle, \langle 0,0,k-2\rangle\}, \{1\}), (21, \emptyset, \{1\})$$

The operator $\mathcal{P}_{21}$ remains unchanged from the $E_{\{123,132\}}$ case, since $(21, \emptyset, \{1\}) \in E_k$ for any $k$. For the operator $\mathcal{P}_{12}$ the gap vector $\langle 0,0,k-2\rangle$ implies the following action on monomials:

$$\mathcal{P}_{12} : z^n x^a \mapsto \begin{cases} z^{n+1}x^a & n-a \leq k-3 \\ 0 & n-a > k-3. \end{cases}$$

We will write $\mathcal{P}_{12}$ in terms of the following operator:

$$\mathcal{Q}_i : z^n x^a \mapsto \begin{cases} z^n x^a & n-a = i \\ 0 & n-a > k-3. \end{cases}$$

Then $\mathcal{P}_{12} = \sum_{i=0}^{k-3} z\,\mathcal{Q}_i$, so it suffices to find a change of variables which acts like $\mathcal{Q}_i$ on formal power series. We resort to the same change of variables $z \to zu$ and $x \to xu^{-1}$ so that each term becomes $z^n x^a u^{n-a}$, and then differentiate $i$ times with respect to $u$ before setting $u = 0$. If $n - a < i$, then the repeated differentiation will annihilate that term, and if $n - a > i$ the substitution $u = 0$ will annihilate that term. If $n - a = i$, we are left with $(n-a)!z^n x^a$. Hence we have arrived at the following formulation for $\mathcal{Q}_i$ as it acts on a formal power series $G(z, x)$:

$$\mathcal{Q}_i \circ G(z,x) = \tfrac{1}{i!} D_u^i \big[G(zu, xu^{-1})\big]\Big|_{u=0}$$

where $D_u$ represents differentiation with respect to $u$. Thus when we let $\mathcal{P}_{12} \circ G(z,x) = \sum_{i=0}^{k-3} z\,\mathcal{Q}_i \circ G(z,x)$, we have our two desired operators.

As a side note, it is worthwhile to note the construction of an operator $\mathcal{R}_i$ similar to $\mathcal{Q}_i$. Suppose $\mathcal{R}_i$ acts as follows on monomials

$$\mathcal{R}_i : z^n x^a \mapsto \begin{cases} z^n x^a & a = i \\ 0 & a \neq i. \end{cases}$$

[4] The scheme for $\{132, 12\cdots k\}^c = \{312, k\cdots 21\}$ is also considered in section 4.7.

We may simulate the action of $\mathcal{R}_i$ on a formal power series $G(z,x)$ by first differentiating $G(z,x)$ $i$ times with respect to $x$ and then substitute $x = 0$. This annihilates any term $z^n x^a$ for $a < i$ via differentiation and annihilates any term $z^n x^a$ for $a > i$ by substitution. We are left with terms of the form $i!z^n$, which are a factor of $\frac{x^i}{i!}$ away from the desired quantity. Thus we see we have the following useful operator:

$$\mathcal{R}_i \circ G(z,x) = \frac{x^i}{i!} D_x^i \big[G(z,x)\big]\Big|_{x=0}$$

where $D_x$ represents differentiation with respect to $x$. For example, an operator $\mathcal{P}$ which maps $z^n x^a$ to $z^{n+1} x^a$ if $a \leq k$ and to 0 otherwise can be written in the form $\mathcal{P} = \sum_{i=1}^{k} z\mathcal{R}_i$. The $\mathcal{R}_i$ operator appears in the construction of $\mathcal{P}_{12}$ for the triple $(12, \{\langle 1,0,0\rangle, \langle 0,1,0\rangle\}, \{1\})$, where $\mathcal{P}_{12} = z\mathcal{R}_i$. This triple appears in the scheme for permutations avoiding $B = \{132, 231\}$ in Table 5.1.

### 5.4.3 Avoiding 123 and 231

We begin with an enumeration scheme for permutations avoiding 123 and 231 simultaneously.

$$E_{\{123,231\}} = \{(\epsilon, \emptyset, \emptyset), (1, \emptyset, \emptyset), (12, \{\langle 1,0,0\rangle, \langle 0,0,1\rangle\}, \{1\}), (21, \{\langle 0,1,1\rangle\}, \{1\})$$

and the implied functional equation for $F(z,x) = W(\mathcal{S}(123, 213))$

$$F(z,x) = xz + \mathcal{P}_{12} \circ F(z,x) + \mathcal{P}_{21} \circ F(z,x)$$

for some operators $\mathcal{P}_{12}$ and $\mathcal{P}_{21}$.

To determine $\mathcal{P}_{12}$, we consider the permutation $\pi$ with weight $z^n x^a$ and insert $b$ into the first position. The gap vector condition $\langle 0,0,1\rangle$ tells us that $a = n$ or else the resulting permutation contains a forbidden pattern (in particular, a copy of 123). Thus we see $\mathcal{P}_{12}$ has the form

$$\mathcal{P}_{12} : z^n x^a \mapsto \begin{cases} \sum_{b=b_0}^{b_1} z^{n+1} x^b & a = n \\ 0 & a \neq n \end{cases}$$

for some $1 \leq b_0 \leq b_1 \leq a$. The gap vector condition $\langle 1,0,0\rangle$ tells us that $b_0 = b_1 = 1$, and so $\mathcal{P}_{12} \circ z^n x^n = z^{n+1}x^1$. Hence

$$\begin{aligned}\mathcal{P}_{12} &= zx\,\mathcal{Q}_0\Big|_{x=1} \\ &= zx\,F(zu, xu^{-1})\Big|_{\substack{u=0\\x=1}}\end{aligned}$$

Moving on to $\mathcal{P}_{21}$, we observe for the first time a gap vector criterion with two nonzero entries, $\langle 0,1,1\rangle$. This implies that given $\pi$ with weight $z^n x^a$ we must insert $b > a$ at the first index so that $b - \pi_1 - 1 < 1$ or $(n+1) - b - 1 < 1$, i.e., we may only insert $b = a+1$ or $b = n+1$. Hence $\mathcal{P}_{21}$ must act on monomials as follows:

$$\begin{aligned}\mathcal{P}_{21} : z^n x^a \mapsto &\begin{cases} z^{n+1}\,x^{a+1} + z^{n+1}\,x^{n+1} & a < n \\ z^{n+1}\,x^{n+1} & a = n \end{cases} \\ &= z^{n+1}\,x^{a+1} + z^{n+1}\,x^{n+1} - \begin{cases} 0 & a < n \\ z^{n+1}\,x^{a+1} & a = n \end{cases}\end{aligned}$$

Using the operator $\mathcal{Q}_0$ to handle this conditioning on $a$ as in the previous section, we see how to extend $\mathcal{P}_{21}$ to formal power series $G(z,x)$ where the only nonzero coefficients are for $z^n\,x^a$ for $1 \leq a \leq n$. The operator $\mathcal{P}_{21}$ acts as follows:

$$\mathcal{P}_{21} \circ G(z,x) = zxG(z,x) + zxG(zx,1) - zx\,G(z\,u, x\,u^{-1})\Big|_{u=0}. \tag{5.6}$$

We can now apply $\mathcal{P}_{12}$ and $\mathcal{P}_{21}$ to $F(z,x)$ to get the desired functional equation:

$$F(z,x) = zx + zxF_x(z,0) + zxF(z,x) + zxF(zx,1) - zx\,F(z\,u, x\,u^{-1})\Big|_{u=0}. \tag{5.7}$$

Again, this functional equation can be used to conjecture, and then prove, the closed form

$$F(z,x) = \frac{zx - 2z^2x^2 + 2z^3x^3 + z^4x^4}{(1-zx)^3(1-z)}. \tag{5.8}$$

### 5.4.4 Avoiding $\{123, 132, 213\}$

When avoiding $B = \{123, 132, 213\}$, we begin with the scheme:

$$\{(\epsilon, \emptyset, \emptyset), (1, \{\langle 0,2\rangle\}, \emptyset), (12, \{\langle 0,1,0\rangle, \langle 0,0,1\rangle\}, \{1\}), (21, \{\langle 0,0,1\rangle, \langle 0,2,0\rangle\}, \{1\})\} \tag{5.9}$$

Observe that the prefix 1 has a gap vector associated to it. This may be ignored, however, since the gap vector criteria implied by $\langle 0, 2\rangle$ is redundant given the criteria for by its children 12 and 21.

For the operator $\mathcal{P}_{12}$, observe that the triple $(12, \{\langle 0,1,0\rangle, \langle 0,0,1\rangle\}, \{1\})$ also appeared in the scheme for permutations avoiding $\{123, 132\}$. Thus the same operator applies:

$$\mathcal{P}_{12} \circ G(z,x) = z\, G(z\,u, x\,u^{-1})\big|_{u=0} \tag{5.10}$$

Now focus on the operator $\mathcal{P}_{21}$. We will insert $b > a$ at the first position of a permutation with weight $z^n x^a$. The gap vector $\langle 0,0,1\rangle$ tells us that $b = n+1$ is the only choice if we are to avoid $B$. The gap vector $\langle 0,2,0\rangle$ tells us that $b - a - 1 < 2$, and so $n - a < 2$. Combining these we get the following action on monomials:

$$\mathcal{P}_{21} : z^n x^a \mapsto \begin{cases} z^{n+1}x^{n+1} & n - a \leq 1 \\ 0 & n - a > 1. \end{cases}$$

From this we get the following extension to formal power series:

$$\begin{aligned} \mathcal{P}_{21} \circ G(z,x) &= zx\big(\mathcal{Q}_0 \circ G + \mathcal{Q}_1 \circ G\big)\Big|_{\substack{x=1\\ z=zx}} \\ &= zx\left( G(zu, xu^{-1})\Big|_{\substack{u=0\\ x=1\\ z=zx}} + D_u G(zu, xu^{-1})\Big|_{\substack{u=0\\ x=1\\ z=zx}} \right). \end{aligned}$$

Hence we see that the weight enumerator $F(z,x)$ satisfies the functional equation

$$F(z,x) = zx + z\,F(z\,u, x\,u^{-1})\big|_{u=0} + zx\left( F(zu, xu^{-1})\Big|_{\substack{u=0\\ x=1\\ z=zx}} + D_u F(zu, xu^{-1})\Big|_{\substack{u=0\\ x=1\\ z=zx}} \right).$$

We can use this functional equation to get the first few terms of $F(z,x)$,

$$F(z,x) = zx + (x + x^2)z^2 + (x^2 + 2x^3)z^3 + (2x^3 + 3x^4)z^4 + \cdots .$$

Specializing $x = 1$ we have have what looks like the generating function for the Fibonacci numbers, $F(z,1) = \frac{1}{1-z-z^2} - 1$. We can also conjecture that $F(z,x) = zx(F(zx,1) + 1) + z^2x(F(zx,1)+1)$. Combining these we can conjecture that

$$F(z,x) = \frac{zx + z^2x}{1 - zx - z^2x^2}.$$

We then check that this conjectured form indeed satisfies the functional equation implied by $\mathcal{P}_{12}$ and $\mathcal{P}_{21}$ to verify that it is correct.

### 5.4.5 Avoiding Other $B \subseteq \mathcal{S}_3$

The preceding arguments are representative of the arguments used for other sets of classical patterns $B \subseteq \mathcal{S}_3$. When a finite scheme exists, it has the form

$$E_B = \{(\epsilon, \emptyset, \emptyset), (1, \emptyset, \emptyset), (12, G_{12}, R_{12}), (21, G_{21}, R_{21})$$

where $R_{12}$ and $R_{21}$ are both nonempty and all vectors appearing in $G_{12}$ and $G_{21}$ have only 0 or 1 as entries. The exceptions to this are when $B \supseteq \{123, 321\}$, but in this case $F(z, x)$ is a polynomial by the Erdős-Szekeres Theorem. In this section we present a summary of the remaining cases, choosing one representative from each trivial Wilf class.

| $B$ | $G_{12}$ | $R_{12}$ | $\mathcal{P}_{12} \circ F(z,x)$ |
|---|---|---|---|
| $\{123\}$ | $\{\langle 0,0,1\rangle\}$ | $\{2\}$ | $zF(z,x)$ |
| $\{132\}$ | $\{\langle 0,1,0\rangle\}$ | $\{1\}$ | $zF(z,x)$ |
| $\{123,132\}$ | $\{\langle 0,1,0\rangle, \langle 0,0,1\rangle\}$ | $\{1\}$ | $z\,F(z\,u, x\,u^{-1})\big\vert_{u=0}$ |
| $\{123,213\}$ | $\{\langle 0,0,1\rangle\}$ | $\{2\}$ | $zF(z,x)$ |
| $\{123,231\}$ | $\{\langle 1,0,0\rangle, \langle 0,0,1\rangle\}$ | $\{1\}$ | $zxF(zu, xu^{-1})\big\vert_{u=0,\, x=1}$ |
| $\{132,213\}$ | $\{\langle 0,0,1\rangle\}$ | $\{1\}$ | $zF(z,x)$ |
| $\{132,231\}$ | $\{\langle 1,0,0\rangle, \langle 0,1,0\rangle\}$ | $\{1\}$ | $zxF_x(z,0)$ |
| $\{123,132,213\}$ | $\{\langle 0,1,0\rangle, \langle 0,0,1\rangle\}$ | $\{1\}$ | $z\,F(z\,u, x\,u^{-1})\big\vert_{u=0}$ |
| $\{123,132,312\}$ | $\{\langle 0,1,0\rangle, \langle 0,0,1\rangle\}$ | $\{1\}$ | $z\,F(z\,u, x\,u^{-1})\big\vert_{u=0}$ |
| $\{123,231,312\}$ | $\{\langle 1,0,0\rangle, \langle 0,0,1\rangle\}$ | $\{1\}$ | $zxF(zu, xu^{-1})\big\vert_{u=0,\, x=1}$ |
| $\{132,213,312\}$ | $\{\langle 0,1,0\rangle\}$ | $\{1\}$ | $zF(z,x)$ |

Table 5.1: Operators $\mathcal{P}_{12}$ for various avoided sets.

## 5.5 Schemes of Depth $\geq$ 3

The previous examples could each be considered as a finite automaton with a single state, the non-empty permutations, with two transfers each corresponding to inserting a letter to create a 12- or 21-prefix. To gain some perspective, let us return to the 123-avoiding permutations and recall that the scheme $E_{123}$ can be represented as a tree-like structure as shown in Figure 5.1. We then translate the scheme into the finite automaton in Figure 5.2.

When we begin with a permutation $z^n x^a$, we then apply either $\mathcal{P}_{12}$ or $\mathcal{P}_{21}$ to make a permutation which starts with the pattern 12 or 21, respectively. The end result is then

| $B$ | $G_{21}$ | $R_{21}$ | $\mathcal{P}_{21} \circ F(z,x)$ |
|---|---|---|---|
| $\{123\}$ | $\emptyset$ | $\{1\}$ | $\frac{zx^2}{x-1}F(zx,1) - \frac{zx}{x-1}F(z,x)$ |
| $\{132\}$ | $\emptyset$ | $\{1\}$ | $\frac{zx^2}{x-1}F(zx,1) - \frac{zx}{x-1}F(z,x)$ |
| $\{123,132\}$ | $\emptyset$ | $\{1\}$ | $\frac{zx^2}{x-1}F(zx,1) - \frac{zx}{x-1}F(z,x)$ |
| $\{123,213\}$ | $\{\langle 0,0,1\rangle\}$ | $\{1\}$ | $zxF(zx,1)$ |
| $\{123,231\}$ | $\{\langle 0,1,1\rangle\}$ | $\{1\}$ | $zxF(z,x) + zxF(zx,1) - zxF(z\,u,x\,u^{-1})\Big\vert_{u=0}$ |
| $\{132,213\}$ | $\{\langle 0,0,1\rangle\}$ | $\{1\}$ | $zxF(zx,1)$ |
| $\{132,231\}$ | $\emptyset$ | $\{1\}$ | $\frac{zx^2}{x-1}F(zx,1) - \frac{zx}{x-1}F(z,x)$ |
| $\{123,132,213\}$ | $\{\langle 0,0,1\rangle,\langle 0,2,0\rangle\}$ | $\{1\}$ | $zx\ F(zu,xu^{-1})\Big\vert_{\substack{u=0\\x=1\\z=zx}} + zx\ \frac{\partial}{\partial u}F(zu,xu^{-1})\Big\vert_{\substack{u=0\\x=1\\z=zx}}$ |
| $\{123,132,312\}$ | $\{\langle 0,0,2\rangle,\langle 0,1,0\rangle\}$ | $\{1\}$ | $zx\big(F(zu,xu^{-1}) + \frac{\partial}{\partial u}F(zu,xu^{-1})\big)\Big\vert_{u=0}$ |
| $\{123,231,312\}$ | $\{\langle 0,1,0\rangle\}$ | $\{1\}$ | $zx\ F(z,x)$ |
| $\{132,213,312\}$ | $\{\langle 0,1,0\rangle,\langle 0,0,1\rangle\}$ | $\{1\}$ | $zx\ F(zu,xu^{-1})\Big\vert_{\substack{u=0\\x=1\\z=zx}}$ |

Table 5.2: Operators $\mathcal{P}_{21}$ for various avoided sets.

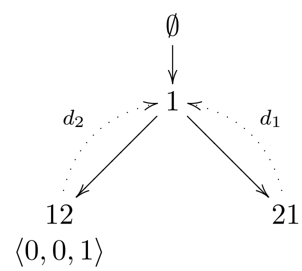


Figure 5.1: Tree representation of the enumeration scheme for $\mathcal{S}_n(123)$

considered to be back in state 1 again, at which point we may repeat the process. The weight of each 123-avoiding permutation arises uniquely from a sequence of applications of $\mathcal{P}_{12}$ and $\mathcal{P}_{21}$ to the initial monomial $zx$, which is the weight of the permutation 1.

