Upper bounds on the rate of quantum ergodicity

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Abstract

We study the semiclassical behaviour of eigenfunctions of quantum systems with ergodic classical limit. By the quantum ergodicity theorem almost all of these eigenfunctions become equidistributed in a weak sense. We give a simple derivation of an upper bound of order $|\ln \hbar|^{-1}$ on the rate of quantum ergodicity if the classical system is ergodic with a certain rate. In addition we obtain a similar bound on transition amplitudes if the classical system is weak mixing. Both results generalise previous ones by Zelditch. We then extend the results to some classes of quantised maps on the torus and obtain a logarithmic rate for perturbed cat-maps and a sharp algebraic rate for parabolic maps.

1 Introduction

The quantum ergodicity theorem by Shnirelman, Zelditch and Colin de Verdière, [Šni74, Zel87, CdV85], states that almost all eigenfunctions of a quantum mechanical Hamilton operator become equidistributed in the semiclassical limit if the underlying classical system is ergodic.

Consider as example an Hamiltonian of the form

$$\mathcal{H} = -\hbar^2 \Delta + V \tag{1}$$

on $L^2(\mathbb{R}^d)$ with a smooth potential satisfying $|\partial^\alpha V(x)| \leq C_\alpha (1 + |x|^2)^{m/2}$ for some $m \in \mathbb{R}$ and all $\alpha \in \mathbb{N}^d$. Assume that for a fixed energy E the classical energy-shell $\Sigma_E := \{(\xi, x) \in \mathbb{R}^d \times \mathbb{R}^d \,;\, \xi^2 + V(x) = E\}$ is compact, then the spectrum of $\mathcal{H}$ is discrete in a neighbourhood of E, and we will denote by $N(I(E, \hbar))$ the number of eigenvalues in the interval $I(E, \hbar) := [E - \alpha\hbar, E + \alpha\hbar]$, $\alpha > 0$. If now the Hamiltonian flow generated by $H = \xi^2 + V(x)$ ergodic on Σ_E then the normalised eigenfunctions ψ_n of $\mathcal{H}$ satisfy

$$\lim_{\hbar \to 0} \frac{1}{N(I(E, \hbar))} \sum_{E_n \in I(E, \hbar)} |\langle \psi_n, \mathrm{Op}[a]\psi_n \rangle - \overline{a}_E|^2 = 0 \tag{2}$$

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with $\overline{a}_E := \frac{1}{\mathrm{vol}(\Sigma_E)} \int_{\Sigma_E} a \, \mathrm{d}\mu_E$ and where a is a smooth bounded function on phase space and $\mathrm{Op}[a]$ its Weyl quantisation (defined below in (4)). This result is the semi-classical version of the quantum ergodicity theorem, which was derived in [HMR87]. It implies that almost all of the expectation values $\langle \psi_n, \mathrm{Op}[a]\psi_n \rangle$ tend to $\overline{a}_E$ in the limit $\hbar \to 0$, so in this sense the eigenfunctions become equidistributed on the energy-shell.

Our aim is to derive an upper bound on the rate by which the left hand side of (2) approaches zero. For the eigenfunctions of the Laplacian on manifolds of negative curvature such a bound has been derived by Zelditch [Zel94]. The bound we give is of the same order, so we do not get an improvement on the rate, but the advantage of our method is that it is simpler and uses only ergodicity with a certain rate as condition on the classical flow. Therefore it applies to a larger class of systems. The main input in the proof is the result on the semiclassical propagation of observables up to Ehrenfest time, [BGP99, BR02].

We will now describe the classes of Hamiltonians and observables we consider, see, e.g., [DS99] for more details. We say $a(\hbar, x, \xi) \in S^m$ for $m \in \mathbb{R}$ if a is smooth, satisfies

$$|\partial_{x,\xi}^\gamma a(\hbar, x, \xi)| \leq C_\gamma (1 + |x|^2 + |\xi|^2)^{m/2} \tag{3}$$

for all $\gamma \in \mathbb{N}^{2d}$ and $\hbar \in (0, 1/2]$, and has an asymptotic expansion $a(\hbar, x, \xi) \sim \sum_{n \in \mathbb{N}} \hbar^n a_n(x, \xi)$, i.e., $(a - \sum_{n=0}^{N-1} \hbar^n a_n)\hbar^{-N}$ satisfies (3) for all $N \in \mathbb{N}$. Now let M be a smooth manifold, the set of operators $\Psi^m(M)$ is given by local Weyl quantisation of these classes, if $a \in S^m$ in some local chart, then $\mathrm{Op}[a]$ is defined as

$$\mathrm{Op}[a]\psi = \frac{1}{(2\pi\hbar)^d} \iint e^{\frac{i}{\hbar}\langle x-y,\xi\rangle} a\big(\hbar, \frac{x+y}{2}, \xi\big) \psi(y) \, \mathrm{d}y\mathrm{d}\xi \ . \tag{4}$$

A general operator $A \in \Psi^m(M)$ is then an operator who is locally of the form (4) with some $a \in S^m$. The function a is called the local symbol of the operator A and the leading term in the asymptotic expansion of a is called the principal symbol

$$\sigma(A) := a_0 \ , \tag{5}$$

the principal symbol can be glued together to a function on T^*M, but the full symbol not. The operators in $\Psi^0(M)$ are bounded on $L^2(M)$ (uniformly in $\hbar$) and will form our basic class of observables.

We will assume that the Hamiltonian $\mathcal{H}$ is a selfadjoint operator in $\mathcal{H} \in \Psi^m(M)$, for some $m > 0$, and denote by Φ^t the Hamiltonian flow on T^*M generated by the principal symbol $H_0 = \sigma(\mathcal{H})$ of $\mathcal{H}$. Let $\Sigma_E := \{(x,\xi) \in T^*M \, ; \, H_0(x,\xi) = E\} \subset T^*M$ denote the energy surface and $\mathrm{d}\mu_E$ the Liouville measure on Σ_E. If E is a regular value of H_0 and Σ_E is compact, then the spectrum of $\mathcal{H}$ is discrete in a neighbourhood of E. If furthermore the set of periodic orbits of Φ^t on Σ_E has measure zero, then the number of eigenvalues close to E satisfies the Weyl estimate

$$N(I(E, \hbar)) = \frac{2\alpha}{(2\pi)^d \hbar^{d-1}} \, \mathrm{vol}(\Sigma_E)(1 + o(1)) \ , \tag{6}$$

where $\mathrm{vol}(\Sigma_E) := \int_{\Sigma_E} \mathrm{d}\mu_E$ and $\mathrm{d}\mu_E$ denotes the Liouville measure on Σ_E, see [PR85, Ivr98, DS99].

The autocorrelation function at energy E of a function a on T^*M is defined as

$$C_E[a](t) := \frac{1}{\text{vol}(\Sigma_E)} \int_{\Sigma_E} a \circ \Phi^t a \, \mathrm{d}\mu_E - \left(\overline{a}_E\right)^2 , \tag{7}$$

where

$$\overline{a}_E := \frac{1}{\text{vol}(\Sigma_E)} \int_{\Sigma_E} a \, \mathrm{d}\mu_E . \tag{8}$$

The flow Φ^t is ergodic on Σ_E if for every $a \in L^1(\Sigma_E, \mathrm{d}\mu_E)$ one has

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T C_E[a](t) \, \mathrm{d}t = 0 , \tag{9}$$

see [Wal82]. We will say that Φ^t is ergodic with rate $\gamma > 0$ on Σ_E if for every $a \in C^\infty(\Sigma_E)$ and $f \in \mathcal{S}(\mathbb{R})$ there is a constant C such that

$$\frac{1}{T} \int f\left(\frac{t}{T}\right) C_E[a](t) \, \mathrm{d}t \leq C(1 + |T|)^{-\gamma} . \tag{10}$$

The rate of ergodicity can be related to the more common rate of mixing, the system is called mixing if $\lim_{t\to\infty} C_E[a](t) = 0$, and if $|C_E[a](t)| \leq C(1 + |t|)^{-\tilde{\gamma}}$, then $\tilde{\gamma}$ is called the rate of mixing. We see from (10) that for $0 < \tilde{\gamma} < 1$ we have at least a rate of ergodicity $\gamma = \tilde{\gamma}$, whereas for $\tilde{\gamma} > 1$ we have at least $\gamma = 1$. So a rate of mixing implies a rate of ergodicity, but the contrary is not true, there are systems which are not mixing but which can have a large rate of ergodicity due to an oscillatory behaviour of $C_E[a](t)$. Examples are easily found among maps, for instance the Kronecker map, and we will discuss some cases in the last section about quantised maps.

Our main result is now

Theorem 1. *Let $\mathcal{H} \in \Psi^m(M)$, for some $m > 0$, be selfadjoint with principal symbol H_0. Assume that E is a regular value of H_0, that Σ_E is compact and denote by E_n, ψ_n the eigenfunctions and eigenvalues of $\mathcal{H}$ in the interval $I(E, \hbar) = [E - \alpha\hbar, E + \alpha\hbar]$, $\alpha > 0$. If the Hamiltonian flow Φ^t generated by H_0 is ergodic with rate $\gamma > 0$ on Σ_E, then for every $A \in \Psi^0(M)$ there exists a $C > 0$ such that*

$$\frac{1}{N(I(E,\hbar))} \sum_{E_n \in I(E,\hbar)} |\langle \psi_n, A\psi_n \rangle - \overline{\sigma(A)}_E|^2 \leq C \begin{cases} |\ln \hbar|^{-\gamma} & \text{if } 0 < \gamma \leq 1 \\ |\ln \hbar|^{-1} & \text{if } \gamma \geq 1 \end{cases} , \tag{11}$$

where $\overline{\sigma(A)}_E$ is defined in (8).

This result is an extension of the previous result by Zelditch, [Zel94], who obtained the same logarithmic bound for $\gamma > 1$ for eigenfunctions of the Laplacian on compact manifolds of negative curvature (in order to connect the two setups one has to rescale the Laplacian with $\hbar$). The improvement lies in the weakening of the assumptions to a rate of ergodicity and in a simpler proof, this is possible because we can use the recent results on propagation of observables up to Ehrenfest time [BGP99, BR02]. A similar result has been stated recently by Robert in the review [Rob04].

Further systems where Theorem 1 applies are Schrödinger operators $\mathcal{H} = -\hbar^2\Delta + V$ on the 2-torus with the smooth potentials V, constructed by Donnay and Liverani [DL91], for which the flow is ergodic and mixing [BT03]. These examples have been recently generalised to higher dimensions, [BT05].

For strongly chaotic systems the bound (11) is far from the conjectured optimal one. For eigenfunctions of the Laplace Beltrami operator on compact surfaces of negative curvature, where the corresponding classical system is the the geodesic flow, which is Anosov, Rudnick and Sarnak [RS94, Sar03] have conjectured that

$$\left| \langle \psi_n, \rho\psi_n \rangle - \int \rho \, \mathrm{d}\nu_g \right| \leq C_\varepsilon E_n^{-1/4+\varepsilon} \tag{12}$$

holds for all $\varepsilon > 0$. Here ρ is a sufficiently nice function on the surface and $\mathrm{d}\nu_g$ is the Riemannian volume element. Translated in our context that would imply a bound $h^{1-\varepsilon}$ in (11). A very precise prediction for the behaviour of the sum on the left hand side of (11) has been derived in [EFK$^+$95], for a compact uniformly hyperbolic system with time reversal invariance and no other symmetry it reads

$$\frac{1}{N(I(E,\hbar))} \sum_{E_n \in I(E,\hbar)} |\langle \psi_n, A\psi_n \rangle - \overline{\sigma(A)}_E|^2$$
$$= 2\frac{(2\pi\hbar)^{d-1}}{\mathrm{vol}\,\Sigma_E} \int_{-\infty}^{\infty} C_E[\sigma(A)](t) \, \mathrm{d}t + o(\hbar^{d-1}) \ . \tag{13}$$

These predictions have been numerically tested in [EFK$^+$95, AT98, BSS98], and confirmed for uniformly hyperbolic systems like manifolds of negative curvature. For non-uniformly hyperbolic systems like Euclidean billiards the findings are less clear and the rate is sometimes slower, at least in the tested energy range. So understanding the rate of quantum ergodicity remains a major open problem. Very recently Luo and Sarnak, see [Sar03], established a result of the form (13) for the discrete spectrum of the Laplacian on the modular surface. But due to the arithmetic nature of the system the right hand side of (13) differs and an additional factor related to L-functions appears.

The reason for the rather large gap between the estimate (11) and the conjectured one is our poor understanding of the quantum time evolution for large times when the underlying classical system is hyperbolic. In our present techniques the hyperbolicity leads to exponentially growing remainder terms and this reduces us to time scales which are logarithmic in $\hbar$. But for systems which are ergodic but not hyperbolic we can hope to get much stronger results. Examples for such systems can be constructed as maps on the torus, and we therefore have added a section on quantised maps. In this section we will first prove an analogue of Theorem 1 for perturbed cat maps using techniques from [BDB04], and then we study the quantised parabolic map introduced in [MR00] and show that we get an algebraic decay of (11), with an optimal rate.

The method we use to prove Theorem 1 can be used as well to get a bound on the off-diagonal matrix elements. We say that the flow Φ^t is weak mixing with rate $\gamma > 0$ on Σ_E if for all smooth a on Σ_E and $f \in \mathcal{S}(\mathbb{R})$ there is a constant C such that for all $\varepsilon \in \mathbb{R}$

$$\frac{1}{T}\int f\left(\frac{t}{T}\right) C_E[a](t)\mathrm{e}^{\mathrm{i}\varepsilon t} \, \mathrm{d}t \leq C(1+|T|)^{-\gamma} \ . \tag{14}$$

That the above quantity tends to 0 for $T \to \infty$ is equivalent to weak mixing, so the above condition quantifies the rate of weak mixing. As for the rate of ergodicity, a rate of mixing implies a similar rate of weak mixing.

Theorem 2. *Under the same conditions as in Theorem 1 we have for $\gamma > 0$*

$$\frac{1}{N(I(E,\hbar))} \underset{\substack{n,m \,;\, E_n \in I(E,\hbar) \\ |E_n - E_m| \leq \hbar/|\ln \hbar|}}{\sum}{}' |\langle \psi_n, A\psi_m \rangle|^2 \leq C \begin{cases} |\ln \hbar|^{-\gamma} & \text{if } 0 < \gamma \leq 1 \\ |\ln \hbar|^{-1} & \text{if } \gamma \geq 1 \end{cases}, \qquad (15)$$

and if the flow is weak mixing with a rate $\gamma > 0$, then for any $\varepsilon \in \mathbb{R}$

$$\frac{1}{N(I(E,\hbar))} \underset{\substack{n,m \,;\, E_n \in I(E,\hbar) \\ |E_n - E_m - \hbar\varepsilon| \leq \hbar/|\ln \hbar|}}{\sum}{}' |\langle \psi_n, A\psi_m \rangle|^2 \leq C \begin{cases} |\ln \hbar|^{-\gamma} & \text{if } 0 < \gamma \leq 1 \\ |\ln \hbar|^{-1} & \text{if } \gamma \geq 1 \end{cases}, \qquad (16)$$

where the prime at the sum indicates that we sum over E_m, E_n with $E_m \neq E_n$.