For deeper enumeration schemes, such as those which occur for longer patterns, more states and transfers are necessary. As a running example, we will illustrate the approach by considering 1234-avoiding permutations. These have the enumeration scheme:

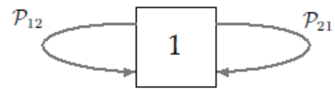


Figure 5.2: The single-state automaton for 123-avoiding permutations

$$E_{1234} = \{(\epsilon, \emptyset, \emptyset), (1, \emptyset, \emptyset), (12, \emptyset, \emptyset), (21, \emptyset, \{1\}),$$
$$(123, \{\langle 0,0,0,1\rangle\}, \{3\}), (132, \emptyset, \{2\}), (231, \emptyset, \emptyset),$$
$$(2413, \emptyset, \{1,2\}), (3412, \emptyset, \{1,2\}), (2314, \{\langle 0,0,0,0,1\rangle\}, \{4\}), (3421, \emptyset, \{3\})\}$$

This has the tree-like structure shown in Figure 5.3, which we will see translates into the automaton shown in 5.4.

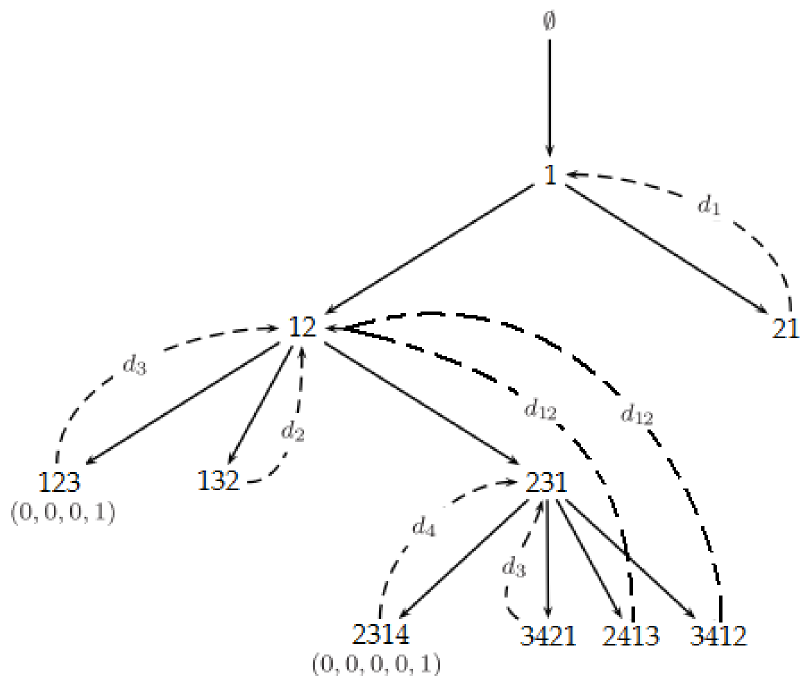


Figure 5.3: The tree-like structure for 1234-avoiding permutations

Given an enumeration scheme $E$ for $B$-avoiding permutations, a prefix $p$ is a *state* of the automaton $A = A(E)$ if the triple $(p, G_p, R_p) \in E$ where $R_p = \emptyset$ and there is some other triple $(p', G_{p'}, R_{p'}) \in E$ such that $d_{R_{p'}}(p') = p$. In the case of depth-2 schemes, we had only the state $p = 1$. For $A(E_{1234})$, the states are 1, 12 and 231. In the tree-like structure, states can be identified as those prefixes with dotted arrows pointing toward them.

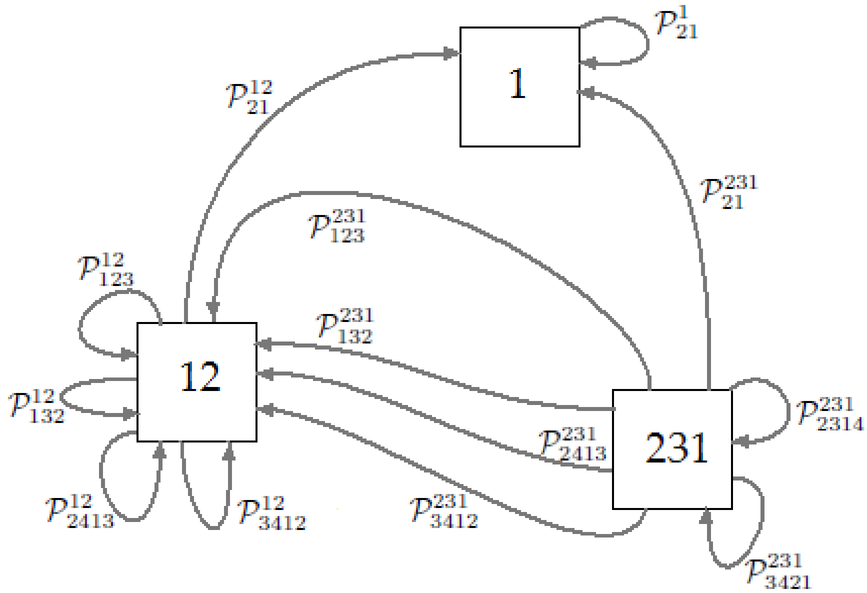


Figure 5.4: The automaton for 1234-avoiding permutations.

The states of $A$ imply a partition of $\mathcal{S}(B) = \bigcup_{n\geq 1} \mathcal{S}_n(B)$ according to prefix. For state $p \in \mathcal{S}_k$ let $\Sigma[p] \subseteq \mathcal{S}(B)$ be the set of permutations $\pi$ such that $\pi_1 \cdots \pi_k \sim p$ *and* no longer prefix is order isomorphic to any other state. In other words, the permutation $\pi$ is placed into the block $\Sigma[p]$ for the longest state $p$ for which $\pi$ has a matching prefix. For the scheme $E_{1234}$, and we see $2314 \in \Sigma[231]$ while $1324 \in \Sigma[12]$ and $3214 \in \Sigma[1]$. More broadly in this example, $\Sigma[1]$ is the set consisting of the permutation 1 and all permutations starting with a descent, $\Sigma[12]$ is the set consisting of the permutation 12 and all permutations starting with a 12 pattern but not a 231 pattern (i.e., those which start with a 123 or 132), and $\Sigma[231]$ is all permutations starting with a 231 pattern. The states of a scheme imply the following partitioning of $\mathcal{S}(B)$:

$$\mathcal{S}(B) = \{\pi : |\pi| < \min\{|p| : p \text{ is a state}\}\} \cup \left( \bigcup_{\text{state } p} \Sigma[p] \right)$$

It will be convenient to abuse language and say that $\pi$ is in state $p$ when $\pi \in \Sigma[p]$.

Suppose $(\tau, G, R) \in E$ for a prefix $\tau$ which is not a state. Then this triple indicates a transfer or transfers between states. In the depth-2 schemes in the previous section we saw two transfers, which corresponded to the prefixes 12 and 21. In $A(E_{1234})$ there will be seven classes of transfers, corresponding to the prefixes 21, 123, 132, 2314, 3421, 2413, and 3412. In the tree-like representation, transfers can be identified as those prefixes with dotted arrows pointing away from them.

These transfers act as follows. Since $(\tau, G, R) \in E$ we have the bijective deletion $d_R : \mathcal{S}_{n+|R|}(B)[\tau; w] \to \mathcal{S}_n(B)[d_R(\tau); d_R(w)]$ for any word $w \sim \tau$ which fails all gap conditions in $G$. Note that $d_R$ is only surjective, not bijective, when the domain is extended to $\mathcal{S}_{n+|R|}(B)[\tau]$; thus $\alpha \in \mathcal{S}_{n+|R|}(B)[\tau; w]$ and $\alpha' \in \mathcal{S}_{n+|R|}(B)[\tau; w']$ may map via $d_R$ to the same permutation $\beta \in \mathcal{S}_n(B)[d_R(\tau); d_R(w)]$ if $d_R(w) = d_R(w')$. Hence we denote the inverse map $i_{R,w} : \mathcal{S}_n(B)[\hat{\tau}; \hat{w}] \to \mathcal{S}_{n+|R|}(B)[\tau; w]$ where $\hat{\tau} = d_R(\tau)$ and $\hat{w} = d_R(w)$ in order to indicate the image's prefix word.

For $\beta \in \mathcal{S}_n(B)[\hat{\tau}; \hat{w}]$ define $I(\beta, \tau)$ to be the set of images of $\beta$ under all insertions $i_{R,w}$:

$$I(\beta, \tau) := \bigcup_{\substack{w \in [n]^{|\tau|}: \\ w \sim \tau, \text{ and} \\ w \text{ fails } G}} i_{R,w}(\beta).$$

Observe that our starting permutation $\beta$ could lie in any one of several states $\Sigma[p]$ such that $p$ has a prefix which is order isomorphic to $\hat{\tau}$. For a given state $p$ define $J(p, \tau)$ to be the images under the insertion maps $i_{R,w}$ of permutations in state $p$:

$$J(p, \tau) := \bigcup_{\substack{n \geq |\tau'| \\ \beta \in \Sigma[p] \cap \mathcal{S}_n(B)[\tau'; w']}} I(\beta, \tau)$$

Observe that the sets $I(\beta, \tau)$ are disjoint; this is important when we consider the weights of these sets since in this case the unions become sums without needing to worry about overlap. Since $i_{R,w}(\beta)$ has prefix pattern $\tau$, we see $J(p, \tau) \subseteq \Sigma[q]$ where $q$ is the state containing the permutation $\tau$. Hence the transfer $\tau$ implies several transfers from several states $p$ to a single state $q$. Also notice that the state $\Sigma[q]$ is partitioned by these $J(p, \tau)$.

To illustrate this, consider the transfer induced by $(\tau = 132, \emptyset, \{2\})$ in $A(E_{1234})$. The insertion maps $i_{R,w}$ are those maps $i_{2,\beta_1 b \beta_2}$ for $b \in \{\beta_2 + 1, \ldots, n + 1\}$.[5] Note that $\beta$ must begin with a $d_2(132) = 12$ pattern, and so $\beta$ could lie in either state 12 or 231,

[5] In the looser notation used for depth-2 schemes, these insertions would be written $i_b : \beta \mapsto \hat{\beta}_1 b \hat{\beta}_2 \cdots \hat{\beta}_n$ where $\hat{\beta}_i = \beta_i$ for $\beta_i < b$ and $\hat{\beta}_i = \beta_i + 1$ otherwise.

while $i_{2,\beta_1 b\beta_2}(\beta)$ has prefix 132 and so must lie in state 12. Hence we see

$$J(231,132) = \{\alpha \in \mathcal{S}(1234) : \alpha_1\alpha_2\alpha_3 \sim 132, \alpha_1\alpha_3\alpha_4 \sim 231\}$$
$$J(12,132) = \{\alpha \in \mathcal{S}(1234) : \alpha_1\alpha_2\alpha_3 \sim 132, \alpha_1\alpha_3\alpha_4 \not\sim 231\}$$

As a more complicated example, consider the transfer induced by $(2413, \emptyset, \{1,2\})$. In this case we insert two letters $b < c$ simultaneously at the first two positions via $i_{bc} := i_{\{1,2\},bc\beta_1(\beta_2+1)} : \beta \mapsto \alpha$ where $\alpha_1 = b$, $\alpha_2 = c$ and for $i \geq 3$ $\alpha_i = \beta_{i-2}$ if $\beta_{i_2} < b$, $\alpha_i = \beta_{i-2} + 1$ if $b \leq \beta_{i_2} < c$, and $\alpha_i = \beta_{i-2} + 2$ if $\beta_{i_2} \geq c$. Hence we see

$$I(\beta, 2413) = \bigcup_{b=\beta_1+1}^{\beta_2} \bigcup_{c=\beta_2+2}^{n+2} i_{bc}(\beta). \tag{5.11}$$

Similar to the 132 transfers, $i_{bc}$ takes in permutations in either state 12 or 231 (since $d_{\{1,2\}}(2413) = 12$ is a prefix for both 12 and 231) and returns permutations in state 231 (since 2413 lies in state 231). Hence we see the following characterizations of $J(231, 2413)$ and $J(12, 2413)$:

$$J(231,2413) = \{\alpha \in \mathcal{S}(1234) : \alpha_1\alpha_2\alpha_3\alpha_4 \sim 2413, \alpha_3\alpha_4\alpha_5 \sim 231\}$$
$$J(12,2413) = \{\alpha \in \mathcal{S}(1234) : \alpha_1\alpha_2\alpha_3\alpha_4 \sim 2413, \alpha_3\alpha_4\alpha_5 \not\sim 231\}.$$

We now consider the weights for our weight-enumerators. For state $p \in \mathcal{S}_k$ and permutation $\pi \in \Sigma[p]$, the weight of $\pi$ is given by the monomial $W(\pi) = z^{|\pi|} \prod_{i=1}^{k} x_i^{\pi(p^{-1}(i))}$. We chose the convention of using $p^{-1}$ since $\pi(p^{-1}(i)) < \pi(p^{-1}(i+1))$, and so the exponent of $x_i$ is the $i^{th}$ smallest letter appearing in the prefix. For example, $3412 \in \Sigma[231]$ and so $W(3412) = z^4 {x_1}^1 {x_2}^3 {x_3}^4$, while $1423 \in \Sigma[12]$ and so $W(1423) = z^4 {x_1}^1 {x_2}^4$. For a general $\pi \in \Sigma[231]$, $W(\pi) = z^{|\pi|} x_1^{\pi_3} x_2^{\pi_1} x_3^{\pi_2}$. As usual, the weight of a set is the sum of the weights of its elements. Rather than work with the weight of $\mathcal{S}(B)$ as a whole, we consider the weight of each state $p$,

$$F_p(z, x_1, \ldots, x_{|p|}) := W(\Sigma[p]) = \sum_{\pi \in \Sigma[p]} W(\pi).$$

We will typically abbreviate $F_p(z, x_1, \ldots, x_k) = F_p(z, \bar{x})$ and $F_p(z, 1, \ldots, 1) = F_p(z, 1)$. Observe that the traditional ordinary generating function $F(z) = \sum_{n \geq 1} s_n(B) z^n$ is the sum $\sum_p F_p(z, 1)$ over all states $p$.

For $\beta \in \Sigma[p]$, $I(\beta,\tau)$ induces an operator $\mathcal{P}^p_\tau \circ W(\beta) = W(I(\beta,\tau))$. Observe $W(\beta)$ is a monomial, and so we may extend linearly to see that

$$\begin{aligned} W\big(J(p,\tau)\big) &= W\Bigg(\bigcup_{\beta\in\Sigma[p]\cap\mathcal{S}(B)[\hat{\tau}]} I(\beta,\tau)\Bigg) \\ &= \sum_{\beta\in\Sigma[p]\cap\mathcal{S}(B)[\hat{\tau}]} W(I(\beta,\tau)) \\ &= \sum_{\beta\in\Sigma[p]\cap\mathcal{S}(B)[\hat{\tau}]} \mathcal{P}^p_\tau \circ W(\beta) \\ &= \mathcal{P}^p_\tau \circ \sum_{\beta\in\Sigma[p]\cap\mathcal{S}(B)[\hat{\tau}]} W(\beta) \\ &= \mathcal{P}^p_\tau \circ W\big(\Sigma[p]\cap\mathcal{S}(B)[\hat{\tau}]\big) \end{aligned} \tag{5.12}$$

To continue the examples above, consider the operators induced by $(132, \emptyset, \{2\})$. For permutation $\beta \in \Sigma[231]$ with weight $z^n {x_1}^{a_1} {x_2}^{a_2} {x_3}^{a_3}$ we see that $a_1 = \beta_3$, $a_2 = \beta_1$, and $a_3 = \beta_2$. Performing the insertion $i_{2,\beta_1 b\beta_2}$ for $b > \beta_2 = a_3$ leaves us a permutation $\alpha$ so that $\alpha_1\alpha_2\alpha_3\alpha_4 = \beta_1 b \beta_2 \beta_3 \sim 1432$. Since $\alpha \in \Sigma[12]$, we get that $W(\alpha) = z^{n+1} {x_1}^{\beta_1} {x_2}^{b} = z^{n+1} {x_1}^{a_2} {x_2}^{b}$. Summing over all valid $b$ gives us the following action on monomials by the operator $\mathcal{P}^{231}_{132}$:

$$\begin{aligned} \mathcal{P}^{231}_{132} : z^n x_1^{a_1} x_2^{a_2} x_3^{a_3} \mapsto & \sum_{b=a_3+1}^{n+1} z^{n+1} x_1^{a_2} x_2^{b} \\ &= z^{n+1} x_1^{a_2} \frac{x_2^{n+2} - x_2^{a_3+1}}{x_2 - 1} \\ &= \frac{z x_2^2}{x_2 - 1} (z x_2)^n x_1^{a_2} - \frac{z x_2}{x_2 - 1} z^n x_1^{a_2} x_2^{a_3} \end{aligned}$$

Similar reasoning for $\beta \in \Sigma[12]$ with weight $z^n {x_1}^{a_1} {x_2}^{a_2} = z^n {x_1}^{\beta_1} {x_2}^{\beta_2}$ gives a formulation for $\mathcal{P}^{12}_{132}$'s action on monomials:

$$\begin{aligned} \mathcal{P}^{12}_{132} : z^n x_1^{a_1} x_2^{a_2} \mapsto & \sum_{b=a_2+1}^{n+1} z^{n+1} x_1^{a_1} x_2^{b} \\ &= z^{n+1} x_1^{a_1} \frac{x_2^{n+2} - x_2^{a_2+1}}{x_2 - 1} \\ &= \frac{z x_2^2}{x_2 - 1} (z x_2)^n x_1^{a_1} - \frac{z x_2}{x_2 - 1} z^n x_1^{a_1} x_2^{a_2} \end{aligned}$$

These operators extend linearly to act on formal power series $G(z,\bar{x})$ by

$$\mathcal{P}_{132}^{231} \circ G(z,x_1,x_2,x_3) = \frac{zx_2^2}{x_2-1} G(z\,x_2,1,x_1,1) - \frac{zx_2}{x_2-1} G(z,1,x_1,x_2),$$
$$\mathcal{P}_{132}^{12} \circ G(z,x_1,x_2) = \frac{zx_2^2}{x_2-1} G(zx_2,x_1,1) - \frac{zx_2}{x_2-1} G(z,x_1,x_2).$$

To summarize, $\mathcal{P}_{132}^{12} \circ F_{12}(z,x_1,x_2)$ gives the weight of the set of permutations (all of which lie in $\Sigma[12]$) which are obtained by inserting into $\beta \in \Sigma[12]$ some letter $b > \beta_2$. Similarly, $\mathcal{P}_{132}^{231} \circ F_{231}(z,x_1,x_2,x_3)$ gives the weight of the set of permutations (again, all of which lie in $\Sigma[12]$) which are obtained by inserting into $\beta \in \Sigma[231]$ some letter $b > \beta_2$.