The behaviour of off-diagonal matrix elements has been studied by Zelditch [Zel90, Zel96] who showed that ergodicity and weak mixing implies that the above sums tend to zero for $\hbar \to 0$. Further results have been derived in [Tat99]. As we will see in Section 4 weak mixing is a necessary condition for (16) to hold.

The plan of the paper is as follows. The next two sections are devoted to the proof of Theorems 1 and 2. In section 2 we collect some preliminaries, and in section 3 we do the proofs. In the final section 4 we then discuss some quantised maps.

2 Preliminaries

The proofs of Theorems 1 and 2 rest on two ingredients, a microlocal version of Weyl's law and a version of Egorov's theorem which is valid up to Ehrenfest time. In this section we will recall these results and present them in the form we need.

The estimates collected in this section will be finally applied to compute

$$\operatorname{Tr} \rho\big((E - \mathcal{H})/\hbar\big) BU^*(t) AU(t) \qquad (17)$$

for $A, B \in \Psi^0(M)$. This quantity can be localised by splitting $B = \sum_j \operatorname{Op}[b_j]$ with b_j supported (modulo $\hbar^\infty$) in local charts. Therefore it is sufficient for us to work in $M = \mathbb{R}^d$, and this will facilitate some of the remainder estimates.

For a function $a \in C^\infty(\mathbb{R}^m)$ we will use the notation

$$|a|_k := \sum_{|\alpha| \leq k} \sup_{x \in \mathbb{R}^m} |\partial^\alpha a(x)| \qquad (18)$$

for $k \in \mathbb{N}$.

Proposition 1. *Assume that $\mathcal{H} \in \Psi^m$ is selfadjoint and has principal symbol H_0. Assume furthermore that E is a regular value of H_0 and that Σ_E is compact. Let ρ be a smooth function on $\mathbb{R}$ such that the Fourier transform $\hat{\rho}$ has compact support in a small neighbourhood of 0 which contains no period of a periodic orbit of Φ^t on Σ_E. Then there is a constant $C > 0$ such that for every $\mathrm{Op}[b] \in \Psi^0$ we have*

$$\left| \sum_{E_n} \rho\left(\frac{E - E_n}{\hbar} \right) \langle \psi_n, \mathrm{Op}[b]\psi_n \rangle - \frac{\hat{\rho}(0)}{(2\pi)^d \hbar^{d-1}} \overline{\sigma(b)}_E \right| \leq C\hbar^{2-d}|\rho|_5 |b|_{2d+8} . \tag{19}$$

The proposition is a standard result and well known in literature, except that the way that the error term depends on b is usually not made explicit. Since the main tool in deriving the formula (19) is the method of stationary phase, or variants thereof, it comes as no surprise that the error term can be estimated by a finite number of derivatives of b. An analogous result for high-energy asymptotics on compact manifolds was derived in [Zel94]. For convenience we will sketch the proof of Proposition 1, for details we frequently refer to [DS99]. .

Proof. We first observe that without loss of generality we can assume that b is supported in a compact neighbourhood of the energy-shell Σ_E. Let $f(E)$ be a smooth function with compact support such that $f(H(x,\xi))$ has compact support and $f(H(x,\xi)) \equiv 1$ on a neighbourhood of Σ_E. By the functional calculus one has then $f(\mathcal{H}) \in \Psi(1)$, see [DS99]. Let $U(t) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}t\mathcal{H}}$ be the time evolution operator, i.e., the solution to $\mathrm{i}\hbar\partial_t U(t) = \mathcal{H}U(t)$ with initial condition $U(0) = I$. One then constructs an approximation to the operator $U_f(t) = U(t)f(\mathcal{H})$ by solving the initial value problem

$$\left(\mathrm{i}\hbar\partial_t - \mathcal{H} \right)U_f(t) = 0 , \quad U_f(0) = f(\mathcal{H}) \tag{20}$$

approximately for small t, i.e., for every $N \in \mathbb{N}$ one can find an $V^{(N)}(t)$ such that

$$\left(\mathrm{i}\hbar\partial_t - \mathcal{H} \right)V^{(N)}(t) = \hbar^{N+1}R_N(t) , \quad V^{(N)}(0) = f(\mathcal{H}) , \tag{21}$$

with $\|R_N(t)\| \leq C$ for $t \in [-T_0, T_0]$ where T_0 is smaller then the period of the shortest periodic orbit on Σ_E. Then Duhamel's principle gives

$$U_f(t) = V^{(N)}(t) + \mathrm{i}\hbar^N \int_0^t U_f(t - t')R_N(t') \, \mathrm{d}t' \tag{22}$$

and therefore

$$\begin{aligned}
|\mathrm{Tr}\, U_f(t)\, \mathrm{Op}[b] - \mathrm{Tr}\, V^{(N)}(t)\, \mathrm{Op}[b]| &\leq \hbar^N |t| \sup_{t' \in [0,t]} |\mathrm{Tr}\, U_f(t - t')R_N(t')\, \mathrm{Op}[b]| \\
&\leq \hbar^N C_N \, \mathrm{Tr}|\mathrm{Op}[b]|
\end{aligned} \tag{23}$$

since $|t| \sup_{t' \in [0,t]} \|U_f(t - t')R_N(t')\| \leq C$ for $t \in [-T_0, T_0]$ and we have used the general relation $|\mathrm{Tr}\, AB| \leq \|A\| \, \mathrm{Tr}|B|$ if A is bounded and B of trace class. Since b is of compact support $\mathrm{Op}[b]$ is of trace class and its trace norm can be estimated as

$$\mathrm{Tr}|\mathrm{Op}[b]| \leq C \frac{1}{(2\pi\hbar)^d} |b|_{2d+1} , \tag{24}$$

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see [DS99, Chapter 9]. The kernel of $V^{(N)}(t)$ satisfying (21) is given by

$$V^{(N)}(t,x,y) = \frac{1}{(2\pi\hbar)^d} \int e^{\frac{i}{\hbar}[\varphi(t,x,\xi)-y\xi]} a^{(N)}(t,x,\xi) \, \mathrm{d}\xi \tag{25}$$

where $\varphi(t,x,\xi)$ is a solution to the Hamilton Jacobi equation

$$\partial_t \varphi(t,x,\xi) + H(x,\varphi_x'(t,x,\xi)) = 0 \tag{26}$$

with initial condition $\varphi(0,x,\xi) = x\xi$, and $a^{(N)}(t,x,\xi) \in C^\infty([-T_0,T_0],S^1)$ is the solution of a corresponding transport equation with initial condition $a^{(N)}(0,x,\xi) = f(H(x,\xi)) + O(\hbar)$ given by the symbol of $f(\mathcal{H})$. See [DS99, Chapter 10] for the proof and more details. If $\tilde{b} = e^{i\hbar\partial_x\partial_\xi} b$ denotes the left symbol of $\mathrm{Op}[b]$ (the case $t=0$ in [DS99, Equation (7.5)]) then we get from (25)

$$\int e^{\frac{i}{\hbar}Et} \mathrm{Tr}\left[V^{(N)}(t)\,\mathrm{Op}[b]\right]\hat{\rho}(t) \, \mathrm{d}t$$
$$= \frac{1}{(2\pi\hbar)^d} \iiint e^{\frac{i}{\hbar}[\varphi(t,x,\xi)-x\xi+Et]} \hat{\rho}(t) a^{(N)}(t,x,\xi) \tilde{b}(x,\xi) \, \mathrm{d}x\mathrm{d}\xi\mathrm{d}t \ . \tag{27}$$