Now consider $(2413, \emptyset, \{1,2\})$. We start with the expression for $I(\beta, 2413)$ in Equation (5.11), take the weight of both sides, and then simplify the resulting geometric sums. This gives us the following operators on monomials:

$$\begin{aligned}
\mathcal{P}_{2413}^{231} : z^n x_1^{a_1} x_2^{a_2} x_3^{a_3} \mapsto & \sum_{b=a_2+1}^{a_3} \sum_{c=a_3+2}^{n+2} z^{n+2} x_1^{a_2} x_2^{b} x_3^{c} \\
&= \frac{z^{n+2}}{(x_2-1)(x_3-1)} \big( {x_1}^{a_2} {x_2}^{a_3+1} {x_3}^{n+3} \\
&\quad - {x_1}^{a_2} {x_2}^{a_2+1} {x_3}^{n+3} - {x_1}^{a_2} {x_2}^{a_3+1} {x_3}^{a_3+2} \\
&\quad + {x_1}^{a_2} {x_2}^{a_2+1} {x_3}^{a_3+2} \big) \\
\mathcal{P}_{2413}^{12} : z^n x_1^{a_1} x_2^{a_2} \mapsto & \sum_{b=a_1+1}^{a_2} \sum_{c=a_2+2}^{n+2} z^{n+2} x_1^{a_1} x_2^{b} x_3^{c} \\
&= \frac{z^{n+2}}{(x_2-1)(x_3-1)} \big( {x_1}^{a_1} {x_2}^{a_2+1} {x_3}^{n+3} \\
&\quad - {x_1}^{a_1} {x_2}^{a_1+1} {x_3}^{n+3} - {x_1}^{a_1} {x_2}^{a_2+1} {x_3}^{a_2+2} \\
&\quad + {x_1}^{a_1} {x_2}^{a_1+1} {x_3}^{a_2+2} \big)
\end{aligned}$$

These operators extend linearly to get us:

$$\begin{aligned}
\mathcal{P}^{231}_{2413} \circ G(z, x_1, x_2, x_3) &= \frac{z^2}{(x_2-1)(x_3-1)}\big(x_2{x_3}^3 G(zx_3, 1, x_1, x_2) \\
&\quad - x_2{x_3}^3 G(zx_3, 1, x_1x_2, 1) - x_2{x_3}^2 G(z, 1, x_1, x_2x_3) \\
&\quad + x_2{x_3}^2 G(z, 1, x_1x_2, x_3)\big) \\
\mathcal{P}^{12}_{2413} \circ G(z, x_1, x_2) &= \frac{z^2}{(x_2-1)(x_3-1)}\big(x_2{x_3}^3 G(zx_3, x_1, x_2) \\
&\quad - x_2{x_3}^3 G(zx_3, x_1x_2, 1) - x_2{x_3}^2 G(z, x_1, x_2x_3) \\
&\quad + x_2{x_3}^2 G(z, x_1x_2, x_3)\big)
\end{aligned} \tag{5.13}$$

Here we see $\mathcal{P}^{12}_{2413} \circ F_{12}(z, x_1, x_2)$ gives the weight of the set of permutations (all of which lie in $\Sigma[231]$) which are obtained by inserting into $\beta \in \Sigma[12]$ some letters $b, c$ such that $\beta_1 < b < \beta_2 + 1 < c$. Similarly, $\mathcal{P}^{231}_{2413} \circ F_{231}(z, x_1, x_2, x_3)$ gives the weight of the set of permutations (again, all of which lie in $\Sigma[231]$) which are obtained by inserting into $\beta \in \Sigma[231]$ some letters $b, c$ such that $\beta_1 < b < \beta_2 + 1 < c$.

To complete the construction of $A(E_{1234})$, we need to develop operators $\mathcal{P}^p_\tau$ for the remaining transfers $\tau \in \{21, 123, 3412, 2314, 3421\}$ in the same manner as we did for $\tau = 132$ and $\tau = 2413$. The thirteen transfer operators $\mathcal{P}^p_\tau$ are summarized in Equation (5.14). The resulting 3-state automaton is illustrated in Figure 5.4

$$
\begin{aligned}
\mathcal{P}^{1}_{21} \circ G(z,x_1) &= \frac{zx_1}{x_1-1}\big(x_1 G(zx_1,1) - G(z,x_1)\big)\\
\mathcal{P}^{12}_{21} \circ G(z,x_1,x_2) &= \frac{zx_1}{x_1-1}\big(x_1 G(zx_1,1,1) - G(z,x_1,1)\big)\\
\mathcal{P}^{231}_{21} \circ G(z,x_1,x_2,x_3) &= \frac{zx_1}{x_1-1}\big(x_1 G(zx_1,1,1,1) - G(z,1,x_1,1)\big)\\
\mathcal{P}^{12}_{123} \circ G(z,x_1,x_2) &= zG(z,x_1,x_2)\\
\mathcal{P}^{231}_{123} \circ G(z,x_1,x_2,x_3) &= zG(z,1,x_1,x_2)\\
\mathcal{P}^{12}_{132} \circ G(z,x_1,x_2) &= \frac{zx_2}{x_2-1}\big(x_2 G(zx_2,x_1,1) - G(z,x_1,x_2)\big)\\
\mathcal{P}^{231}_{132} \circ G(z,x_1,x_2,x_3) &= \frac{zx_2}{x_2-1}\big(x_2 G(z\,x_2,1,x_1,1) - G(z,1,x_1,x_2)\big)\\
\mathcal{P}^{12}_{2413} \circ G(z,x_1,x_2) &= \frac{z^2 x_2 {x_3}^2}{(x_2-1)(x_3-1)}\big(x_3 G(zx_3,x_1,x_2)\\
&\qquad - x_3 G(zx_3,x_1x_2,1) - G(z,x_1,x_2x_3)\\
&\qquad + G(z,x_1x_2,x_3)\big)\\
\mathcal{P}^{231}_{2413} \circ G(z,x_1,x_2,x_3) &= \frac{z^2 x_2 {x_3}^2}{(x_2-1)(x_3-1)}\big(x_3 G(zx_3,1,x_1,x_2)\\
&\qquad - x_3 G(zx_3,1,x_1x_2,1) - G(z,1,x_1,x_2x_3)\\
&\qquad + G(z,1,x_1x_2,x_3)\big)\\
\mathcal{P}^{12}_{3412} \circ G(z,x_1,x_2) &= \frac{z^2 x_2 {x_3}^2}{x_3-1}\Bigg(\frac{x_2x_3}{x_2-1} G(zx_2x_3,x_1,1)\\
&\qquad - \frac{x_2x_3}{x_2x_3-1} G(zx_2x_3,x_1,1)\\
&\qquad + \frac{1}{x_2x_3-1} G(z,x_1,x_2x_3)\\
&\qquad - \frac{x_3}{x_2-1} G(zx_3,x_1,x_2)\Bigg)\\
\mathcal{P}^{231}_{3412} \circ G(z,x_1,x_2,x_3) &= \frac{z^2 x_2 {x_3}^2}{x_3-1}\Bigg(\frac{x_2x_3}{x_2-1} G(zx_2x_3,1,x_1,1)\\
&\qquad - \frac{x_2x_3}{x_2x_3-1} G(zx_2x_3,1,x_1,1)\\
&\qquad + \frac{1}{x_2x_3-1} G(z,1,x_1,x_2x_3)\\
&\qquad - \frac{x_3}{x_2-1} G(zx_3,1,x_1,x_2)\Bigg)\\
\mathcal{P}^{231}_{2314} \circ G(z,x_1,x_2,x_3) &= zG(z,x_1,x_2,x_3)\\
\mathcal{P}^{231}_{3421} \circ G(z,x_1,x_2,x_3) &= \frac{zx_1x_2x_3}{x_1-1}\big(G(z,1,x_1x_2,x_3) - G(z,x_1,x_2,x_3)\big)
\end{aligned}
\tag{5.14}
$$

Define the operator $\mathcal{U}_{p,q} \circ F_q(z,\bar{x}) = \sum_\tau \mathcal{P}^p_\tau \circ F_p(z,\bar{x})$, where the sum ranges over all transfers $\tau$ from state $p$ to state $q$. Note that $\mathcal{U}_{p,q} = 0$ if there are no transfers from $p$ to $q$. It follows that the weight enumerator of the state $\Sigma[q]$, $F_q(z,\bar{x})$, satisfies the functional equation

$$F_q(z,\bar{x}) = W(q) + \sum_p \mathcal{U}_{p,q} \circ F_p(z,\bar{x}) \tag{5.15}$$

where the sum ranges over all states $p$. Note for any state $q$, $W(q) = zx_1{}^1 x_2{}^2 \cdots x_{|q|}{}^{|q|}$. Thus we see that our state-weights $F_q(z,\bar{x})$ satisfy a system of functional equations as determined by $\mathcal{U}_{p,q}$.

Returning to $A(E_{1234})$ for illustration, we see that

$$\mathcal{U}_{231,12} = \mathcal{P}^{231}_{123} + \mathcal{P}^{231}_{132} + \mathcal{P}^{231}_{2413} + \mathcal{P}^{231}_{3412}$$

while $\mathcal{U}_{12,231} = 0$. We may use these $\mathcal{U}_{p,q}$ to write the system of equations in matrix form in the following way:

$$\begin{bmatrix} F_1(z,x_1) \\ F_{12}(z,x_1,x_2) \\ F_{231}(z,x_1,x_2,x_3) \end{bmatrix} = \begin{bmatrix} zx_1 \\ zx_1x_2{}^2 \\ zx_1x_2{}^2x_3{}^3 \end{bmatrix} + \begin{bmatrix} \mathcal{U}_{1,1} & \mathcal{U}_{12,1} & \mathcal{U}_{231,1} \\ \mathcal{U}_{1,12} & \mathcal{U}_{12,12} & \mathcal{U}_{231,12} \\ \mathcal{U}_{1,231} & \mathcal{U}_{12,231} & \mathcal{U}_{231,231} \end{bmatrix} \circ \begin{bmatrix} F_1(z,x_1) \\ F_{12}(z,x_1,x_2) \\ F_{231}(z,x_1,x_2,x_3) \end{bmatrix}. \tag{5.16}$$

The $\mathcal{U}_{p,q}$ translate into the following system of functional equations:

$$\begin{aligned} F_1(z,x_1) &= zx_1 + \mathcal{P}^1_{21} \circ F_1(z,x_1) \\ &\qquad + \mathcal{P}^{12}_{21} \circ F_{12}(z,x_1,x_2) \\ &\qquad + \mathcal{P}^{231}_{21} \circ F_{231}(z,x_1,x_2,x_3) \\ F_{12}(z,x_1,x_2) &= zx_1x_2^2 \\ &\quad + (\mathcal{P}^{12}_{123} + \mathcal{P}^{12}_{132} + \mathcal{P}^{12}_{2413} + \mathcal{P}^{12}_{3412}) \circ F_{12}(z,x_1,x_2) \\ &\qquad + (\mathcal{P}^{231}_{123} + \mathcal{P}^{231}_{132} + \mathcal{P}^{231}_{2413} + \mathcal{P}^{231}_{3412}) \circ F_{231}(z,x_1,x_2,x_3) \\ F_{231}(z,x_1,x_2,x_3) &= zx_1x_2^2x_3^3 + (\mathcal{P}^{231}_{2314} + \mathcal{P}^{231}_{3421}) \circ F_{231}(z,x_1,x_2,x_3). \end{aligned} \tag{5.17}$$

As the example for 1234-avoiding permutations illustrates, these systems of equations get very complicated very quickly. A scheme with $k$ states and $t$ transfers will require a system of $k$ equations, which altogether requires the application of at least $t$ transfers. Further, any transfer prefix $(\tau, G, R) \in E$ can imply as many as $k$ different operators $\mathcal{P}^p_\tau$, depending on the number of states $p$ which partially match $d_R(\tau)$.

## 5.6 Refinements

One should also be able to adapt this approach to include the refinements according to permutation statistics discussed in Chapter 4. As one inserts letters, one must consider the number of inversions (or other patterns) *added.* As a proof-of-concept, we will present one example where we keep track of inversion number.

Consider the operators developed for 123-avoiding permutations:

$$\mathcal{P}_{21} : z^n x^a \mapsto \sum_{b=a+1}^{n+1} z^{n+1} x^b = z^{n+1} \frac{x^{n+2} - x^{a+1}}{x-1}$$

$$\mathcal{P}_{12} : z^n x^a \mapsto z^{n+1} x^a$$

Suppose $F(z,x,q)$ is the weight-enumerator for $\bigcup_{n\geq 1} \mathcal{S}_n(123)$ with the weight $W(\pi) = z^{|\pi|} x^{\pi_1} q^{\mathrm{INV}(\pi)}$. The map corresponding to the transfer 21 inserts the letter $b$ at the beginning of the permutation, thus adding $b-1$ inversions. The map corresponding to the transfer 12 inserts an $n+1$ at the second index, thus adding $n-1$ inversions. Hence we get the operators for $F(z,x,q)$ which act on monomials $z^n x^a q^k$ as follows:

$$\mathcal{P}_{21} : z^n x^a q^k \mapsto \sum_{b=a+1}^{n+1} z^{n+1} x^b q^{k+b-1} = q^k z^{n+1} \frac{(xq)^{n+2} - (xq)^{a+1}}{q(xq-1)}$$

$$\mathcal{P}_{12} : z^n x^a q^k \mapsto z^{n+1} x^a q^{k+n-1}$$

These extend to formal power series in indeterminates $z, x, q$ as expected. Thus we get a functional equation for the weight-enumerator $F(z,x,q)$.

$$F(z,x,q) = zx + zq^{-1} F(zq,x,q) + \frac{zx^2q^2}{q(qx-1)} F(zxq,1,q) - \frac{zxq}{q(qx-1)} F(z,xq,q).$$

The first half of [15] describes the application of the Umbral Transfer Matrix Method in the special case of counting occurrences of consecutive patterns as per section 4.8 of Chapter 4. While it does not make explicit mention of schemes or the refinement, it implicitly invokes Theorems 34 and 37 to modify the scheme for unrestricted permutations. We then apply the methods above to the resulting scheme. Zeilberger provides a Maple implementation, SERGI.

## 5.7 Conclusions and Future Directions

One of the criticisms of enumeration schemes is that they could only compute the sequences for $s_n(B)$, but could not be used to prove much else about those sequences. The methods used in this chapter are a strong response to this criticism, since the resulting systems of functional equations can indeed prove qualities of the sequences, such as closed forms for their generating functions. We also saw that while $\mathcal{S}_n(123)$ and $\mathcal{S}_n(132)$ have different schemes, these schemes imply the same functional equation, thus proving that 123 and 132 are Wilf-equivalent. In [91] Vatter raises the question to provide criteria for when different schemes produce the same sequence, and here we see a partial answer forming. These methods could prove some of the conjectured Wilf-equivalences in Conjecture 24 in Chapter 3. In that same work Vatter also asks whether every sequence produced by a finite enumeration scheme is holonomic. Again these methods may provide a route to such an answer, which schemes alone have not.

The examples above illustrate many of the operators which appear in translating an enumeration scheme into a functional equation. Admittedly, this is not necessarily an exhaustive list. There could be triples $p \in \mathcal{S}_k$, $G \subseteq \mathbb{N}^{k+1}$, $R \subseteq [k]$ which require operators which cannot be realized by the sequence of changing variables, differentiating, specializing variables, and multiplying by a rational function. On the other hand there is no classification of triples $(p, G, R)$ which appear in schemes, either, so such an exhaustive description may be more than necessary.

As mentioned in the introduction of this chapter, the above arguments apply equally well to the schemes developed in Chapter 3 for dashed patterns. These patterns tend to require deeper schemes, however, and thus result in more complicated functional equations. For example, Theorem 21 in Chapter 3 gives a construction of depth $t$ for permutations avoiding a consecutive pattern of length $t$. The automaton based on the scheme for permutations avoiding the consecutive pattern $12\cdots t$ has $t-1$ states (which are 1, 12, $\ldots$, $12\cdots(t-1)$) and $\binom{t}{2}+1$ transfers ($12\cdots t$ itself, plus every permutation of length between 2 and $t$ which has exactly one descent formed by the last two letters). Thus we are left with a system of $t-1$ functional equations.

A Maple package UmbralScheme is currently in development to apply the methods of this chapter.

# Chapter 6

# Wilf Equivalence and Sign

*This chapter represents joint work with Aaron Jaggard.*

## 6.1 Introduction

In this chapter we focus on questions of Wilf-equivalence when one restricts their attention to only the even permutations. In particular we are interested in when the results of classical Wilf-equivalence do and do not restrict to even-Wilf-equivalence. Similar studies have been done in the case of pattern-avoiding involutions, in particular [81, 56, 38].

We will restrict our attention only to classical patterns, and will omit the dashes between letters. Thus we write 132 to represent what would be the dashed pattern 1–3–2.