The main contributions to this integral come from the points where the phase is stationary, the stationary phase condition reads

$$\partial_t \varphi(t,x,\xi) + E = 0 \ , \quad \partial_x \varphi(t,x,\xi) - \xi = 0 \quad \text{and} \quad \partial_\xi \varphi(t,x,\xi) - x = 0 \ . \tag{28}$$

In view of (26) the first equation means that $H(x,\xi) = E$ and the second and third mean that $\Phi^t(x,\xi) = (x,\xi)$, i.e., (x,ξ) lie on a periodic orbit with period t. Since by assumption the support of $\hat{\rho}$ does not contain any period of a periodic orbit, the only stationary points left are at $t=0$, and consist of the whole energy shell Σ_E. Because E is assumed to be a non-degenerate energy level we can choose new coordinates (E',z) in a neighbourhood of Σ_E such that $H(E',z) = E'$, when we use furthermore that $\varphi(t,x,\xi) = x\xi - tH(x,\xi) + r(t,x,\xi)$ with $r(t,x,\xi) = O(t^2)$ which follows from (26), then the above integral becomes

$$\frac{1}{(2\pi\hbar)^d} \iiint e^{\frac{i}{\hbar}[(E-E')t+r(t,E',z)]} \hat{\rho}(t) a^{(N)}(t,E',z) \tilde{b}(E',z) J(E',z) \, \mathrm{d}E'\mathrm{d}t\mathrm{d}z \ , \tag{29}$$

where $J(E',z)$ denotes the Jacobian of the change of coordinates. We can now apply the stationary phase theorem with remainder estimate to the t, E' integrals and get

$$\frac{1}{2\pi\hbar} \iint e^{\frac{i}{\hbar}[(E-E')t+r(t,E',z)]} \hat{\rho}(t) a^{(N)}(t,E',z) \tilde{b}(E',z) J(E',z) \, \mathrm{d}E'\mathrm{d}t$$
$$= \hat{\rho}(0) a^{(N)}(0,E,z) \tilde{b}(E,z) J(E,z) + O(\hbar|\rho|_5|\tilde{b}|_5) \ , \tag{30}$$

where the implied constant does only depend on a and φ. With the initial condition $a^{(N)}(0,E,z) = 1 + (\hbar^\infty)$ and $|\partial^\alpha b - \partial^\alpha \tilde{b}| \le C|b|_{|\alpha|+2d+3}$ we then finally obtain

$$\left| \int e^{\frac{i}{\hbar}Et} \mathrm{Tr}(V^{(N)}(t)\,\mathrm{Op}[b])\hat{\rho}(t) \, \mathrm{d}t - \frac{\hat{\rho}(0)}{(2\pi\hbar)^{d-1}} \int_{\Sigma_E} \sigma(b) \, \mathrm{d}\mu_E \right| \le C\hbar^{d-2}|\rho|_5|b|_{2d+8} \ . \tag{31}$$

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On the other hand side, by the spectral resolution of $U(t)$ we have

$$\int e^{\frac{i}{\hbar}Et} \operatorname{Tr}(U_f(t) \operatorname{Op}[b]) \hat{\rho}(t) \, dt = 2\pi \sum_{E_n} \rho\left(\frac{E - E_n}{\hbar}\right) \langle \psi_n, \operatorname{Op}[b]\psi_n \rangle \tag{32}$$

and so finally we get

$$\sum_{E_n} \rho\left(\frac{E - E_n}{\hbar}\right) \langle \psi_n, \operatorname{Op}[b]\psi_n \rangle$$
$$= \frac{\hat{\rho}(0)}{(2\pi)^d \hbar^{d-1}} \overline{\sigma(b)}_E + O(\hbar^{d-2}|\rho|_5|b|_{2d+8}) + O(\hbar^{d-N}|\rho|_0|b|_{2d+1}) \tag{33}$$

where the implied constants do only depend on a, φ and f. $\qquad\square$

We want to use this Proposition with $\operatorname{Op}[b] = \operatorname{Op}[a]U^*(t) \operatorname{Op}[a]U(t)$ where $\operatorname{Op}[a] \in \Psi^0$. In order to do so we will use the Theorem of Egorov with remainder estimate from [BGP99] and [BR02, Proposition 2.7].

Theorem 3 ([BR02]). *Assume that $\mathcal{H} \in \Psi^m$ is selfadjoint and let $U(t) := e^{-\frac{i}{\hbar}t\mathcal{H}}$. Then for any compact $\Omega \subset \mathbb{R}^d \times \mathbb{R}^d$ there exists a constant $\Gamma_1 > 0$ such that for every $\operatorname{Op}[a] \in \Psi^0$ with $\operatorname{supp} a \subset \Omega$ there is a $C > 0$ with*

$$\|U^*(t) \operatorname{Op}[a]U(t) - \operatorname{Op}[a \circ \Phi^t]\| \leq C\hbar e^{\Gamma_1|t|} \tag{34}$$

From this we get

Corollary 1. *Under the assumption in Theorem 3 there exists a constant $\Gamma > 0$ such that for every $\operatorname{Op}[a] \in \Psi^0$ with support in Ω there is a $C > 0$ with*

$$\|\operatorname{Op}[a]^*U^*(t) \operatorname{Op}[a]U(t) - \operatorname{Op}[a^*a \circ \Phi^t]\| \leq C\hbar e^{\Gamma|t|} \tag{35}$$

Proof. Using the triangle inequality and Egorov's theorem we get

$$\|\operatorname{Op}[a]^*U^*(t) \operatorname{Op}[a]U(t) - \operatorname{Op}[a^*a \circ \Phi^t]\|$$
$$\leq \|\operatorname{Op}[a]^*U^*(t) \operatorname{Op}[a]U(t) - \operatorname{Op}[a]^* \operatorname{Op}[a \circ \Phi^t]\|$$
$$+ \|\operatorname{Op}[a]^* \operatorname{Op}[a \circ \Phi^t] - \operatorname{Op}[a^*a \circ \Phi^t]\| \tag{36}$$
$$\leq C\hbar\|\operatorname{Op}[a]\|e^{\Gamma_1|t|} + \|\operatorname{Op}[a]^* \operatorname{Op}[a \circ \Phi^t] - \operatorname{Op}[a^*a \circ \Phi^t]\|$$

and since $\operatorname{Op}[a]$ is bounded we only have to estimate the second term. By the product formula for pseudo-differential operators and the Calderon Vallaincourt Theorem there exists a $k \in \mathbb{N}$ such that

$$\|\operatorname{Op}[a] \operatorname{Op}[b] - \operatorname{Op}[ab]\| \leq C\hbar|a|_k|b|_k \tag{37}$$

where C does not depend on a and b. We use this estimate with $b = a \circ \Phi^t$ and that for some $\Gamma_k > 0$

$$|a \circ \Phi^t|_k \leq Ce^{\Gamma_k|t|} , \tag{38}$$

see [BR02, Lemma 2.4]. This proves the Corollary with $\Gamma = \max\{\Gamma_1, \Gamma_k\}$. $\qquad\square$

Using Corollary 1 together with Proposition 1 we obtain

Corollary 2. *There exists $C > 0$, $\Gamma > 0$ and $k \in \mathbb{N}$ such that for every selfadjoint* $\mathrm{Op}[a] \in \Psi^0$

$$\sum_{E_n, E_m} \rho\left(\frac{E - E_n}{\hbar}\right) \mathrm{e}^{\frac{\mathrm{i}}{\hbar} t(E_n - E_m)} \big|\langle \psi_n, \mathrm{Op}[a]\psi_m\rangle - \overline{\sigma(a)}_E\big|^2$$
$$= \frac{\hat{\rho}(0)}{(2\pi)^d \hbar^{d-1}} C_E[\sigma(a)](t) + O(\hbar^{2-d}|\rho|_5|a|_k \mathrm{e}^{\Gamma|t|}) . \tag{39}$$

This kind of relationship between transition amplitudes and the autocorrelation function is well known, the only new piece is that we have an explicit estimate on the time dependence of the remainder term. In fact if we multiply with a function $f(t)$ of compact support and integrate over t we obtain

$$\sum_{E_n, E_m} \rho\left(\frac{E - E_n}{\hbar}\right) \hat{f}\left(\frac{E_m - E_n}{\hbar}\right) \big|\langle \psi_n, \mathrm{Op}[a]\psi_m\rangle - \overline{\sigma(a)}_E\big|^2$$
$$= \frac{\hat{\rho}(0)}{(2\pi)^d \hbar^{d-1}} \int C_E[\sigma(a)](t)\, f(t)\, \mathrm{d}t + O(\hbar^{2-d}) , \tag{40}$$

which was derived in [FP86, Wil87] and proved in [CR94].

3 Proofs of Theorems 1 and 2

The proof of Theorem 1 will rely on the fact that by Corollary 2 we can let the support of f in (40) become larger with $\hbar$.

Proof of Theorem 1. We will assume in the following that $\overline{a}_E = 0$, this can always be achieved by subtracting $\overline{a}_E$ from a. Choose ρ such that $\rho \geq 0$, $\rho(\frac{E-E'}{\hbar}) \geq 1$ for $E' \in I(E, \hbar)$. Choose furthermore f such that $\hat{f} \in C^\infty([-1,1])$ and $f \geq 0$ and $f(0) = 1$ and set $f_T(\tau) := f(T\tau)$ so that $\widehat{f_T}(t) = \hat{f}(t/T)/T$. Then we have

$$\sum_{E_n \in I(E, \hbar)} |\langle \psi_n, \mathrm{Op}[a]\psi_n\rangle|^2 \leq \sum_{E_n, E_m} \rho\left(\frac{E - E_n}{\hbar}\right) f_T\left(\frac{E_m - E_n}{\hbar}\right) |\langle \psi_n, \mathrm{Op}[a]\psi_m\rangle|^2 . \tag{41}$$

and with Corollary 2 we get

$$\sum_{E_n, E_m} \rho\left(\frac{E - E_n}{\hbar}\right) f_T\left(\frac{E_m - E_n}{\hbar}\right) |\langle \psi_n, \mathrm{Op}[a]\psi_m\rangle|^2$$
$$= \frac{\hat{\rho}(0)}{(2\pi)^d \hbar^{d-1}} \int C_E[\sigma(a)](t)\widehat{f_T}(t)\, \mathrm{d}t + O\left(\hbar^{2-d}|\rho|_5|a|_k \int \mathrm{e}^{\Gamma|t|}\widehat{f_T}(t)\, \mathrm{d}t\right) . \tag{42}$$

Now we have

$$\left|\int \mathrm{e}^{\Gamma|t|}\widehat{f_T}(t)\, \mathrm{d}t\right| \leq |\hat{f}|_0 \frac{1}{\Gamma T}\mathrm{e}^{\Gamma T} \tag{43}$$

and with (10) we obtain

$$\left| \int C_E[\sigma(a)](t)\widehat{f_T}(t) \, \mathrm{d}t \right| \leq \begin{cases} C\frac{1}{T} & \text{for } \gamma \geq 1 \\ C\frac{1}{T^\gamma} & \text{for } 0 < \gamma \leq 1 \end{cases} , \tag{44}$$

for large T, since $\overline{a}_E = 0$ by assumption. If we choose

$$T = \frac{1}{\Gamma} |\ln(\hbar)| \tag{45}$$

then $\hbar e^{\Gamma T} = 1$, and therefore we get

$$\sum_{E_n, E_m} \rho\left(\frac{E - E_n}{\hbar}\right) f_T\left(\frac{E_m - E_n}{\hbar}\right) |\langle \psi_n, \mathrm{Op}[a]\psi_m \rangle|^2 \leq C\hbar^{d-1} \begin{cases} |\ln \hbar|^{-\gamma} & \text{if } 0 < \gamma \leq 1 \\ |\ln \hbar|^{-1} & \text{if } \gamma \geq 1 \end{cases} . \tag{46}$$

Combining this inequality with the estimate (41) and the asymptotic for the number of eigenvalues in $I(E, \hbar)$, (6), finally gives

$$\frac{1}{N(I(E,\hbar))} \sum_{E_n \in I(E,\hbar)} |\langle \psi_n, \mathrm{Op}[a]\psi_n \rangle|^2 \leq C \begin{cases} |\ln \hbar|^{-\gamma} & \text{if } 0 < \gamma \leq 1 \\ |\ln \hbar|^{-1} & \text{if } \gamma \geq 1 \end{cases} \tag{47}$$

and the proof is complete. $\qquad\qquad\square$

Theorem 2 is proved along the same lines.

Proof of Theorem 2. The proof is based on relation (46), notice that the only assumption on ρ and f which entered the derivation are that $\hat{f}$ has compact support and $\hat{\rho}$ is supported in $(-T_0, T_0)$. We choose now ρ as before and f such that

$$f \geq \chi_{[-\Gamma, \Gamma]} \tag{48}$$

where $\chi_{[-\Gamma, \Gamma]}$ is the characteristic function of the interval $[-\Gamma, \Gamma]$. Then we get using (46)

$$\frac{1}{N(I(E,\hbar))} \sum_{\substack{n,m \,:\, E_n \in I(E,\hbar) \\ |E_n - E_m| \leq \hbar/|\ln \hbar|}} |\langle \psi_n \, \mathrm{Op}[a]\psi_m \rangle|^2 \leq C \begin{cases} |\ln \hbar|^{-\gamma} & \text{if } 0 < \gamma \leq 1 \\ |\ln \hbar|^{-1} & \text{if } \gamma \geq 1 \end{cases} \tag{49}$$

if $\overline{a}_E = 0$. Together with (47) this gives

$$\frac{1}{N(I(E,\hbar))} \sideset{}{'}\sum_{\substack{n,m \,:\, E_n \in I(E,\hbar) \\ |E_n - E_m| \leq \hbar/|\ln \hbar|}} |\langle \psi_n \, \mathrm{Op}[a]\psi_m \rangle|^2 \leq C \begin{cases} |\ln \hbar|^{-\gamma} & \text{if } 0 < \gamma \leq 1 \\ |\ln \hbar|^{-1} & \text{if } \gamma \geq 1 \end{cases} \tag{50}$$

and since $\langle \psi_m, \overline{a}_E \psi_n \rangle = 0$ if $E_m \neq E_n$, this estimate is true for all $\mathrm{Op}[a] \in \Psi^0$.