Recall from 1.1.3 of Chapter 1 that two patterns $\sigma, \tau$ are [classically] *Wilf-equivalent* if $s_n(\sigma) = s_n(\tau)$ for all $n \geq 0$. We denote this $\sigma \equiv \tau$. A pair of indices $i < j$ forms an *inversion* in permutation $\pi$ if $\pi_i > \pi_j$. Let $\textsc{inv}(\pi)$ denote the number of inversions in $\pi$, and let $\operatorname{sgn}(\pi) = (-1)^{\textsc{inv}(\pi)}$ be the *sign* of $\pi$. If $\operatorname{sgn}(\pi) = 1$ (i.e., $\textsc{inv}(\pi)$ is even) we say that $\pi$ is *even* and otherwise $\pi$ is *odd*.[1] Let $\mathcal{E}_n \subset \mathcal{S}_n$ be the set of even permutations of length $n$ and $\mathcal{O}_n \subset \mathcal{S}_n$ be the set of odd permutations of length $n$. Let $\mathcal{E}_n(\sigma) = \mathcal{S}_n(\sigma) \cap \mathcal{E}_n$ be the set of even permutations avoiding $\sigma$ and let $E_n(\sigma) := \#\mathcal{E}_n(\sigma)$, and similarly for $\mathcal{O}_n(\sigma)$ and $O_n(\sigma)$. We say that two permutations $\sigma$ and $\tau$ are *even-Wilf-equivalent* if $E_n(\sigma) = E_n(\tau)$ for all $n \geq 0$; we then write $\sigma \equiv_{\mathcal{E}_n} \tau$. When we need contrast with classical Wilf-equivalence, we denote classical Wilf-equivalence $\sigma \equiv_{\mathcal{S}_n} \tau$.

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[1] It is easily seen that this corresponds to the algebraic definition of even and odd, since $\textsc{inv}(\pi)$ is the number of adjacent transpositions necessary to return $\pi$ to the identity $12 \cdots n$.

## 6.2 Equivalences via Symmetry

In this section we present two useful lemmas connecting classical Wilf-equivalence to even-Wilf-equivalence. First we exhibit a case that shows where it is is clear that $\sigma$ is *not* even-Wilf-equivalent to $\tau$.

**Lemma 38.** *If $\sigma, \tau \in \mathcal{S}_k$ but $\mathrm{sgn}(\sigma) \neq \mathrm{sgn}(\tau)$, then $\sigma \not\equiv_{\mathcal{E}_n} \tau$.*

*Proof.* If $\sigma$ is even and $\tau$ is odd, then $\mathcal{E}_k(\sigma) = \mathcal{E}_n \setminus \{\sigma\}$ while $\mathcal{E}_k(\tau) = \mathcal{E}_k$. Hence $E_k(\sigma) = E_k(\tau) - 1$. □

Next we consider the trivial symmetries induced by the symmetry of the square. Recall the *reverse* of $\pi = \pi_1\pi_2 \ldots \pi_n$ is the horizontal reflection of $\pi$, denoted

$$\pi^r := \pi_n\pi_{n-1} \cdots \pi_1.$$

Similarly, the *complement* of $\pi \in \mathcal{S}_n$ is the vertical reflection

$$\pi^c := (n+1-\pi_1)(n+1-\pi_2)\cdots(n+1-\pi_n).$$

The inverse of $\pi$ is denoted as usual by $\pi^{-1}$. The following lemma summarizes how these reflections affect the sign of $\pi$.

**Lemma 39.** *The sign of a permutation $\pi \in S_n$ is affected by reflections in the following ways:*

*(a.)* $\mathrm{sgn}(\pi) = \mathrm{sgn}(\pi^r)$ *if and only if* $n \equiv 0, 1 \pmod 4$.

*(b.)* $\mathrm{sgn}(\pi) = \mathrm{sgn}(\pi^c)$ *if and only if* $n \equiv 0, 1 \pmod 4$.

*(c.)* $\mathrm{sgn}(\pi) = \mathrm{sgn}(\pi^{-1})$

*Proof.* For each pair of indices $i < j$, $\pi_i > \pi_j$ if and only if $(\pi^r)_i < (\pi^r)_j$. That is, the reversal map swaps the sites of inversions and non-inversions. Therefore $\mathrm{INV}(\pi^r) = \binom{n}{2} - \mathrm{INV}(\pi)$. Since $\binom{n}{2}$ is even if and only if $n \equiv 0, 1 \pmod 4$, part (a) is proven. Part (b) is proven similarly since it is also the case that $\mathrm{INV}(\pi^c) = \binom{n}{2} - \mathrm{INV}(\pi)$. Part (c) follows from the fact that for any permutation, $\mathrm{INV}(\pi) = \mathrm{INV}(\pi^{-1})$. □

In the classical case, $\sigma \equiv_{\mathcal{S}_n} \sigma^r$, $\sigma \equiv_{\mathcal{S}_n} \sigma^c$, and $\sigma \equiv_{\mathcal{S}_n} \sigma^{-1}$. Parts (a) and (b) of the lemma above, however, show that even-Wilf-equivalence for $\sigma$ and $\sigma^r$ is not guaranteed, and similarly for $\sigma^c$. For example $123 \not\equiv_{\mathcal{E}_n} 321$, since $E_3(123) = 2$ and $E_3(321) = 3$. Part (c) confirms, however, that we still have $\sigma \equiv_{\mathcal{E}_n} \sigma^{-1}$.

The next lemma demonstrates that while we lose the equivalences from reversal and complement, we may use symmetric versions of any even-Wilf-equivalences discovered.

**Lemma 40.** *If $\sigma \equiv_{\mathcal{E}_n} \tau$ and $\sigma \equiv_{\mathcal{S}_n} \tau$, then $\sigma^r \equiv_{\mathcal{E}_n} \tau^r$ and $\sigma^c \equiv_{\mathcal{E}_n} \tau^c$.*

*Proof.* We will prove $\sigma^r \equiv_{\mathcal{E}_n} \tau^r$. The proof for $\sigma^c \equiv_{\mathcal{E}_n} \tau^c$ is analogous. First observe that if $s_n(\sigma) = s_n(\tau)$ and $E_n(\sigma) = E_n(\tau)$, then $O_n(\sigma) = O_n(\tau)$. We continue by cases. If $n \equiv 0$ or $1 \pmod 4$, then

$$E_n(\sigma^r) = E_n(\sigma) = E_n(\tau) = E_n(\tau^r),$$

where the first and third equalities follow from Lemma 39 and the second equality by our assumptions. If $n \equiv 2$ or $3 \pmod 4$, then we see

$$E_n(\sigma^r) = O_n(\sigma) = O_n(\tau) = E_n(\tau^r),$$

where again the first and third equalities follow from Lemma 39 and the second equality by the observation above. □

It is worth stating the following lemma regarding the trivial equivalence classes for even-Wilf-equivalences. Its proof is similar to those above and is left to the reader.

**Lemma 41.** *For a pattern $\sigma$, we have the following trivial equivalences:*

- $\sigma \equiv_{\mathcal{E}_n} \sigma^{-1} \equiv_{\mathcal{E}_n} \sigma^{rc} \equiv_{\mathcal{E}_n} \left(\sigma^{-1}\right)^{rc}$
- $\sigma^r \equiv_{\mathcal{E}_n} \sigma^c \equiv_{\mathcal{E}_n} \left(\sigma^{-1}\right)^r \equiv_{\mathcal{E}_n} \left(\sigma^{-1}\right)^c$

## 6.3 Short Patterns

In this section we turn to the question of classifying patterns of a given length according to even-Wilf-equivalence.

Lemma 38 immediately implies $12 \not\equiv_{\mathcal{E}_n} 21$, completing the classification of $\mathcal{S}_2$.

Moving on to patterns of length three, we turn to Simion and Schmidt's observations in [81]. Their enumerations of $E_n(\sigma) - O_n(\sigma)$ for each $\sigma \in \mathcal{S}_3$ imply the following

**Theorem 42** (Simion and Schmidt, 1985)**.** *There are two distinct even-Wilf-equivalence classes for patterns of length 3:*

- $123 \equiv_{\mathcal{E}_n} 312 \equiv_{\mathcal{E}_n} 231$
- $321 \equiv_{\mathcal{E}_n} 213 \equiv_{\mathcal{E}_n} 132$

This suggests that if $\sigma \equiv_{\mathcal{S}_n} \tau$ and $\operatorname{sgn}(\sigma) = \operatorname{sgn}(\tau)$, then $\sigma \equiv_{\mathcal{E}_n} \tau$. This is not the case, however, as demonstrated by $1234 \not\equiv_{\mathcal{E}_n} 4321$: $E_6(1234) = 258$, while $E_6(4321) = 255$.

To classify patterns of length 4 or more, we use tools developed in the next section.

## 6.4 An Infinite Class of Non-trivial Equivalences

In this section we discuss an extension of the celebrated "prefix reversal" result for classical Wilf-equivalence, as proven by Backelin, West, and Xin in [10]. We follow and adapt their notation, aside from a change in convention: we reflect everything vertically. Backelin et al. state their results in terms of (permutation) matrices avoiding other (permutation) matrices. Hence the permutation 132, for example, is written as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

They then proceed to consider pattern avoidance in non-attacking rook placements in Young diagrams. These rook placements correspond to permutation matrices with some of the southeast cells of the matrix absent.

We choose to illustrate our permutations as graphs of functions as in Figure 1.1 of the introduction. Hence our graph of 132 looks like Figure 6.1.

As a result of this new convention, we orient our Young diagrams $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ such that the largest part $\lambda_1$ forms the *bottom* row of cells (boxes), then $\lambda_2$

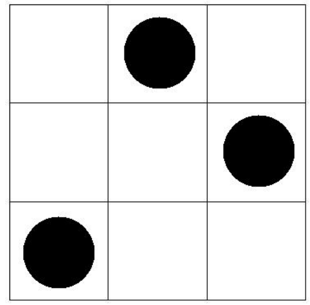

Figure 6.1: The graph of the permutation 132

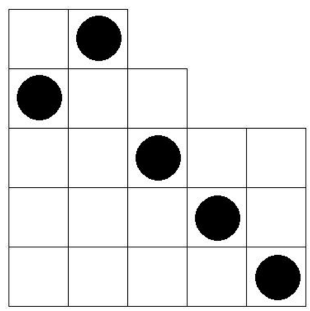

Figure 6.2: $\pi = 45321$ is a transversal of $\lambda = (5, 5, 5, 3, 2)$.

cells lie above this bottom layer, and so on as per the French custom. Cells of $\lambda$ are indexed from the lower-left corner by rows and columns, so $(r, c)$ is the cell in the $r^{th}$ row (increasing from the bottom) and $c^{th}$ column (increasing from the left). Hence $(r', c')$ is *above* $(r, c)$ if $r' > r$ and *to the the right* if $c' > c$.

A transversal of $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a permutation $\pi \in \mathcal{S}_n$ such that each point in the graph of $\pi$ lies inside some cell of $\lambda$ (i.e., $\pi_i^{-1} \leq \lambda_i$ for $1 \leq i \leq n$). Figure 6.2 illustrates that $\pi = 45321$ is a transversal of $\lambda = (5, 5, 3, 2, 2)$. Let $\mathcal{S}_\lambda$ denote all transversals of $\lambda$.

Pattern containment is stricter for transversals than it is for permutations. A transversal $\pi \in \mathcal{S}_\lambda$ contains $\sigma \in \mathcal{S}_k$ if there exists a subsequence $i_1 < i_2 < \cdots < i_k$ such that $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_k} \sim \sigma$ and the cell $(\max\{\pi_{i_1}, \pi_{i_2}, \cdots, \pi_{i_k}\}, i_k)$ lies in $\lambda$. In other words, the rows and columns of $\lambda$ containing $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_k}$ must form a full $k \times k$ square. In Figure 6.2 we see the transversal 45321 in $(5, 5, 5, 3, 2)$ contains 321 in the last three entries. Further, the transversal 45321 avoids 231 even though the *permutation* 45321 does not. We let $\mathcal{S}_\lambda(\sigma)$ denote the set of all transversals of $\lambda$ which do not contain $\sigma$, and $s_\lambda(\sigma) := \#\mathcal{S}_\lambda(\sigma)$. Two patterns $\sigma$ and $\tau$ are called *shape-Wilf-equivalent* if

$s_\lambda(\sigma) = s_\lambda(\tau)$ for all shapes $\lambda$; we denote this $\sigma \overset{s}{\equiv}_{\mathcal{S}_n} \tau$. Clearly shape-Wilf-equivalence implies Wilf-equivalence, since Wilf-equivalence considers only the shapes $\lambda$ which are $n \times n$ squares.

We adapt these concepts for even permutations as follows. A transversal $\pi \in \mathcal{S}_\lambda$ is *even* if the underlying permutation $\pi$ is even. Note that the presence/absence of an inversion is independent of $\lambda$, that is, an inversion is *not necessarily* a copy of a 21 pattern in the sense of transversals. Let $\mathcal{E}_\lambda$ be the even transversals in $\mathcal{S}_\lambda$, $\mathcal{E}_\lambda(\sigma)$ be the even transversals in $\lambda$ avoiding $\sigma$, and $E_\lambda(\sigma) := \#\mathcal{E}_\lambda(\sigma)$. We may do the same for odd transversals, using $\mathcal{O}_\lambda$, $\mathcal{O}_\lambda(\sigma)$, and $O_\lambda(\sigma)$. If $E_\lambda(\sigma) = E_\lambda(\tau)$, then we say $\sigma$ and $\tau$ are *even-shape-Wilf-equivalent* and we write $\sigma \overset{s}{\equiv}_{\mathcal{E}_n} \tau$.

Recall the direct sum of two permutations, $\alpha \in \mathcal{S}_k$ and $\beta \in \mathcal{S}_\ell$, is the length-$(k+\ell)$ permutation $\alpha_1\alpha_2\cdots\alpha_k(\beta_1+k+1)(\beta_2+k+1)\cdots(\beta_\ell+k+1)$. This is most easily seen as placing $\beta$ above and to the right of $\alpha$. See Figure 6.3 where we illustrate $312 \oplus 2413 = 3125746$.

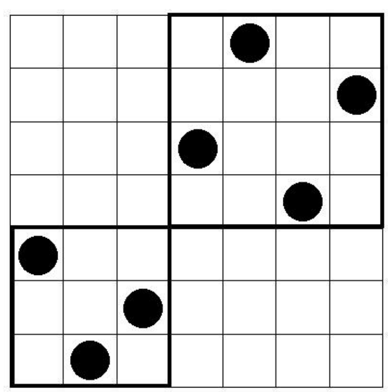

Figure 6.3: The direct sum $312 \oplus 2413 = 3125746$

We now re-state Proposition 2.3 from [10] as a lemma.

**Lemma 43** (Backelin, West, and Xin, [10]). *For patterns $\alpha$ and $\beta$, $\alpha \overset{s}{\equiv}_{\mathcal{S}_n} \beta$ implies $\alpha \oplus \sigma \overset{s}{\equiv}_{\mathcal{S}_n} \beta \oplus \sigma$.*

We summarize the proof here, as it will be useful for the following lemma.

*Proof.* For any shape $\lambda$, let $f_\lambda : \mathcal{S}_\lambda(\alpha) \to \mathcal{S}_\lambda(\beta)$ be the bijection implied by the hypothesis. Now fix $\lambda$ and let $\pi \in \mathcal{S}_\lambda(\alpha \oplus \sigma)$. We will color the cells of $\lambda$ either white or gray by a two-step procedure, then transform within the white cells while leaving the gray cells fixed. In this way we create a bijection $\mathcal{S}_\lambda(\alpha \oplus \sigma) \to \mathcal{S}_\lambda(\beta \oplus \sigma)$. We illustrate these steps in Figures 6.4 – 6.6.

Step 1. Color cell $(r, c)$ white if the part of $\pi$ lying in the subboard above and to the right of it contains $\sigma$ (as a transversal). Otherwise color $(r, c)$ gray.

Step 2. For each point in the graph of $\pi$ which lies in a gray cell, color gray the remaining cells in its row and column.

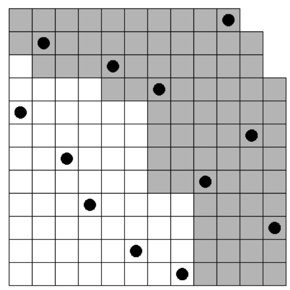

Figure 6.4: Executing step 1 for given $\pi$, $\lambda$ for $\sigma = 12$

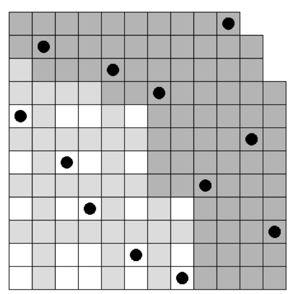

Figure 6.5: Executing step 2 for given $\pi$, $\lambda$ for $\sigma = 12$

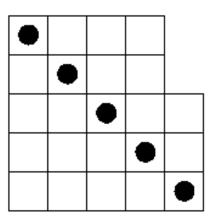

Figure 6.6: The resulting $\bar{\pi}$ and $\bar{\lambda}$

Denote the white cells by $\bar{\lambda}$ and the subtransversal of $\pi$ lying in $\bar{\lambda}$ by $\bar{\pi}$. By step 2, $\bar{\lambda}$ is itself a Young diagram, and further $\bar{\pi}$ is a transversal of $\bar{\lambda}$. Further, since $\pi$ avoids

$\alpha \oplus \sigma$, step 1 implies $\bar{\pi}$ avoids $\alpha$. We apply $f_{\bar{\lambda}}$ to $\bar{\pi}$ within the white cells, and so $f_{\bar{\lambda}}(\bar{\pi})$ avoids $\beta$. Restoring the gray portions of $\lambda$ and $\pi$, we finish with a transversal avoiding $\beta \oplus \sigma$. The inverse map is identical, except that $f_{\bar{\lambda}}$ is replaced by its inverse $f_{\bar{\lambda}}^{-1}$. □

We are now ready to state the even-shape-Wilf-equivalence analogue.