With the same choices of ρ and f and by shifting f_T,

$$f_T^{(\varepsilon)}(\tau) := f_T(\tau - \varepsilon) , \tag{51}$$

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we get from (42) and (43)

$$
\sum_{E_n, E_m} \rho\left(\frac{E - E_n}{\hbar}\right) f_T\left(\frac{E_m - E_n - \hbar\varepsilon}{\hbar}\right) |\langle \psi_n, \mathrm{Op}[a]\psi_m\rangle|^2
$$
$$
= \frac{\hat{\rho}(0)}{(2\pi)^d \hbar^{d-1}} \int C_E[\sigma(a)](t)\widehat{f_T}(t)\mathrm{e}^{\mathrm{i}\varepsilon t} \, \mathrm{d}t + O\left(\hbar^{2-d}|\rho|_5|a|_k|\hat{f}|_0 \frac{1}{\Gamma T}\mathrm{e}^{\Gamma T}\right) .
\tag{52}
$$

And with the choice (45) and the rate of weak mixing (14) the second relation in Theorem 2 follows. $\qquad\square$

4   Quantum maps

In this final section we study the application of the ideas from the previous sections to the quantisation of some maps on the torus. We will study two classes of maps, we begin with perturbed cat maps which are Anosov, and for which we derive the same results as for the flows. The second class of examples is given by maps which are ergodic but not hyperbolic. This means that we have better control on the remainder term in the Egorov theorem, and in turn our method gives for these maps sometimes optimal bounds on the rate of quantum ergodicity.

Let us first quickly review the setup for quantised maps on the 2-torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$, see [DB01, DES03, MO05] for some recent and more complete treatments.

Instead of one fixed Hilbert space we have now a sequence of Hilbert spaces of dimension $1/\hbar$. For each $N \in \mathbb{N}$ the N-dimensional Hilbert-space H_N will be identified with $L^2(\mathbb{Z}_N)$ with the inner product

$$
\langle \psi, \phi \rangle = \frac{1}{N} \sum_{q=1}^{N} \psi^*(q)\phi(q) .
\tag{53}
$$

The semiclassical parameter $\hbar$ is identified with $1/N$, so the semiclassical limit is $N \to \infty$.

Operators can be defined again by a Weyl quantisation prescription. For $n = (n_1, n_2) \in \mathbb{Z}^2$ define the translation operators on H_N as

$$
T_N(n)\psi(q) = \mathrm{e}_N(n_1 n_2/2)\mathrm{e}_N(n_2(q + n_1))\psi(q + n_1)
\tag{54}
$$

where $\mathrm{e}_N(x) := \exp(\frac{2\pi\mathrm{i}}{N}x)$. They satisfy

$$
T_N(m)T_N(n) = \mathrm{e}_N(\omega(m, n)/2)T_N(m + n) ,
\tag{55}
$$

where $\omega(m, n) = m_1 n_2 - m_2 n_1$, and

$$
\mathrm{Tr}\, T_N(n) = \begin{cases} N & \text{if } n = 0 \mod N \\ 0 & \text{otherwise} \end{cases} .
\tag{56}
$$

Now for $a \in C^\infty(T^2)$ one defines the Weyl quantisation as

$$
\mathrm{Op}_N[a] := \sum_{n \in \mathbb{Z}^2} \hat{a}(n)T_N(n)
\tag{57}
$$

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with the Fourier coefficients $\hat{a}(n) = \int_{T^2} a(x)\mathrm{e}(nx)\ \mathrm{d}x$, where $\mathrm{e}(x) = \exp(2\pi\mathrm{i}x)$.

The analogue of Proposition 1 is very simple and we state it immediately in a form containing products of operators, which we will need later on.

Lemma 1. *Let $a, b \in C^\infty(T^2)$, then for all $L \geq 3$ we have*

$$\frac{1}{N}\operatorname{Tr}\operatorname{Op}_N[a]\operatorname{Op}_N[b] = \int_{T^2} a(x)b(x)\ \mathrm{d}x + O\left(\frac{|a|_L |b|_{3+L}}{N^L}\right) . \tag{58}$$

Proof. Using the definition (57) and equations (55), (56) we obtain

$$\frac{1}{N}\operatorname{Tr}\operatorname{Op}_N[a]\operatorname{Op}_N[b] = \sum_{n\in\mathbb{Z}^2}\sum_{m\in\mathbb{Z}}\hat{a}(n)\hat{b}(-n+mN)\mathrm{e}(\omega(m,n)/2)$$

$$= \sum_{n\in\mathbb{Z}^2}\hat{a}(n)\hat{b}(-n) + \sum_{n\in\mathbb{Z}^2}\sum_{m\in\mathbb{Z}^2\setminus\{0\}}\hat{a}(n)\hat{b}(-n+mN)\mathrm{e}(\omega(m,n)/2) . \tag{59}$$

Now we have for the first term

$$\int_{T^2} a(x)b(x)\ \mathrm{d}x = \sum_{n\in\mathbb{Z}^2}\hat{a}(n)\hat{b}(-n) \tag{60}$$

and by partial integration for $m \neq 0$

$$|\hat{a}(n)| \leq C|a|_k(1+|n|)^{-k} , \quad |\hat{b}(-n+mN)| \leq C|b|_L(1+|n|)^L(N|m|)^{-L} , \tag{61}$$

so with $k = L+3$ and $L \geq 3$ the remainder term converges and the result follows. $\qquad\square$

The quantisation of a classical volume preserving map $\Phi : T^2 \to T^2$ is now defined to be a sequence of unitary operators $\{U_N\}_{N\in\mathbb{N}}$ such that for all $a \in C^\infty(T^2)$ an Egorov Theorem holds,

$$\lim_{N\to\infty}\|U_N\operatorname{Op}_N[a]U_N^* - \operatorname{Op}_N[a \circ \Phi]\| = 0 . \tag{62}$$

In case that the classical map is ergodic this property allows to prove a quantum ergodicity theorem. And as before, in case that we have more detailed information on how the remainder in the Egorov Theorem behaves under iteration of the map we can get a bound on the rate of quantum ergodicity. We will discuss this now for two examples.

Perturbed cat maps: We begin with a class of Anosov maps studied recently by Bouclet and De Bièvre in [BDB04]. Let $A \in Sp(2,\mathbb{Z})$ be a cat map and $g \in C^\infty(T^2)$ a real valued function, and consider the Hamiltonian flow $\phi^t : T^2 \to T^2$ generated by g. One can define then

$$\Phi_\varepsilon := \phi^\varepsilon \circ A : T^2 \to T^2 \tag{63}$$

which for small ε is a small perturbation of the Anosov map A, and hence by structural stability will be Anosov, too. The quantisation of Φ_ε is now defined as

$$U_N := \mathrm{e}^{-\mathrm{i}N\varepsilon\operatorname{Op}_N[g]}M_N(A) \tag{64}$$

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where $M_N(A)$ is the standard metaplectic quantisation of A, see, e.g., [DB01, KR00, DES03]. In [BDB04] it is now shown that there is a constant $\Gamma > 0$ such that for $t \in \mathbb{Z}$

$$\|U_N^t \operatorname{Op}_N[a] U_N^{*\,t} - \operatorname{Op}_N[a \circ \Phi_\varepsilon^t]\| \leq C_a \frac{1}{N} e^{\Gamma|t|} . \tag{65}$$

In fact the estimates in [BDB04] are more precise, and Γ is estimated quite explicitly, but the estimate (65) is sufficient for our purpose.

Using (65) and the trace estimates in Lemma 1 we can apply our strategy from the proof of Theorem 1 and obtain

Theorem 4. *Let U_N be the sequence of quantum maps* (64) *and ψ_j^N, $j = 1, \ldots, N$ an orthonormal basis of eigenfunctions of U_N for every $N \in \mathbb{N}$. Then for every $a \in C^\infty(T^2)$ there is a constant C_a such that*

$$\frac{1}{N} \sum_{j=1}^N |\langle \psi_j^N, \operatorname{Op}[a]\psi_j^N\rangle - \overline{a}|^2 \leq C_a \frac{1}{\ln N} , \tag{66}$$

where $\overline{a} = \int_{T^2} a \, \mathrm{d}x$.

The same result has been recently proved for the baker's map too, see [DENW04]. For cat maps much stronger results are known, due to their arithmetic nature, see [KR00, KR05].

Proof. We will assume that $\overline{a} = 0$. Let ψ_j^N, $e(\theta_j^N)$, $j = 1, \ldots, N$, be the eigenfunctions and eigenvalues of U_N, then we have

$$\operatorname{Tr} \operatorname{Op}_N[a] U_N^{-t} \operatorname{Op}_N[a] U_N^t = \sum_{i,j=1}^N |\langle \psi_j^N, \operatorname{Op}[a]\psi_i^N\rangle|^2 e\bigl(t(\theta_j^N - \theta_i^N)\bigr) . \tag{67}$$

Now choose $f \in \mathcal{S}(\mathbb{R})$ such that $\operatorname{supp} \hat{f} \in [-1, 1]$, $f \geq 0$ and $f(0) = 1$, we have by the Poisson summation formula

$$\sum_{t \in \mathbb{Z}} \frac{1}{T} \hat{f}\left(\frac{t}{T}\right) e\bigl(t(\theta_j^N - \theta_i^N)\bigr) = \sum_{n \in \mathbb{Z}} f\bigl(T(\theta_j^N - \theta_i^N - n)\bigr) \tag{68}$$

for any $T > 0$. By the positivity of f and since $f(0) = 1$ we find then

$$\sum_{i,j=1}^N |\langle \psi_j^N, \operatorname{Op}[a]\psi_i^N\rangle|^2 \sum_{t \in \mathbb{Z}} \frac{1}{T} \hat{f}\left(\frac{t}{T}\right) e\bigl(t(\theta_j^N - \theta_i^N)\bigr) \geq \sum_{j=1}^N |\langle \psi_j^N, \operatorname{Op}[a]\psi_j^N\rangle|^2 , \tag{69}$$

and so we have the estimate

$$\frac{1}{N} \sum_{j=1}^N |\langle \psi_j^N, \operatorname{Op}[a]\psi_j^N\rangle|^2 \leq \sum_{t \in \mathbb{Z}} \frac{1}{T} \hat{f}\left(\frac{t}{T}\right) \frac{1}{N} \operatorname{Tr} \operatorname{Op}_N[a] U_N^{-t} \operatorname{Op}_N[a] U_N^t . \tag{70}$$

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The Egorov estimate (65) and Lemma 1 give

$$\frac{1}{N} \operatorname{Tr} \operatorname{Op}_N[a] U_N^{-t} \operatorname{Op}_N[a] U_N^t = \frac{1}{N} \operatorname{Tr} \operatorname{Op}_N[a] \operatorname{Op}_N[a \circ \Phi_\varepsilon^t] + O\left(\frac{e^{\Gamma|t|}}{N}\right)$$

$$= \int_{T^2} a \, a \circ \Phi_\varepsilon^t \, \mathrm{d}x + O\left(\frac{e^{3\Gamma'|t|}}{N^3}\right) + O\left(\frac{e^{\Gamma|t|}}{N}\right) , \tag{71}$$

where we have used in addition that there is a constant $\Gamma' > 0$ such that $|a \circ \Phi_\varepsilon^t|_3 \leq C e^{3\Gamma'|t|}$. Now we can proceed as before in the proof of Theorem 1 with the choice $T \sim \ln N$ and we use that the map Φ_ε^t is mixing with an exponential rate, since it is Anosov. $\qquad\square$

An analogue of Theorem 2 could be derived easily with the same methods.

Parabolic maps: Our second example will be the parabolic map studied by Marklof and Rudnick in [MR00]. Let $\alpha \in \mathbb{R}$, then the map $\Psi_\alpha : T^2 \to T^2$ is defined by

$$\Psi_\alpha : \begin{pmatrix} p \\ q \end{pmatrix} \mapsto \begin{pmatrix} p + \alpha \\ q + 2p \end{pmatrix} \quad \mod 1 . \tag{72}$$

If α is irrational this map is uniquely ergodic but not mixing and not hyperbolic. This map is quantised in [MR00] and it is shown that its quantisation U_N satisfies the Egorov estimate

$$\|U_N^{-t} \operatorname{Op}_N[a] U_N^t - Op_N[a \circ \Psi_\alpha^t]\| \leq C_a \frac{|t|}{N} \tag{73}$$

for $t \in \mathbb{Z}$.

In order to study the rate of quantum ergodicity, we need an estimate on the rate of classical ergodicity.

Lemma 2. *Let $a \in C^\infty(T^2)$ and $C[a](t)$ be the autocorrelation function of the map (72) and assume that α satisfies a Diophantine condition, i.e., there are $C, \gamma > 0$ such that $|k\alpha - l| \geq C/|k|^\gamma$ for all $k, l \in \mathbb{Z}\backslash\{0\}$. Then we have for $f \in \mathcal{S}(\mathbb{R})$*

$$\sum_{t \in \mathbb{Z}} \frac{1}{T} \hat{f}\left(\frac{t}{T}\right) C[a](t) = O\left(\frac{1}{T}\right) , \tag{74}$$

where $\hat{f}$ denotes the Fourier-transform of f. Furthermore, if a depends only on p then

$$\sum_{t \in \mathbb{Z}} \frac{1}{T} \hat{f}\left(\frac{t}{T}\right) C[a](t) = O_M\left(\frac{1}{T^M}\right) , \quad \textit{for all } M \in \mathbb{N} . \tag{75}$$

Proof. We have

$$\Psi_\alpha^t : \begin{pmatrix} p \\ q \end{pmatrix} \mapsto \begin{pmatrix} p + t\alpha \\ q + 2tp + \alpha t(t-1) \end{pmatrix} , \tag{76}$$

and with $a(x) = \sum_{n \in \mathbb{Z}^2} \hat{a}(n) e(nx)$ we get

$$C[a](t) = \sum_{n,m \in \mathbb{Z}^2\backslash\{0\}} \hat{a}(n)\hat{a}(m) \int_{T^2} e(nx) e\big(m\Psi_\alpha^t(x)\big) \, \mathrm{d}x . \tag{77}$$

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Then we find

$$\int_{T^2} e(nx)e\big(m\Psi_\alpha^t(x)\big) \ dx = \delta(-n_1, m_1 + 2tm_2)\delta(-n_2, m_2)e(m_1\alpha t + m_2\alpha t(t-1)) \ , \tag{78}$$

where $\delta(m,n)$ denotes the Kronecker delta, and therefore

$$C[a](t) = \sum_{(m_1,m_2)\in\mathbb{Z}^2\setminus\{0\}} \hat{a}(-m_1 - 2tm_2, -m_2)\hat{a}(m_1, m_2)e(m_1\alpha t + m_2\alpha t(t-1)) \ . \tag{79}$$