**Lemma 44.** *For patterns $\alpha$ and $\beta$, if $\alpha \stackrel{s}{\equiv}_{\mathcal{E}_n} \beta$ and $\alpha \stackrel{s}{\equiv}_{\mathcal{S}_n} \beta$ then $\alpha \oplus \sigma \stackrel{s}{\equiv}_{\mathcal{E}_n} \beta \oplus \sigma$.*

*Proof.* We will adapt notation from Lemma 43 above. Let $g_\lambda : \mathcal{E}_\lambda(\alpha) \to \mathcal{E}_\lambda(\beta)$ be the bijection implied by the hypothesis. By reasoning similar to that in Lemma 40, we see that we may also construct a bijection for odd transversals $h_\lambda : \mathcal{O}_\lambda(\alpha) \to \mathcal{O}_\lambda(\beta)$.

The map $\mathcal{E}_\lambda(\alpha \oplus \sigma) \to \mathcal{E}_\lambda(\beta \oplus \sigma)$ is constructed in the same way as for Lemma 43. Color cells of $\lambda$ white or gray by the same rules, and isolate $\bar{\lambda}$ and $\bar{\pi}$. Now $\bar{\pi}$ is either even or odd, so we apply the appropriate map $g_{\bar{\lambda}}$ or $h_{\bar{\lambda}}$. Observe that these maps preserve sign and so correspond to multiplying the original $\pi$ by some even permutation. Hence the image of the transversal $\pi$ is also even and we have our bijection. □

Backelin et al. also prove $J_t \stackrel{s}{\equiv}_{\mathcal{S}_n} I_t$, where $J_t$ is the decreasing permutation $t(t-1)\cdots 21$ and $I_t$ is the increasing permutation $12\cdots t$. By Lemma 43 above, this implies the well-known "prefix reversal" maneuver for Wilf-equivalence, namely $12\ldots k \oplus \sigma \equiv_{\mathcal{S}_n} k(k-1)\ldots 1 \oplus \sigma$. They prove[2] $J_t \stackrel{s}{\equiv}_{\mathcal{S}_n} I_t$ via their Proposition 3.1, that $J_t \stackrel{s}{\equiv}_{\mathcal{S}_n} F_t$ for all $t > 0$, where $F_t = J_{t-1} \oplus 1 = (t-1)(t-2)\cdots 21t$. Iterating this proves $J_t \stackrel{s}{\equiv}_{\mathcal{S}_n} J_{t-k} \oplus I_k$ for all $0 \le k \le t$. They provide a bijection $\phi_t^* : \mathcal{S}_\lambda(F_t) \to \mathcal{S}_\lambda(J_t)$, which we will show preserves sign. Here we will construct the map only; the proof of its correctness can be found in [10].

The map from $\mathcal{S}_\lambda(F_t)$ to $\mathcal{S}_\lambda(J_t)$ uses the following transformation. At its heart, it systematically converts all occurrences of $J_t$ into occurrences of $F_t$. Suppose $\pi \in \mathcal{S}_\lambda(J_t)$. Then we apply the following algorithm:

**Algorithm 45.**

*Step 1. Find all occurrences of $J_t$ in $\pi$ (as a transversal). If $\pi$ contains no $J_t$, then stop and return $\pi$.*

[2]They actually provide two proofs of $J_t \stackrel{s}{\equiv}_{\mathcal{S}_n} I_t$. Here we discuss only their first proof.

*Step 2. Find the smallest letter $\pi(i_1)$ such that $\pi(i_1)$ is the leftmost letter in an copy of $J_t$.*

*Step 3. Find the leftmost letter $\pi(i_2)$ such that $i_1 < i_2$ and there is an occurrence of $J_t$ such that $\pi(i_1)$ and $\pi(i_2)$ are the leftmost letters.*

*Step 4. Find indices $i_3 < i_4 < \cdots < i_t$ one by one as described in step 3. This yields a subpermutation $\pi(i_1)\pi(i_2)\cdots\pi(i_t)$, which is a copy of $J_t$. See Figure 6.7.*

*Step 5. Form a new permutation $\pi'$ by moving $\pi(i_1)$ to the ${i_t}^{th}$ position, and each other $\pi(i_j)$ to the ${i_{j-1}}^{th}$ position. Call this transformation $\theta(\pi) = \pi'$. Observe that $\pi'(i_1)\pi'(i_2)\cdots\pi'(i_t)$ is a copy of $F_t$. See 6.8*

*Step 6. Return to step 1.*

We denote a single application of steps 2 through 5 by $\phi_t(\pi)$. As described in steps 1 and 6, we compose $\phi_t$ with itself repeatedly until all copies of $J_t$ are eliminated. We denote this repeated composition $\phi_t^*$.

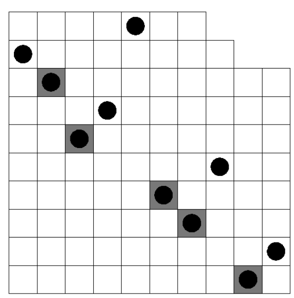

Figure 6.7: Selecting a copy of $J_t$

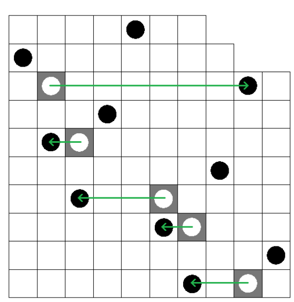

Figure 6.8: Applying the transformation $\theta$

For completeness we present the inverse map from $\mathcal{S}_\lambda(J_t)$ to $\mathcal{S}_\lambda(F_t)$. It operates on the same principle, converting copies of $F_t$ into copies of $J_t$.

**Algorithm 46.**

*Step 1. Find all occurrences of $F_t$ in $\pi$ (as a transversal). If $\pi$ contains no $F_t$, then stop and return $\pi$.*

*Step 2. Find the largest letter $\pi(i_t)$ such that $\pi(i_t)$ is the rightmost letter in an copy of $F_t$.*

*Step 3. Find the largest letter $\pi(i_{t-1})$ such that $i_{t-1} < i_t$ and there is an occurrence of $F_t$ such that $\pi(i_{t-1})$ and $\pi(i_t)$ are the rightmost letters.*

*Step 4. Find indices $i_{t-2} > i_{t-3} > \cdots > i_1$ one by one as described in step 3. This yields a subpermutation $\pi(i_1)\pi(i_2)\cdots\pi(i_t)$, which is a copy of $F_t$.*

*Step 5. Form a new permutation $\pi'$ by moving $\pi(i_t)$ to the ${i_1}^{th}$ position, and each other $\pi(i_j)$ to the ${i_{j+1}}^{th}$ position. Call this transformation $\theta'(\pi) = \pi'$. Observe that $\pi'(i_1)\pi'(i_2)\cdots\pi'(i_t)$ is a copy of $J_t$.*

*Step 6. Return to step 1.*

We denote the application of steps 2 through 5 by $\psi_t(\pi)$. We compose $\psi_t$ with itself a certain number of times as outlined in steps 1 and 6, yielding a map $\psi_t^* : \mathcal{S}_\lambda(J_t) \to \mathcal{S}_\lambda(F_t)$. Backelin et al. then show that $\phi_t$ and $\psi_t$ are inverses of one another, and hence so are $\phi_t^*$ and $\psi_t^*$.

We are now ready to prove our main result.

**Theorem 47.** *$J_t \overset{s}{\equiv}_{\mathcal{E}_n} F_t$ for all odd $t$.*

*Proof.* Fix $t$ odd. The theorem follows from the claim that $\phi_t$ preserve sign. If $\phi_t$ preserves sign, then so does $\phi_t^*$. Hence $\phi_t^*$ restricts to the map $\phi_t^* : \mathcal{E}_\lambda(F_t) \to \mathcal{E}_\lambda(J_t)$. Since $\psi_t$ is the inverse of $\phi_t$, $\psi_t^*$ must also preserve sign and hence we have our desired bijection.

Thus it remains to show that $\phi_t$ preserves sign when $t$ is odd. Careful inspection reveals that the map $\theta$ in Step 5 is merely multiplying by the cycle $(i_1\, i_2\, \cdots\, i_t)$. Since an odd cycle is an even permutation, applying $\theta$ preserves sign. $\square$

It should be noted that $\phi_t$ reverses sign when $t$ is even. This follows from the fact that $\theta$ is an even cycle and hence its application reverses sign. Since $\phi_t$ may be composed with itself either an even or odd number of times in the application of $\phi_t^*$, however, the composition $\phi_t^*$ neither preserves nor reverses sign for the entirety of $\mathcal{E}_\lambda(F_t)$.

The restriction that $t$ be odd prevents the iteration which implies $J_t \overset{s}{\equiv}_{\mathcal{S}_n} J_{t-k} \oplus I_k$ for all $0 \leq k \leq t$. Applying the theorem once gets us $J_t \overset{s}{\equiv}_{\mathcal{E}_n} J_{t-1} \oplus 1$, at which point $t-1$ is even and the theorem no longer applies. Note that the general prefix reversal result is not true for even-Wilf-equivalence: for example, $1234 \not\equiv_{\mathcal{E}_n} 4321$.

## 6.5 Classifications

This section makes the classification of 4-patterns under $\equiv_{\mathcal{E}_n}$ explicit, as well as the partial classifications of patterns of length 5 and 6.

### 6.5.1 Classification of $\mathcal{S}_4$

With Theorem 47 above and sufficient numerical computation, we may classify all patterns $\sigma \in \mathcal{S}_4$. There are eleven equivalence classes in total. The values of each $E_n(\sigma)$ are listed for $n \leq 10$ in Table 6.1

In $\mathcal{S}_4$, only two non-trivial equivalences appear. Since $\sigma \equiv_{\mathcal{E}_n} \sigma^{rc}$ we get that $3214 \equiv_{\mathcal{E}_n} 1432$ and $2134 \equiv_{\mathcal{E}_n} 1243$. Applying Theorem 47 and Lemma 44, we get that $3214 = J_3 \oplus 1 \equiv_{\mathcal{E}_n} F_3 \oplus 1 = 2134$ to complete the class. The reverses of these patterns comprise the other non-trivial class, as per Lemma 40: $3421 \equiv_{\mathcal{E}_n} 4312 \equiv_{\mathcal{E}_n} 4123 \equiv_{\mathcal{E}_n} 2341$. Thus we obtain the classification shown in Table 6.1. Horizontal lines separate the even-Wilf classes: patterns in the same even-Wilf class appear in adjacent rows with no separating line.

| $\sigma$ | sgn($\sigma$) | $E_4(\sigma)$ | $E_5(\sigma)$ | $E_6(\sigma)$ | $E_7(\sigma)$ | $E_8(\sigma)$ | $E_9(\sigma)$ | $E_{10}(\sigma)$ |
|---|---|---|---|---|---|---|---|---|
| 2134 | -1 | 12 | 52 | 257 | 1381 | 7885 | 47181 | 293297 |
| 3214 | -1 | 12 | 52 | 257 | 1381 | 7885 | 47181 | 293297 |
| 1243 | -1 | 12 | 52 | 257 | 1381 | 7885 | 47181 | 293297 |
| 1432 | -1 | 12 | 52 | 257 | 1381 | 7885 | 47181 | 293297 |
| 4312 | -1 | 12 | 52 | 256 | 1380 | 7885 | 47181 | 293293 |
| 4123 | -1 | 12 | 52 | 256 | 1380 | 7885 | 47181 | 293293 |
| 3421 | -1 | 12 | 52 | 256 | 1380 | 7885 | 47181 | 293293 |
| 2341 | -1 | 12 | 52 | 256 | 1380 | 7885 | 47181 | 293293 |
| 2314 | 1 | 11 | 51 | 257 | 1371 | 7742 | 45622 | 277826 |
| 1423 | 1 | 11 | 51 | 257 | 1371 | 7742 | 45622 | 277826 |
| 3124 | 1 | 11 | 51 | 257 | 1371 | 7742 | 45622 | 277826 |
| 1342 | 1 | 11 | 51 | 257 | 1371 | 7742 | 45622 | 277826 |
| 4132 | 1 | 11 | 51 | 255 | 1369 | 7742 | 45622 | 277836 |
| 3241 | 1 | 11 | 51 | 255 | 1369 | 7742 | 45622 | 277836 |
| 4213 | 1 | 11 | 51 | 255 | 1369 | 7742 | 45622 | 277836 |
| 2413 | 1 | 11 | 51 | 255 | 1369 | 7742 | 45622 | 277836 |
| 2413 | -1 | 12 | 52 | 256 | 1370 | 7743 | 45623 | 277831 |
| 3142 | -1 | 12 | 52 | 256 | 1370 | 7743 | 45623 | 277831 |
| 1234 | 1 | 11 | 51 | 258 | 1382 | 7879 | 47175 | 293311 |
| 4321 | 1 | 11 | 51 | 255 | 1379 | 7879 | 47175 | 293279 |
| 2143 | 1 | 11 | 51 | 256 | 1380 | 7885 | 47181 | 293301 |
| 3412 | 1 | 11 | 51 | 257 | 1381 | 7885 | 47181 | 293289 |
| 1324 | -1 | 12 | 52 | 258 | 1382 | 7903 | 47393 | 296002 |
| 4231 | -1 | 12 | 52 | 255 | 1380 | 7903 | 47393 | 295948 |

Table 6.1: The classification of $\mathcal{S}_4$ into $\equiv_{\mathcal{E}_n}$-classes with values of $E_n(\sigma)$ for $\sigma \in \mathcal{S}_4$ and $n \leq 10$.

### 6.5.2 Partial Classification of $\mathcal{S}_5$

The techniques of the previous section imply a partial classification of $\mathcal{S}_5$. Based on computations for $E_n(\sigma)$ for $n \leq 11$, there appear to be four non-trivial equivalence classes, listed in Table 6.2. Each row represents one trivial equivalence class with a chosen representative. Two rows written adjacently with no separating line are proven above to be even-Wilf-equivalent, as discussed below. Two rows written adjacently and separated by a dotted line are conjectured to be even-Wilf-equivalent based on numerical data for $n \leq 11$. Solid lines separate rows which are not even-Wilf-equivalent.

| $\sigma$ | sgn($\sigma$) | $E_7(\sigma)$ | $E_8(\sigma)$ | $E_9(\sigma)$ | $E_{10}(\sigma)$ | $E_{11}(\sigma)$ |
|---|---|---|---|---|---|---|
| 12345 | 1 | 2293 | 16662 | 130897 | 1095344 | 9659368 |
| 23451 | 1 | 2293 | 16662 | 130897 | 1095344 | 9659368 |
| 45312 | 1 | 2293 | 16662 | 130897 | 1095344 | 9659368 |
| 34512 | 1 | 2293 | 16662 | 130897 | 1095344 | 9659368 |
| 15432 | 1 | 2289 | 16662 | 130897 | 1095344 | 9659320 |
| 54321 | 1 | 2289 | 16662 | 130897 | 1095344 | 9659320 |
| 21354 | 1 | 2289 | 16662 | 130897 | 1095344 | 9659320 |
| 21543 | 1 | 2289 | 16662 | 130897 | 1095344 | 9659320 |
| 12354 | -1 | 2291 | 16662 | 130907 | 1095344 | 9659344 |
| 12543 | -1 | 2291 | 16662 | 130907 | 1095344 | 9659344 |
| 45321 | -1 | 2291 | 16662 | 130907 | 1095344 | 9659344 |
| 34521 | -1 | 2291 | 16662 | 130907 | 1095344 | 9659344 |
| 13524 | -1 | 2290 | 16627 | 130145 | 1081965 | 9450267 |
| 42531 | -1 | 2290 | 16627 | 130145 | 1081965 | 9450267 |

Table 6.2: The classification of $\mathcal{S}_5$ into $\equiv_{\mathcal{E}_n}$-classes with values of $E_n(\sigma)$ for $\sigma \in \mathcal{S}_5$ and $n \leq 11$.

The proven equivalences are each a corollary to Theorem 47 in conjunction with the symmetries in Lemmas 40 and 41. For example, $12345 \equiv_{\mathcal{E}_n} 23451$ since $12345^c = 54321$, $54321 \overset{s}{\equiv}_{\mathcal{E}_n} \phi_5^*(54321) = 43215$, and $43215^c = 23451$. Thus we see that $\pi \mapsto \phi_5^*(\pi^c)^c$ provides the bijection $\mathcal{E}_n(12345) \to \mathcal{E}_n(23451)$.

- $12345 \equiv_{\mathcal{E}_n} 23451$ (under $\phi_5^*(\pi^c)^c$)
- $45312 \equiv_{\mathcal{E}_n} 34512$ (under $\psi_3^*(\pi^c)^c$)
- $15432 \equiv_{\mathcal{E}_n} 54321$ (under $\phi_5^*(\pi)^{rc}$)
- $21354 \equiv_{\mathcal{E}_n} 21543$ (under $\psi_3^*(\pi^{rc})^{rc}$)

- $12354 \equiv_{\mathcal{E}_n} 12543$ (under $\psi_3^*(\pi^{rc})^{rc}$)
- $45321 \equiv_{\mathcal{E}_n} 34521$ (under $\psi_3^*(\pi^{c})^{c}$)

This leaves the following conjectured equivalences:

**Conjecture 48.** *The following equivalences hold:*

- $12345 \equiv_{\mathcal{E}_n} 45312$
- $54321 \equiv_{\mathcal{E}_n} 21354$
- $12354 \equiv_{\mathcal{E}_n} 45321$
- $13524 \equiv_{\mathcal{E}_n} 42531$

Observe that Lemma 40 implies that the first and second conjectured equivalences follow from one another.

The second conjectured equivalence class contains all patterns of the form $J_r \oplus J_s$ for all $r + s = 5$ and $r, s \geq 0$, together with 21354. The first conjectured class contains the reverses of these. A similar pattern seems to emerge in patterns of length 7, although again conjecturally. This suggests the following more general statement:

**Conjecture 49.** *For odd $t$, $J_r \oplus J_s \equiv_{\mathcal{E}_n} J_t$ for any $r + s = t$.*

Also notice that the third and fourth conjectured equivalences in Conjecture 48 have been written in the form $\sigma \equiv_{\mathcal{E}_n} \sigma^r$. In the classical case this is trivial under the reversal map, but when $n = 3, 4 \pmod 4$ $\mathcal{E}_n(\sigma)^r \cap \mathcal{E}_n(\sigma^r)$ since $\mathcal{E}_n(\sigma)^r$ would contain only odd permutations by Lemma 39.