Now we split $C[a](t)$ into two parts, $C[a](t) = C^0[a](t) + C^1[a](t)$, such that $C^0[a](t)$ contains only the terms with $m_2 = 0$

$$C^0[a](t) = \sum_{m\in\mathbb{Z}\setminus\{0\}} \hat{a}(-m,0)\hat{a}(m,0)e(m\alpha t) \ . \tag{80}$$

The second term satisfies

$$|C^1[a](t)| \leq C_K(1 + |t|)^{-K} \tag{81}$$

for all $K \in \mathbb{N}$ since the Fourier-coefficients $\hat{a}(n)$ are quickly decreasing and therefore

$$\sum_{t\in\mathbb{Z}} \frac{1}{T}\hat{f}\left(\frac{t}{T}\right)C^1[a](t) = O\left(\frac{1}{T}\right) \ . \tag{82}$$

For the first term we find

$$\sum_{t\in\mathbb{Z}} \frac{1}{T}\hat{f}\left(\frac{t}{T}\right)C^0[a](t) = \sum_{m\in\mathbb{Z}\setminus\{0\}} |\hat{a}(m,0)|^2 \sum_{t\in\mathbb{Z}} \frac{1}{T}\hat{f}\left(\frac{t}{T}\right)e(m\alpha t) \tag{83}$$

and by the Poisson summation formula we obtain

$$\sum_{t\in\mathbb{Z}} \frac{1}{T}\hat{f}\left(\frac{t}{T}\right)e(m\alpha t) = \sum_{n\in\mathbb{Z}} f(T(m\alpha - n)) = O_M(|m|^{\gamma M}T^{-M}) \tag{84}$$

since $f \in \mathcal{S}(\mathbb{R})$ and by the Diophantine condition on α. And since the Fourier-coefficients $\hat{a}(n)$ are quickly decreasing we find

$$\sum_{t\in\mathbb{Z}} \frac{1}{T}\hat{f}\left(\frac{t}{T}\right)C^0[a](t) = O_M(T^{-M}) \tag{85}$$

Combining the two estimates for $C^0[a](t)$ and $C^1[a](t)$ gives the lemma. $\quad\square$

Combining the Egorov estimate and this lemma we then obtain

Theorem 5. *Let U_N be the quantisation of the map (72) due to [MR00] with a Diophantine α, and ψ_j^N, $j = 1, \ldots N$, a orthonormal basis of eigenfunctions. Then we have*

$$\frac{1}{N}\sum_{j=1}^N |\langle\psi_j^N, \mathrm{Op}[a]\psi_j^N\rangle - \bar{a}|^2 \leq C_a \frac{1}{N^{1/2}} \ , \tag{86}$$

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and if a depends on p only then we have the stronger estimate

$$\frac{1}{N}\sum_{j=1}^{N}|\langle\psi_j^N, \mathrm{Op}[a]\psi_j^N\rangle - \bar{a}|^2 \le C_{a,\varepsilon}\frac{1}{N^{1-\varepsilon}} \ , \tag{87}$$

for every $\varepsilon > 0$.

Proof. Using the estimate (70) from the proof of Theorem 4 we have to estimate

$$\sum_{t\in\mathbb{Z}}\frac{1}{T}\hat{f}\left(\frac{t}{T}\right)\frac{1}{N}\,\mathrm{Tr}\,\mathrm{Op}_N[a]U_N^{-t}\,\mathrm{Op}_N[a]U_N^t \ . \tag{88}$$

Now $|a \circ \Psi_\alpha^t|_k \le C_k |t|^k$, so with Lemma 1 and the Egorov estimate (73) we get

$$\frac{1}{N}\,\mathrm{Tr}\,\mathrm{Op}_N[a]U_N^{-t}\,\mathrm{Op}_N[a]U_N^t = \frac{1}{N}\,\mathrm{Tr}\,\mathrm{Op}_N[a]\,\mathrm{Op}_N[a\circ\Psi_\alpha^t] + O\left(\frac{|t|}{N}\right)$$
$$= C[a](t) + O\left(\frac{|t|^3}{N^3}\right) + O\left(\frac{|t|}{N}\right) \ . \tag{89}$$

If we use then (74) we obtain

$$\sum_{t\in\mathbb{Z}}\frac{1}{T}\hat{f}\left(\frac{t}{T}\right)\frac{1}{N}\,\mathrm{Tr}\,\mathrm{Op}_N[a]U_N^{-t}\,\mathrm{Op}_N[a]U_N^t = O\left(\frac{1}{T}\right) + O\left(\frac{T^3}{N^3}\right) + O\left(\frac{T}{N}\right) \ , \quad (90)$$

and so the choice $T = N^{1/2}$ gives (86). If we have instead the faster decay (75) we get

$$\sum_{t\in\mathbb{Z}}\frac{1}{T}\hat{f}\left(\frac{t}{T}\right)\frac{1}{N}\,\mathrm{Tr}\,\mathrm{Op}_N[a]U_N^{-t}\,\mathrm{Op}_N[a]U_N^t = O_M\left(\frac{1}{T^M}\right) + O\left(\frac{T^3}{N^3}\right) + O\left(\frac{T}{N}\right) \ , \quad (91)$$

for every $M \in \mathbb{N}$ and so by choosing $T = N^{\varepsilon'}$, with ε' small enough, and M large enough we obtain (87). $\square$

The results in [MR00] show that the estimate (86) is optimal, so in this case we obtain a sharp estimate. The analysis in [MR00] is much more detailed and they have sharp estimates for the rate of quantum ergodicity for individual eigenfunctions. But Theorem 5 might still be of some interest because the proof is of a more dynamical nature, and therefore may be easier to extend to more general cases.

One further class of systems where one could apply the same methods is given by perturbed Kronecker maps, which were recently studied by Rosenzweig, [Ros05]. Here the proof would be very similar to the one of (87), and we would get the same rate $O_\varepsilon(1/N^{1-\varepsilon})$. But in [Ros05] an stronger bound on individual eigenfunctions is given, so our method does not give an optimal result.

The results of Theorem 2 do not hold for these maps since they are not weakly mixing. In particular, using the same methods as in the proof of Lemma 2, one finds for Ψ_α^t that for $\varepsilon = k\alpha$, $k \in \mathbb{Z}\backslash\{0\}$,

$$\sum_{t\in\mathbb{Z}}\frac{1}{T}\hat{f}\left(\frac{t}{T}\right)C[a](t)\mathrm{e}(\varepsilon t) = |\hat{a}(k,0)|^2 + O\left(\frac{1}{T}\right) \ . \tag{92}$$

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From this result together with the techniques used in the proof of Theorem 5 one can derive

$$\lim_{N\to\infty} \frac{1}{N} \sum_{|\theta_i-\theta_j-\varepsilon/N|\leq 1/N^{1/2}}^{N} |\langle \psi_i^N, \mathrm{Op}_N[a]\psi_j^N\rangle|^2 = |\hat{a}(k,0)|^2 \ , \tag{93}$$

where ψ_i, $\mathrm{e}(\theta_i)$, $i = 1,\ldots,N$ are the eigenvectors and eigenvalues of U_N, and $\varepsilon = k\alpha$. So weak mixing is a necessary condition for the validity of (16) in Theorem 2.

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