### 6.5.3 Partial Classification of $\mathcal{S}_6$

For $\mathcal{S}_6$ there are 10 non-trivial even-Wilf classes, plus two more conjectured based on numerical results. These are listed below in Table 6.3. Each of these equivalences follows from Theorem 47 and its symmetries. In the classical case, classifying the length 6 patterns required an additional result provided by Stankova and West in [82]. They prove that $312 \overset{s}{\equiv}_{\mathcal{S}_n} 231$, which in combination with Lemma 43 provides the equivalence

312564 $\equiv_{\mathcal{S}_n}$ 231564. We have checked computationally for all Ferrers shapes $\lambda$ which lie in an $9 \times 9$ box that $E_\lambda(312) = E_\lambda(231)$, and that $E_n(231 \oplus \alpha) = E_n(312 \oplus \alpha)$ for all $\alpha \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4$ and $n \leq 11$. This naturally leads to Conjecture 50, which would imply 312564 $\overset{s}{\equiv}_{\mathcal{E}_n}$ 231564 (and 465312 $\overset{s}{\equiv}_{\mathcal{E}_n}$ 465132 by Lemma 44).

**Conjecture 50.** 312 *is even-shape-Wilf-equivalent to* 231.

This analogue of the Stankova-West result, combined with those discussed in the previous sections, would complete the classification of the length 6 patterns.

| $\sigma$ | sgn($\sigma$) | $E_7(\sigma)$ | $E_8(\sigma)$ | $E_9(\sigma)$ | $E_{10}(\sigma)$ | $E_{11}(\sigma)$ |
|---|---|---|---|---|---|---|
| 543216 | 1 | 2501 | 19713 | 172417 | 1645790 | 16917552 |
| 432156 | 1 | 2501 | 19713 | 172417 | 1645790 | 16917552 |
| 612345 | -1 | 2502 | 19713 | 172417 | 1645800 | 16917562 |
| 651234 | -1 | 2502 | 19713 | 172417 | 1645800 | 16917562 |
| 213564 | -1 | 2502 | 19714 | 172392 | 1644933 | 16895077 |
| 321564 | -1 | 2502 | 19714 | 172392 | 1644933 | 16895077 |
| 465312 | 1 | 2501 | 19714 | 172392 | 1644930 | 16895074 |
| 465123 | 1 | 2501 | 19714 | 172392 | 1644930 | 16895074 |
| 213456 | -1 | 2502 | 19714 | 172418 | 1645799 | 16917561 |
| 321456 | -1 | 2502 | 19714 | 172418 | 1645799 | 16917561 |
| 654312 | 1 | 2501 | 19714 | 172418 | 1645791 | 16917553 |
| 654123 | 1 | 2501 | 19714 | 172418 | 1645791 | 16917553 |
| 213546 | 1 | 2501 | 19712 | 172417 | 1645814 | 16918707 |
| 321546 | 1 | 2501 | 19712 | 172417 | 1645814 | 16918707 |
| 645312 | -1 | 2502 | 19712 | 172417 | 1645838 | 16918725 |
| 645123 | -1 | 2502 | 19712 | 172417 | 1645838 | 16918725 |
| 213465 | 1 | 2501 | 19713 | 172417 | 1645791 | 16917553 |
| 321465 | 1 | 2501 | 19713 | 172417 | 1645791 | 16917553 |
| 321654 | 1 | 2501 | 19713 | 172417 | 1645791 | 16917553 |
| 564312 | -1 | 2502 | 19713 | 172417 | 1645799 | 16917561 |
| 564123 | -1 | 2502 | 19713 | 172417 | 1645799 | 16917561 |
| 456123 | -1 | 2502 | 19713 | 172417 | 1645799 | 16917561 |
| 231564 | 1 | 2501 | 19716 | 172388 | 1644575 | 16882865 |
| 312564 | 1 | 2501 | 19716 | 172388 | 1644575 | 16882865 |
| 465132 | -1 | 2502 | 19716 | 172388 | 1644588 | 16882878 |
| 465213 | -1 | 2502 | 19716 | 172388 | 1644588 | 16882878 |

Table 6.3: The classification of $\mathcal{S}_6$ into $\equiv_{\mathcal{E}_n}$-classes with values of $E_n(\sigma)$ for $\sigma \in \mathcal{S}_6$ and $n \leq 11$.

## 6.6 Conclusions and Future Directions

In this chapter we have established the foundation for a theory of even-Wilf-equivalence, parallel to the classical theory of Wilf-equivalence. The general trend appears to be that results in Wilf-equivalence have weaker versions for even-Wilf-equivalence. For example, $\sigma \not\equiv_{\mathcal{E}_n} \sigma^r$ but if $\sigma \equiv_{\mathcal{E}_n} \tau$ then $\sigma^r \equiv_{\mathcal{E}_n} \tau^r$. Similarly, we have proven that the Backelin et al. result that $J_t \equiv_{\mathcal{S}_n} F_t$ only holds for $\equiv_{\mathcal{E}_n}$ when $t$ is odd. Even with these weakened versions we could classify $\mathcal{S}_4$ according to even-Wilf-equivalence, and partially classify $\mathcal{S}_k$ for larger $k$.

Theoretical and numerical results thus far suggest the following conjecture that even-Wilf-equivalence is a refinement of Wilf-equivalence:

**Conjecture 51.** *If $\sigma \equiv_{\mathcal{E}_n} \tau$, then $\sigma \equiv_{\mathcal{S}_n} \tau$.*

The conjecture may be skewed, however, as much of the work in this paper was motivated by first looking at classical equivalences and looking for refinements. Even so, the available numerical data was thoroughly examined for a counterexample and none was found. The analogous conjecture for involution-Wilf-equivalence[3] remains open.

Examining the equivalence classes under $\equiv_{\mathcal{E}_n}$ suggests that even-Wilf-equivalence is a very strong condition. Table 6.6 summarizes the number of equivalence classes under classical and even-Wilf-equivalence. Values for the number of equivalence classes under Wilf equivalence are taken from OEIS sequence A099952, [55]. Lower bounds for the even-Wilf-equivalence classes for 5- and 6- patterns are based on avoidance by permutations of length $n \leq 11$. Upper bounds were obtained by assuming all conjectures above are false.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---:|---|---|---|---|---|---|
| Wilf-equivalence | 1 | 1 | 1 | 3 | 16 | 91 |
| even-Wilf-equivalence | 1 | 1 | 2 | 11 | $[35, 39]$ | $\{216, 218\}$ |

Table 6.4: The number of equivalence classes for patterns of length $n$.

[3]Two patterns $\sigma$ and $\tau$ are said to be involution-Wilf-equivalent if $\#(\mathcal{S}_n(\sigma) \cap \mathcal{I}_n) = \#(\mathcal{S}_n(\tau) \cap \mathcal{I}_n)$ for all $n$, where $\mathcal{I}_n$ is the set of involutions of length $n$.

There are many more trivial equivalence classes under $\equiv_{\mathcal{E}_n}$ than in the classical case. A possible weakening of even-Wilf-equivalence is perhaps to require that $E_n(\sigma) = E_n(\tau)$ only for "most" $n$. For example, the results of Simion and Schmidt in [81] imply that $E_n(123) = E_n(132)$ for any $n \neq 0 \pmod 4$. In other instances, data suggests pairs $(\sigma, \tau)$ such that $E_{2n}(\sigma) = E_{2n}(\tau)$ for all $n$. For example, the enumeration schemes in Chapter 4 verify that $E_{2n}(12345) = E_{2n}(54321)$ is such a pair for $n \leq 7$.

# Chapter 7

# Asymptotic Independence of Inversion Number and Major Index

*This chapter represents joint work with Doron Zeilberger.*

## 7.1 Introduction

This chapter explores the joint distribution of the inversion number and the major index[1] over permutations of length $n$ as $n \to \infty$. When MacMahon introduced the major index, denoted by MAJ, in [64], he also showed that it was equidistributed with the inversion number INV, i.e.

$$\sum_{\pi \in \mathcal{S}_n} q^{\mathrm{INV}(\pi)} = \sum_{\pi \in \mathcal{S}_n} q^{\mathrm{MAJ}(\pi)} = \mathbf{n}_q! \tag{7.1}$$

where $\mathbf{n}_q = (1 + q + \cdots + q^{n-1})$ and $\mathbf{n}_q! = \mathbf{n}_q(\mathbf{n-1})_q \cdots \mathbf{1}_q$ is the $q$-factorial. It is not obvious, however, to what degree INV and MAJ depend on one another. In other words, how much information can one draw about $\mathrm{MAJ}(\pi)$ if one knows $\mathrm{INV}(\pi)$, and vice versa? The main result of this chapter is to show that as $n \to \infty$, the joint distribution of random variables $\mathrm{MAJ}(\pi)$ and $\mathrm{INV}(\pi)$ over $\pi \in \mathcal{S}_n$ approach a joint-indendent normal distribution.

A sequence of distributions $\{X_n\}$ is *asymptotically normal* if $X_n$ converges in distribution to the normal distribution $\mathcal{N}$. If we let $X_n$ be the distribution of the random variable $\mathrm{INV}(\pi)$ over $\pi \in \mathcal{S}_n$, then Feller shows in [44] that $X_n$ is asymptotically normal. In Section 7.2 we present an alternate proof of this fact, using the arguments from [103].

Similarly, we say that a sequence of bivariate distributions $\{X_n\}$ is *asymptotically*

[1] The major index $\mathrm{MAJ}(\pi) = \sum_{i:\pi_i > \pi_{i+1}} i$ was introduced in section 1.2 of Chapter 1

*joint-independent normal* if $X_n$ converges in distribution to the joint-independent normal distribution $\mathcal{N} \times \mathcal{N}$. In Section 7.3 we will use the similar methods to those of Section 7.2 to prove that the joint-distribution of $\textsc{inv}(\pi)$ and $\textsc{maj}(\pi)$ over $\pi \in \mathcal{S}_n$ is asymptotically joint-independent normal.

Since we are no longer concerned with order isomorphism, we may safely use the notation $f(n) \sim g(n)$ to denote asymptotic equivalence, that is, $f(n) \sim g(n)$ if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1.$$

## 7.2 Asymptotic Normality of Inversion Number

Let $\mathbb{E}(X)$ denote the expected value of random variable $X$. Then the $r^{th}$ *moment about the mean* $\mu = \mathbb{E}(X)$ is given by $\mathbb{E}((X-\mu)^r)$. Note that the second moment is the well-known *variance* of the distribution, $\sigma^2$. The moments of $X$ can be used to prove facts about the distribution in the *method of moments*, which we summarize in Lemma 52.

**Lemma 52** (Method of Moments)**.** *Suppose that $X$ and $Y$ are two distributions each with mean $\mu$ . If $\mathbb{E}((X-\mu)^r) = \mathbb{E}((Y-\mu)^r)$ for all $r \geq 0$, then $X = Y$.*

For the normal distribution $\mathcal{N}$ with mean $\mu$ and variance $\sigma^2$, the moments are given by

$$\mathbb{E}((X-\mu)^r) = \begin{cases} \sigma^r \frac{r!}{2^{r/2}\ (r/2)!} & r \text{ even}, \\ 0 & r \text{ odd}. \end{cases} \tag{7.2}$$

Combining Lemma 52 and Equation (7.2), we have a method for proving asymptotic normality:

**Lemma 53.** *Let $\{X_n\}$ be a sequence of distributions such that $m(n,r)$ is the $r^{th}$ moment about the mean for $X_n$. Then $\{X_n\}$ is asymptotically normal if*

$$\lim_{n\to\infty} \frac{m(n,r)}{m(n,2)^{r/2}} = \begin{cases} \frac{r!}{2^{r/2}\ (r/2)!} & r \text{ even}, \\ 0 & r \text{ odd}. \end{cases} \tag{7.3}$$

In this work we will find it computationally advantageous to use the $r^{th}$ *factorial* moment about the mean for a distribution $X$. This is given by $\mathbb{E}((X-\mu)^{(r)})$, where $a^{(r)} = a(a-1)(a-2)\cdots(a-r+1)$ is the $r^{th}$ falling factorial. Note that if $m(n,r)$ is the $r^{th}$ moment of $X_n$ and $f(n,r)$ is the $r^{th}$ factorial moment of $X_n$, then $m(n,r) \sim f(n,r)$ as $n \to \infty$ since $n^r \sim n^{(r)}$. Therefore Lemma 53 still holds if the moments $m(n,r)$ are taken to be factorial moments.

Let $F_n(q)$ be the probability generating function for the inversion number of a randomly chosen permutation of length $n$, i.e.,

$$F_n(q) := \frac{\mathbf{n}_q!}{n!}. \tag{7.4}$$

We wish to use Lemma 53 to prove this distribution is asymptotically normal.

Observe that the mean is given by $\mu_n = {F_n}'(1) = n(n-1)/4$. To simplify matters, we will centralize $F_n(q)$ to form $G_n(q)$,

$$G_n(q) := q^{-\mu_n} F_n(q). \tag{7.5}$$

Since $\mathbf{n}_q! = (1+q+\cdots+q^{n-1})(\mathbf{n-1})_q!$, we see that $F_n(q)$ satisfies the functional equation

$$F_n(q) = \frac{(1+q+\cdots+q^{n-1})}{n} F_{n-1}(q). \tag{7.6}$$

This in turn implies the following functional equation for the centralized version:

$$\begin{aligned} G_n(q) &= q^{-\mu_n} \frac{(1+q+\cdots+q^{n-1})}{n}\, q^{\mu_{n-1}} G_{n-1}(q) \\ &= \frac{q^n - 1}{q^{(n-1)/2}\ (q-1)\ n} G_{n-1}(q) \end{aligned} \tag{7.7}$$

If we compute the Taylor expansion of $G_n(q)$ about $q = 1$, then the coefficient of $(q-1)^r/r!$ is the $r^{th}$ factorial moment. To ease computation we make the substitution $q = 1 + z$ and expand about $z = 0$. Let $G_n(1+z) = \sum_{r\geq 0} g(n,r)\frac{z^r}{r!}$, so that $g(n,r)$ is the $r^{th}$ factorial moment. Computation and induction on $n$ reveals that $g(n,0) = 1$, $g(n,1) = 0$, and $g(n,2) = \frac{2n^3+3n^2-5n}{72}$. These will be useful later. Exact expressions for further terms $g(n,r)$ are useful for estimating the rate of convergence, but we will not concern ourselves with that level of detail at the moment. Asymptotic expressions for $g(n,r)$ will suffice for the application of Lemma 53.

Refer to equation (7.7) and define $a(n,r)$ as:

$$\left.\frac{q^n-1}{(q-1)\ q^{(n-1)/2}\ n}\right|_{q=1+z} = \frac{(1+z)^n-1}{(z)\ (1+z)^{(n-1)/2}\ n}$$
$$= \frac{1}{nz}\left((1+z)^{(n+1)/2}-(1+z)^{-(n-1)/2}\right) \tag{7.8}$$
$$= \sum_{r\geq 0} a(n,r)\frac{z^r}{r!}$$

Computation reveals that $a(n,0) = 1$, $a(n,1) = 0$, $a(n,2) = \frac{1}{12}(n^2-1)$ and $a(n,3) = \frac{1}{4}(1-n^2)$. Further the generalized binomial theorem implies that for fixed $r$, $a(n,r)$ is a polynomial in $n$ of degree $r$ if $r$ is even and degree $r-1$ if $r$ is odd. These degree bounds will be important later. Further terms $a(n,r)$ are useful for estimating the rate of convergence, but we will not concern ourselves with that level of detail. See [103] for an example of getting rates of convergence to the normal distribution.

Using Equation (7.7) and comparing coefficients we get

$$G_n(1+z) = \left(\sum_{r\geq 0} a(n,r)\frac{z^r}{r!}\right)\left(\sum_{r\geq 0} g(n-1,r)\frac{z^r}{r!}\right) \tag{7.9a}$$

$$g(n,r) = \sum_{k=0}^{r}\binom{r}{k} a(n,k)\ g(n-1,r-k) \tag{7.9b}$$

Since we know that $a(n,0) = 1$ and $a(n,1) = 0$, we rearrange the terms of Equation (7.9b) into a slightly more workable form.

$$g(n,r) - g(n-1,r) = \sum_{k=2}^{r}\binom{r}{k} a(n,k)\ g(n-1,r-k) \tag{7.10}$$

Equation (7.10) gives us useful information about the structure of $g(n,r)$. For fixed $r$, $g(n,r)$ is a polynomial in $n$ of degree $3\lfloor r/2\rfloor$. This can be shown as follows. Suppose $r$ is chosen minimally such that $g(n,r) = p(n,r)+q(n,r)$ where $p(n,r)$ is a polynomial of degree $3\lfloor r/2\rfloor$ and $q(n,r)$ is either a polynomial where the lowest-order term has degree greater than $3\lfloor r/2\rfloor$ or is not a polynomial at all. By minimality, the right-hand side of (7.10) is necessarily a polynomial of degree $3\lfloor r/2\rfloor - 1$ based on our observations on the degree of $a(n,k)$ above. Then $g(n,r)-g(n-1,r) = p(n,r)-p(n-1,r)+q(n,r)-q(n-1,r)$ must be a polynomial of degree at most $3\lfloor r/2\rfloor - 1$ and so $q(n,r)-q(n-1,r) = 0$ for any positive integer $n$. Hence $q(n,r)$ is constant, contradicting our assumption that $q(n,r)$ is not a small-degree polynomial. Thus it follows that $g(n,r)$ must be a polynomial in $n$ of degree at most $3\lfloor r/2 \rfloor$.

We wish to show that the factorial moments $g(n,r)$ satisfy the conditions in Lemma 53. We will do this by forming a secondary function $f(n,r)$ which satisfies the conditions in Lemma 53 and such that $f(n,r) \sim g(n,r)$ as $n \to \infty$. We construct $f(n,r)$ so that it has the following structure:

$$f(n,2r) = \frac{1}{36^r}\frac{(2r)!}{2^r r!}P(n,r) \tag{7.11a}$$

$$f(n,2r+1) = \frac{1}{36^r}\frac{(2r)!}{2^r r!}Q(n,r) \tag{7.11b}$$

$$P(n,r) = p_0(r)n^{3r} + p_1(r)n^{3r-1} \tag{7.11c}$$

$$Q(n,r) = q_0(r)n^{3r} + q_1(r)n^{3r-1} \tag{7.11d}$$

where $p_i(r)$ and $q_i(r)$ are polynomials in $r$.[2] To get explicit expressions for $p_i(r)$ and $q_i(r)$, we employ a bit of curve-fitting. Use (7.10) to compute $g(n,r) \cdot \frac{36^r 2^r r!}{(2r)!}$ for, say, $n \le 30$ and $r \le 10$. For fixed $r = 2r_0$ one fits a degree $3r_0$ polynomial in $n$, then one looks at the coefficients of $n^{3r_0-i}$ and fits another polynomial $p_i(r)$ to that sequence. Using the same procedure for odd $r = 2r_0+1$ gets us $q_i(r)$. Executing these procedures gets us the following polynomials:

$$\begin{aligned}
p_0(r) &= 1\\
p_1(r) &= -\frac{9}{25}r^2 + \frac{93}{50}r\\
q_0(r) &= -r(2r+1)\\
q_1(r) &= \frac{18}{25}r^4 - \frac{84}{25}r^3 - \frac{93}{50}r^2.
\end{aligned}$$

By construction $f(n,2)^r \sim \frac{n^{3r}}{36^r}$, so $m(n,r) = f(n,r)$ will satisfy Equation (7.3). Now it remains to prove $f(n,r) \sim g(n,r)$ and we accomplish this by proving $f(n,r)$ satisfies a recurrence similar to Equation (7.10), where $g$ is replaced by $f$ and equality by asymptotic equivalence. This in turn implies $f(n,r) \sim g(n,r)$, and hence $G_n(1+z)$ (and $F_n(q)$) approaches the normal distribution as $n \to \infty$.

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[2]To get information on *rate* of convergence we could extend the expressions for $P(n,r)$ and $Q(n,r)$ and consider terms $p_i(r)n^{3r-i}$ and $q_i(r)n^{3r-i}$ for larger $i$, but we will not concern ourselves with that level of detail here.

We first assume $r$ is even. This will amount to proving:

$$f(n,2r) - f(n-1,2r) \sim \sum_{k=2}^{2r} \binom{2r}{k} a(n,k)\ f(n-1,2r-k) \tag{7.12}$$

Looking at the left-hand side we see:

$$f(n,2r) - f(n-1,2r) = \frac{(2r)!}{2^r r!} \left(P(n,r) - P(n-1,r)\right) \tag{7.13}$$

Since $\deg_n(P(n,r)) = 3r$, we see that $\deg_n(P(n,r) - P(n-1,r)) \leq 3r-1$ and so we focus on finding the coefficient of $n^{3r-1}$

$$\begin{aligned}[n^{3r-1}]\left(P(n,r) - P(n-1,r)\right) &= p_1(r) - \left(p_0(r)\binom{3r}{1}(-1) + p_1(r)\binom{3r}{0}\right)\\ &= 3rp_0(r)\\ &= 3r.\end{aligned} \tag{7.14}$$

Therefore we have shown

$$f(n,2r) - f(n-1,2r) \sim \frac{(2r)!}{2^r r!} 3rp_0(r)n^{3r-1}. \tag{7.15}$$

We now move to the right-hand side of (7.12). Recall that for fixed $k$, $a(n,k)$ is a polynomial in $n$ of degree $k$ if $k$ is even and $k-1$ if $k$ is odd. Similarly for fixed $k=2j$, $f(n-1,2r-k)$ is a polynomial in $n$ with degree $3(r-k/2)$, and for $k=2j+1$ $f(n-1,2r-k)$ is a polynomial in $n$ with degree $3(r-(k-1)/2)$. Hence the asymptotic behavior of the right-hand side of (7.12) is determined solely by the $k=2$ term. This implies the following asymptotic behavior of the right-hand side:

$$\begin{aligned}&\sum_{k=2}^{2r} \binom{2r}{k} a(n,k)\ f(n-1,2r-k)\\ &\quad\sim \binom{2r}{2} a(n,2)\ f(n-1,2r-2)\\ &\quad= \frac{1}{36^{r-1}} \frac{(2r-2)!}{2^{r-1}(r-1)!} \frac{(2r)(2r-1)}{2} \frac{1}{12}(n^2-1)P(n-1,r-1)\\ &\quad= \frac{3}{36^r} \frac{(2r)!}{2^r(r-1)!}(n^2-1)(p_0(r)(n-1)^{3(r-1)} + p_1(r)(n-1)^{3(r-1)-1})\\ &\quad\sim \frac{3}{36^r} \frac{(2r)!}{2^r(r-1)!} p_0(r)n^{3r-1}\\ &\quad= \frac{1}{36^r} \frac{(2r)!}{2^r(r)!} 3rp_0(r)n^{3r-1}.\end{aligned} \tag{7.16}$$

Comparing (7.15) and (7.16), we see that we have proven that Equation (7.12) holds.

An alternate approach with less tedious algebra can be accomplished via computer. Suppose we have not pre-computed the $p_i(r)$ and $q_i(r)$. Divide both sides of (7.12) by $\frac{1}{36^r}\frac{(2r)!}{2^r(r)!}$ and make the same observations that both sides are polynomials in $n$ with degree $3r-1$ (one can even keep the full summation in the right-hand side). To show these two sides are asymptotically equivalent, compute both sides for, say, $n \leq 30$ and $r \leq 10$ and compute the difference of the two sides which we know to be a polynomial of degree at most $3r-1$ with a coefficient $p(r)$ which itself is a polynomial in $r$ which has degree at most 2. The 300 terms we computed are more than sufficient to prove that $p(r) = 0$, in a similar fashion to how we computed $p_i(r)$ and $q_i(r)$ above. This in turn implies the correctness of (7.12). Of course in this case it turns out that the algebra above shows $p(r) = 3rp_0(r) - 3rp_0(r)$, which is obviously 0, but we will see below the analogue in the case for odd moments is less obviously 0. Also observe this method precludes performing the algebra that gets us this formulation of $p(r)$. Zeilberger uses this alternate approach in [103] to prove that a certain $q$-analogue of the Catalan numbers also tends toward the normal distribution, and we will follow a similar tack in Section 7.3 below.

We now move to the case of odd moments, although much of the same reasoning will apply. We wish to prove

$$f(n, 2r+1) - f(n-1, 2r+1) \sim \sum_{k=2}^{2r+1} \binom{2r+1}{k} a(n,k)\ f(n-1, 2r+1-k). \quad (7.17)$$

By the same calculations (with $P$ replaced by $Q$ and $p_i$ by $q_i$) as shown in (7.14), we see that

$$f(n, 2r+1) - f(n-1, 2r+1) \sim \frac{(2r)!}{2^r r!} 3rq_0(r)n^{3r-1}. \quad (7.18)$$

For the right-hand side of (7.17), observe as above that for fixed $k$ $a(n,k)f(n, 2r+1-k)$ is a polynomial in $n$ with degree $3r-k/2$ if $k$ is even and degree $3r-(k-1)/2$ if $k$ is odd. Hence the summands for both $k=2$ and $k=3$ (and no others) will contribute terms of order $n^{3r-1}$

$$\begin{aligned}
&\sum_{k=2}^{2r+1}\binom{2r+1}{k}a(n,k)\ f(n-1,2r+1-k)\\
&\quad\sim\binom{2r+1}{2}a(n,2)\ f(n-1,2r-1)+\binom{2r+1}{3}a(n,3)\ f(n-1,2r-2)\\
&\quad=\frac{1}{36^{r-1}}\frac{(2r-2)!}{2^{r-1}(r-1)!}\Bigg(\binom{2r+1}{2}a(n,2)Q(n-1,r-1)+\\
&\qquad\binom{2r+1}{3}a(n,3)P(n-1,r-1)\Bigg)
\end{aligned}\tag{7.19}$$

We now substitute for the known values of $a(n,2)$ and $a(n,3)$ and expand the $(n-1)^{3(r-1)}$ and $(n-1)^{3(r-1)-1}$ terms which appear in $P(n-1,r-1)$ and $Q(n-1,r-1)$. Extracting the leading terms we get

$$\begin{aligned}
&\binom{2r+1}{2}a(n,2)Q(n-1,r-1)+\binom{2r+1}{3}a(n,3)P(n-1,r-1)\\
&\qquad\sim\frac{1}{24}(2r+1)(2r)\left(q_0(r-1)-(2r-1)p_0(r-1)\right)n^{3r-1}
\end{aligned}\tag{7.20}$$

Returning the right-hand side of (7.17) we get

$$\begin{aligned}
&\sum_{k=2}^{2r+1}\binom{2r+1}{k}a(n,k)\ f(n-1,2r+1-k)\\
&\qquad\sim\frac{1}{36^r}\frac{(2r)!}{2^r r!}\left(3r(2r+1)\left(\frac{q_0(r-1)}{2r-1}-p_0(r-1)\right)\right)n^{3r-1}
\end{aligned}\tag{7.21}$$

Using the values for $p_0(r)$ and $q_0(r)$ computed previously, we see that $3r\ q_0(r) = 3r(2r+1)\left(\frac{q_0(r-1)}{2r-1}-p_0(r-1)\right)$, thus proving (7.17).

We could have proven (7.17) via the same "alternate approach" as described above for (7.12). Divide both sides of (7.17) by $\frac{1}{36^r}\frac{(2r)!}{2^r(r)!}$ and again observe that both sides are polynomials in $n$ with degree $3r-1$. To show these two sides are asymptotically equivalent, compute both sides for $n \leq 30$ and $r \leq 10$ and compute the difference of the two sides which we know to be a polynomial of degree at most $3r-1$ with a coefficient $p(r)$ which itself is a polynomial in $r$ which has degree at most 4. In this case we would get $p(r) = 3rq_0(r) - \frac{3}{4r-2}((2r+1)(2r)q_0(r-1)-(2r-1)p_0(r-1))$. The 300 terms we computed are more than sufficient to prove that $p(r) = 0$. This in turn implies the correctness of (7.17).

Hence we have proven equation (7.17), which combines with (7.12) to form equation (7.22):

$$f(n,r) - f(n-1,r) \sim \sum_{k=2}^{r} \binom{r}{k} a(n,k)\ f(n-1,r-k) \tag{7.22}$$

Since $f(n,0) = g(n,0) = 1$ and $f(n,1) = g(n,1) = 0$ for any $n$, we know that $f(n,r) - f(n-1,r) \sim g(n,r) - g(n-1,r)$ for any fixed $r$. Since we know that $f(n,r)$ and $g(n,r)$ are both polynomials in $n$ of the same degree, expanding binomials shows us that $f(n,r) - f(n-1,r) \sim g(n,r) - g(n-1,r)$, which implies $f(n,r) \sim g(n,r)$. Hence we have proven that the distribution of inversion number over permutations in $\mathcal{S}_n$ is asymptotically normal.

## 7.3 Joint Distribution of Inversion Number and Major Index

We now may move on to the main result of this chapter, the asymptotic independence of the number of inversions and the major index.

Let $F_n(p,q)$ be the weight enumerator of $\mathcal{S}_n$ where each permutation has weight $p^{\mathrm{INV}(\pi)} q^{\mathrm{MAJ}(\pi)}$, i.e.,

$$F_n(p,q) := \sum_{\pi \in \mathcal{S}_n} p^{\mathrm{INV}(\pi)}\, q^{\mathrm{MAJ}(\pi)}$$

As shown in the previous sections $F_n(p,1) = F_n(1,p)$ is asymptotically normal. We wish to show that $F_n(p,q)$ is asymptotically joint-independently normal.

We will need to define the joint-moments for two sequences, often called *mixed moments* or *covariances.*

**Definition 54.** *Let $X$ and $Y$ be two probability distributions with means $\mu_X, \mu_Y$, respectively, and standard deviations $\sigma_X, \sigma_Y$, respectively. The $(r,s)$-covariance (or $(r,s)$-moment) of the pair $(X,Y)$ is given by*

$$\mathrm{Cov}_{r,s}(X,Y) = \mathbb{E}((X-\mu_X)^r\,(Y-\mu_Y)^s)$$

*The normalized $(r,s)$-covariance of $(X,Y)$ is given by*

$$\frac{\mathrm{Cov}_{r,s}(X,Y)}{{\sigma_X}^r\ {\sigma_Y}^s}$$

Observe that $\mathrm{Cov}_{r,0}(X,Y)$ is just the $r^{th}$ moment of $X$ and is $\mathrm{Cov}_{0,s}(X,Y)$ the $s^{th}$ moment of $Y$. To apply the method of moments to prove asymptotic independence, we will need the moments for such a distribution.

**Lemma 55.** *Suppose that $X$ and $Y$ are joint-independent normal distributions each with mean 0 and standard deviations $\sigma_x = \sqrt{\mathrm{Cov}_{2,0}(X,Y)}$, $\sigma_y = \sqrt{\mathrm{Cov}_{0,2}(X,Y)}$. Then the normalized covariances are given by*

$$\frac{\mathrm{Cov}_{r,s}(X,Y)}{\sigma^{r+s}} = \begin{cases} \frac{r!}{2^{r/2}\ (r/2)!} \frac{s!}{2^{s/2}\ (s/2)!} & r \text{ and } s \text{ are even,} \\ 0 & \text{otherwise.} \end{cases}$$

Now we may write the essential lemma for proving asymptotic joint-independent normality.

**Lemma 56.** *Let $\{X_n\}$ and $\{Y_n\}$ be two sequences of distributions, each with mean 0 and standard deviation $\sigma_n$. Then if*

$$\lim_{n\to\infty} \frac{\mathrm{Cov}_{r,s}(X_n,Y_n)}{{\sigma_n}^{r+s}} = \begin{cases} \frac{r!}{2^{r/2}\ (r/2)!} \frac{s!}{2^{s/2}\ (s/2)!} & r \text{ and } s \text{ are even,} \\ 0 & \text{otherwise.} \end{cases}$$

*then $\{X_n\}$ and $\{Y_n\}$ are asymptotically joint-independently normal.*

As above in the univariate case, we will use *factorial* moments

$$\mathrm{FCov}_{r,s}(X,Y) = \mathbb{E}\big((X-\mu_X)^{(r)}\ (Y-\mu_Y)^{(s)}\big)$$

where $x^{(r)}$ is the falling factorial. Since moments and factorial moments exhibit the same asymptotic behavior, the above lemmas hold when Cov is replaced by FCov.

As a precursor to this work, Ekhad shows in [40] that the correlation coefficient for INV and MAJ over $\mathcal{S}_n$ is given by

$$\rho_n = \frac{\mathrm{Cov}_{1,1}(\text{INV}, \text{MAJ})}{{\sigma_n}^2} = \frac{9}{2n+5}. \tag{7.23}$$

Hence $\rho_n \to 0$ as $n \to \infty$ so INV and MAJ are asymptotically uncorrelated. This is not enough to prove independence, but it is enough to warrant further investigation.

We now move on to prove $F_n(p,q)$ is asymptotically joint-independent normal. To get an analogue of equation (7.6), we will need to refine $\mathcal{S}_n$ according to the last letter. Let $F(n,i,p,q)$ be the weight enumerator of $\{\pi \in \mathcal{S}_n : \pi_n = i\}$ with weight $p^{\text{INV}(\pi)}\, q^{\text{MAJ}(\pi)}$:

$$F(n,i,p,q) := \sum_{\substack{\pi\in\mathcal{S}_n \\ \pi_n = i}} p^{\text{INV}(\pi)}\, q^{\text{MAJ}(\pi)} \tag{7.24}$$

For brevity we will omit the arguments $p, q$ from $F(n, i, p, q)$ when no confusion will arise.

If $\pi$ ends with $n$, then that last letter cannot be involved in any inversions (and hence does not contribute to INV) nor is it involved with any descents (and hence does not contribute to MAJ). Hence

$$F(n, n) = \sum_{i=1}^{n-1} F(n-1, i). \tag{7.25}$$

In particular, observe that $F(n+1, n+1) = \sum_{\pi \in \mathcal{S}_n} p^{\text{INV}(\pi)} q^{\text{MAJ}(\pi)}$.

Now suppose $\pi \in \mathcal{S}_n$ such that $\pi_{n-1} = j$ and $\pi_n = i < n$. Let $\pi' = \pi_1 \cdots \pi_{n-2}$, so we have decomposed $\pi = \pi' \, j \, i$. Deleting $\pi_n = i$ will remove $n - i$ inversions from $\pi$. Similarly, if $j > i$ then removing $\pi_n$ will remove a descent and decrease MAJ by $(n-1)$. If $j < i$ then removing $\pi_n$ will have no effect on major index. Hence we see:

$$\begin{aligned} \text{INV}(\pi' \, j \, i) &= (n-i) + \text{INV}(\pi' \, j) \\ \text{MAJ}(\pi' \, j \, i) &= \begin{cases} \text{MAJ}(\pi' \, j) & \text{if } j < i \\ \text{MAJ}(\pi' \, j) + (n-1) & \text{if } j > i. \end{cases} \end{aligned} \tag{7.26}$$

Observe for fixed $i, j$ such that $i < j$ there is a natural correspondence between the length-$n$ permutations $\pi = \pi' \, j \, i$ and length-$(n-1)$ permutations which end with $(j-1)$ ("delete $i$ and decrement each $\pi_k > i$"). It then follows that if $i < n$ then $F(n, i, p, q)$ satisfies the following recurrence:

$$F(n, i) = p^{n-i} \left( \sum_{j=1}^{i-1} F(n-1, j) + q^{n-1} \sum_{j=i}^{n-1} F(n-1, j) \right). \tag{7.27}$$

We can change (7.27) into a more desirable "sum-less" form by considering the following difference when $i < n$:

$$\begin{aligned} F(n, i) - p \, F(n, i+1) = p^{n-i} &\left( \sum_{j=1}^{i-1} F(n-1, j) + q^{n-1} \sum_{j=i}^{n-1} F(n-1, j) \right) + \\ &- p \, p^{n-i+1} \left( \sum_{j=1}^{i} F(n-1, j) + q^{n-1} \sum_{j=i+1}^{n-1} F(n-1, j) \right) \\ &= p^{n-1} \left( -F(n-1, i) + q^{n-1} \, F(n-1, i) \right) \\ &= p^{n-i} \, (q^{n-1} - 1) \, F(n-1, i). \end{aligned}$$

Rearranging terms, we get the following recurrence when $i < n$:

$$F(n,i) = p\,F(n,i+1) + p^{n-i}\,(q^{n-1}-1)\,F(n-1,i). \tag{7.28}$$

As in the univariate case, we would like to centralize $F(n,i,p,q)$ to form $G(n,i,p,q)$. To do this, we will need to know the average values of INV and MAJ over the set of $n$-permutations ending with $i$. For INV it was stated above that removing a terminal $i$ leaves a $(n-1)$-permutation with $(n-i)$ fewer inversions. The average number of inversions over $(n-1)$-permutations is $\frac{1}{2}\binom{n-1}{2}$, so linearity of expectation gets us that the average number of inversions of $n$-permutations ending with $i$ is $\mu_{n,i} = (n-i)+\frac{1}{2}\binom{n-1}{2}$. To find the average value of MAJ, Foata proved in [45] that $F(n,i,p,q) = F(n,i,q,p)$, that is, INV and MAJ are equidistributed over the set of permutations of length $n$ ending with a fixed letter $i$. Hence the average value of MAJ is also $(n-i)+\frac{1}{2}\binom{n-1}{2}$.[3]

We now construct the centralized probability generating function $G(n,i,p,q)$,

$$G(n,i,p,q) := \frac{1}{(n-1)!}\,\frac{F(n,i,p,q)}{(pq)^{n-i+\binom{n-1}{2}/2}}. \tag{7.29}$$

As we did for $F(n,i,p,q)$, we will omit the arguments $p, q$ when no confusion will arise. We may now use (7.28) and (7.25) to get recurrences for $G(n,i)$,

$$G(n,i) = q^{-1}G(n,i+1) + \frac{p^{n-i}\,(q^{n-1}-1)}{(pq)^{n/2}(n-1)}\,G(n-1,i) \quad \text{if } i < n \tag{7.30a}$$

$$G(n,n) = \frac{1}{n-1}\sum_{j=1}^{n-1}(pq)^{n/2-j}G(n-1,j). \tag{7.30b}$$

We also need the initial conditions that $G(1,1) = 1$ and $G(n,i) = 0$ if $i < 1$ or $i > n$.

Equations (7.30a) and (7.30b) stand as analogues to (7.7), and we will proceed in a similar fashion. We perform a Taylor expansion of $G(n,i,1+x,1+y)$ about $(x,y) = (0,0)$, and denote by $g(n,i,r,s)$ the coefficient of $\frac{x^r}{r!}\frac{y^s}{s!}$:

$$G(n,i,1+x,1+y) = \sum_{r,s} g(n,i,r,s)\frac{x^r}{r!}\frac{y^s}{s!}. \tag{7.31}$$

As in the univariate case, these $g(n,i,r,s)$ have a meaning in terms of mixed moments. In particular, $g(n,i,r,s)$ is the factorial $(r,s)$-covariance over the set $\{\pi \in \mathcal{S}_n :$

[3] Alternately one could use (7.26) in conjunction with mathematical induction on the length of the permutations to derive the average value of MAJ.

$\pi_n = i\}$:

$$g(n,i,r,s) = \mathrm{FCov}_{r,s}(\textsc{inv}, \textsc{maj}).$$

We will show that $(\textsc{inv}, \textsc{maj})$ is asymptotically joint-independent by proving the following proposition:

**Proposition 57.**

$$\lim_{n\to\infty} \frac{g(n,n,r,s)}{g(n,n,2,0)^{r/2}\, g(n,n,0,2)^{s/2}} = \begin{cases} \frac{r!}{2^{r/2}\ (r/2)!} \frac{s!}{2^{s/2}\ (s/2)!} & r \text{ and } s \text{ are even,} \\ 0 & \text{otherwise.} \end{cases}$$

Combining Proposition 57 with Lemma 56 will prove that $\textsc{maj}$ and $\textsc{inv}$ are asymptotically joint-independent normal, since $g(n,n,r,s)$ is equal to the factorial $(r,s)$-covariance of $\textsc{maj}$ and $\textsc{inv}$ over all of $\mathcal{S}_{n-1}$.

To prove Proposition 57 we will proceed as in the univariate case by conjecturing some secondary function $f(n,i,r,s)$ that we know has the desired limits as $n \to \infty$ when $i = n$, and then show that $f(n,i,r,s) \sim g(n,i,r,s)$ showing that each asymptotically satisfy the same recurrence. We will first construct the recurrence, which we can then use to generate sample values for many $n, i, r, s$ to inform our conjecture for $f(n,i,r,s)$.

We will need a more complicated recurrence than that of (7.9b), since the corresponding functional equations (7.30a) and (7.30b) are more complicated than (7.7). Making the changes of variables $p = 1 + x$ and $q = 1 + y$, equations (7.30a) and (7.30b) implies:

$$\begin{aligned} g(n,i,r,s) = \sum_{k,\ell\geq 0} \binom{r}{k}\binom{s}{\ell} a(n,i,k,\ell)\, g(n,i+1,r-k,s-\ell) \\ + \sum_{k,\ell\geq 0} \binom{r}{k}\binom{s}{\ell} b(n,i,k,\ell)\, g(n-1,i,r-k,s-\ell) \text{ if } i<n \quad (7.32) \\ g(n,n,r,s) = \frac{1}{n-1} \sum_{j=1}^{n-1} \sum_{k,\ell\geq 0} \binom{r}{k}\binom{s}{\ell} c(n,j,k,\ell)\, g(n-1,j,r-k,s-\ell) \end{aligned}$$

where $a(n,i,k,\ell), b(n,i,k,\ell)$, and $c(n,j,k,\ell)$ are the following coefficients:

$$\begin{aligned} \frac{1}{1+y} &= \sum_{k,\ell\geq 0} a(n,i,k,\ell)\frac{x^k}{k!}\frac{y^\ell}{\ell!} \\ \frac{(1+x)^{n-i}\left((1+y)^{n-1}-1\right)}{((1+x)(1+y))^{n/2}(n-1)} &= \sum_{k,\ell\geq 0} b(n,i,k,\ell)\frac{x^k}{k!}\frac{y^\ell}{\ell!} \\ ((1+x)(1+y))^{n/2-j} &= \sum_{k,\ell\geq 0} c(n,j,k,\ell)\frac{x^k}{k!}\frac{y^\ell}{\ell!}. \end{aligned} \tag{7.33}$$

The generalized binomial theorem yields explicit forms for $a(n,i,k,\ell)$, $b(n,i,k,\ell)$, and $c(n,i,k,\ell)$ in terms of $n,i,k,\ell$. First we see that

$$a(n,i,k,\ell) = \begin{cases} (-1)^\ell \ell! & k=0 \\ 0 & k>0. \end{cases} \tag{7.34}$$

Next consider $b(n,i,k,\ell)$. Taylor expansion will show that $b(n,i,k,0)=0$ for any $k\geq 0$ and that $b(n,i,1,0)=b(n,i,0,1)=1$. For fixed $k,\ell$, the generalized binomial theorem tells us that $b(n,i,k,\ell)$ is a polynomial in $n$ and $i$ with total degree $k+\ell-1$ if $\ell$ is odd and total degree $k+\ell-2$ if $\ell\geq 2$ is even.

Last, the generalized binomial theorem gets an explicit expression for $c(n,i,k,\ell)$

$$c(n,i,k,\ell) = \prod_{j=0}^{k-1}(\frac{n}{2}-i-j)\prod_{j=0}^{\ell-1}(\frac{n}{2}-i-j) \tag{7.35}$$

In particular, observe that $c(n,i,0,0)=1$, and that $c(n,i,k,\ell)$ is a polynomial in $(n,i)$ of total degree $k+\ell$.

We can combine the above observations and collect similar terms to refine equation (7.32) as follows:

$$\begin{aligned} &g(n,i,r,s) - g(n,i+1,r,s) \\ &\quad = \sum_{\ell\geq 1}\binom{s}{\ell}(-1)^\ell \ell! g(n,i+1,r,s-\ell) \\ &\qquad\qquad + \sum_{k=0}^{r}\sum_{\ell=1}^{s}\binom{r}{k}\binom{s}{\ell} b(n,i,k,\ell) g(n-1,i,r-k,s-\ell) \text{ if } i<n \\ &g(n,n,r,s) - \frac{1}{n-1}\sum_{j=1}^{n-1} g(n-1,j,r,s) \\ &\quad = \sum_{\substack{(k,\ell)\in\mathbb{N}^2 \\ (k,\ell)\neq(0,0)}}\binom{r}{k}\binom{s}{\ell}\sum_{j=1}^{n-1} c(n,j,k,\ell) g(n-1,j,r-k,s-\ell) \end{aligned} \tag{7.36}$$

It can be seen that for fixed $r, s$ $g(n,i,r,s)$ is a polynomial in $(n,i)$ of total degree:

$$\deg g(n,i,r,s) = \begin{cases} \frac{3}{2}(r+s) & r \text{ and } s \text{ even} \\ \frac{3}{2}(r+s)-1 & r \text{ and } s \text{ odd} \\ \frac{3}{2}(r+s-1) & \text{otherwise} \end{cases}$$

Since we are only interested in satisfying (7.32) asymptotically, we can ignore most of the low-degree terms which result from the summations. In particular, we can restrict our attention to the summation indices $k = 0, 1$ and $\ell = 0, 1$ and get the following information about $g(n,i,r,s)$:

$$\begin{aligned} & g(n,i,r,s) - g(n,i+1,r,s) \\ & \qquad = -s\, g(n,i+1,r,s-1) + s\, g(n-1,i,r,s-1) \\ & \qquad\quad + \tfrac{rs}{2}(n-2i)g(n-1,i,r-1,s-1) \\ & \qquad\quad + \cdots \text{(lower order terms)} \\ & g(n,n,r,s) - \frac{1}{n-1}\sum_{j=1}^{n-1} g(n-1,j,r,s) \\ & \qquad = \frac{s}{n-1}\sum_{j=1}^{n-1}\left(\frac{n}{2}-j\right) g(n-1,j,r,s-1) \\ & \qquad\quad + \frac{r}{n-1}\sum_{j=1}^{n-1}\left(\frac{n}{2}-j\right) g(n-1,j,r-1,s) \\ & \qquad\quad + \frac{rs}{n-1}\sum_{j=1}^{n-1}\left(\frac{n}{2}-j\right)^2 g(n-1,j,r-1,s-1) \\ & \qquad\quad + \cdots \text{(lower order terms).} \end{aligned} \tag{7.37}$$

The "lower order terms" each have total degree in $(n,i)$ at most $\lfloor \frac{3}{2} \rfloor - 2$.

We now use the exact recurrences (7.32) to compute $g(n,i,r,s)$ for $1 \le i \le n \le 30$ and $1 \le r, s \le 10$. This is enough to form the conjectured asymptotics, which we make into a secondary function $f(n,i,r,s)$.

$$f(n,i,2r,2s) := \frac{(2r)!}{2^r\, r!}\,\frac{(2s)!}{2^s\, s!}\left(\frac{1}{36}\right)^{r+s} Q_{EE}(n,i,r,s) \tag{7.38a}$$

$$f(n,i,2r+1,2s) := \frac{(2r)!}{2^r\, r!}\,\frac{(2s)!}{2^s\, s!}\left(\frac{1}{36}\right)^{r+s} Q_{OE}(n,i,r,s) \tag{7.38b}$$

$$f(n,i,2r,2s+1) := \frac{(2r)!}{2^r\, r!}\,\frac{(2s)!}{2^s\, s!}\left(\frac{1}{36}\right)^{r+s} Q_{EO}(n,i,r,s) \tag{7.38c}$$

$$f(n,i,2r+1,2s+1) := \frac{(2r)!}{2^r\, r!}\,\frac{(2s)!}{2^s\, s!}\left(\frac{1}{36}\right)^{r+s} Q_{OO}(n,i,r,s) \tag{7.38d}$$

where $Q_{EE}(n,i,r,s)$, $Q_{OE}(n,i,r,s)$, and $Q_{EO}(n,i,r,s)$ are polynomials in $n,i$ of total degree $3r+3s$ and $Q_{OO}$ is a polynomial in $n,i$ of total degree $3r+3s+2$. We will dictate the leading terms to be:

$$Q_{EE}(n,i,r,s) := n^{3r+3s} + q_{EE}(n,i,r,s) \tag{7.39a}$$

$$Q_{OE}(n,i,r,s) := (-r)(2r+1)n^{3r+3s} + q_{OE}(n,i,r,s) \tag{7.39b}$$

$$Q_{EO}(n,i,r,s) := (2s+1)n^{3r+3s-3}\big(-sn^3 - 6rn^2\, i + 18rn\, i^2 - 12ri^3\big) + q_{EO}(n,i,r,s) \tag{7.39c}$$

$$Q_{OO}(n,i,r,s) := \frac{1}{8}n^{3r+3s}(n-2i)^2 + q_{OO}(n,i,r,s), \tag{7.39d}$$

where in each case $q_{**}$ is a polynomial in $n,i$ of total degree strictly less than that of $Q_{**}$.

Observe that $f(n,i,2,0) = f(n,i,0,2) = \frac{n^3}{36}$, and so we get the following limits:

$$\lim_{n\to\infty} \frac{f(n,n,r,s)}{f(n,n,2,0)^{r/2}\, f(n,n,0,2)^{s/2}} = \begin{cases} \frac{r!}{2^{r/2}\,(r/2)!}\frac{s!}{2^{s/2}\,(s/2)!} & r \text{ and } s \text{ are even,} \\ 0 & \text{otherwise.} \end{cases}$$

Thus we see that the form given by $f(n,n,r,s)$ is desirable, since it satisfies the conditions of Proposition 57. It remains to show that this form is *accurate* however, i.e., that $f(n,n,r,s) \sim g(n,n,r,s)$. This is accomplished by showing that $f(n,i,r,s)$ satisfies the same large-scale behavior as $g(n,i,r,s)$, in particular it suffices to show that $f(n,i,r,s)$ satisfies the same relations as illustrated in (7.37).

Showing that $f(n,i,r,s)$ satisfies the relations in (7.37) requires considerable case analysis: in particular one must consider the pair of equations in four cases depending on whether $(r, s)$ is (even, even), (even, odd), (odd, even), or (odd, odd). Each of these eight computations is done in the same way: substitute the formulas for $f(n, i, r, s)$ given in (7.38) into the left- and right-hand sides of (7.37) and check that the leading terms match. One does not need to compute $q_{EE}$, $q_{EO}$, $q_{OE}$, or $q_{OO}$, since the coefficients of the leading terms in $f(n, i, r, s)$ are themselves polynomials in $(r, s)$ of degree at most 2. The relations in (7.37) themselves introduce $(r, s)$-terms of total degree at most 2. Hence in each of these 8 equations to check, it suffices to check that (7.37) holds for finitely many numerical values of $r$ and $s$. The computations are left to the computer. The Maple code to verify this is contained in the Maple package MajInv, available for download at `http://www.math.rutgers.edu/~zeilberg/tokhniot/InvMaj`.

Thus we have proven Proposition 57 that the joint distribution of INV and MAJ over $\mathcal{S}_n$ is asymptotically joint-independently normal.

## 7.4 Conclusions and Future Directions

In this chapter we proved that the inversion number and major index are asymptotically joint-independent normal. Hence as $n \to \infty$ one cannot draw conclusions about $\text{MAJ}(\pi)$ given the value of $\text{INV}(\pi)$. More importantly, the method should be general enough that it will work for other pairs of permutation statistics.

Imitating [40], one can show that the correlation between the number of descents and the number of inversions is

$$\rho_n^{\text{des},\text{INV}} = \frac{\text{Cov}_{1,1}(\text{des}, \text{INV})}{\sqrt{\text{Var}(\text{des})\text{Var}(\text{INV})}} = \frac{18(n-1)}{\sqrt{12\,n^4 + 30\,n^3 - 12\,n^2 - 30\,n}}. \tag{7.40}$$

Since $\lim_{n\to\infty} \rho_n^{\text{des},\text{INV}} = 0$, we see des and INV are asymptotically uncorrelated. Since both des and INV are asymptotically normal, this suggests that perhaps together they are asymptotically joint-independent normal.

In [9], Babson and Steingrímsson list fourteen different Mahonian statistics (those with weight-enumerator $\mathbf{n}_q!$), including INV and MAJ. This suggests a systematic study of pairwise independence. Such a study would require analogues of equations (7.25) and (7.28), although the tools developed in [48] would yield many of these recurrences automatically.

One may also consider the statistics over certain subclasses of permutations, such as the $B$-avoiding permutations for a set of patterns $B$. The refinements of enumeration schemes discussed in Chapter 4 can yield recurrences similar to that in Equation (7.6). Perhaps INV is asymptotically normal over certain classes $\mathcal{S}_n(B)$ as well, and even more interesting would be sets $B$ such that INV and MAJ are asymptotically joint-independently normal over certain classes $\mathcal{S}_n(B)$.

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