ON DISTANCE-TRANSITIVE GRAPHS

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An undirected graph Γ is called distance-transitive if for each $i \in \mathbb{N}$, the group $\operatorname{aut}(\Gamma)$ acts transitively on the set of ordered pairs of vertices (u, v) such that $\partial(u, v) = i$. Here $\partial(u, v)$ denotes the distance from u to v. Cameron proved the following in [**1**]:

THEOREM 1. *For given $k \geqslant 3$ there are, up to isomorphism, only finitely many finite distance-transitive graphs of valency k.*

Cameron's proof of this result is based on Sims' Conjecture, which has only been shown to hold using the classification of finite simple groups. In the final section of [**1**], Cameron indicates how Theorem 1 might be proved in an elementary fashion using Macpherson's classification of infinite distance-transitive graphs of finite valency [**4**]. Corollary 1 below provides the missing portion of this elementary proof. In fact, using in place of [**4**] a result of A. A. Ivanov, which yields the classification of infinite distance-regular graphs of finite valency as a corollary, we obtain a proof of Theorem 1 which also avoids the use of the Compactness Theorem suggested by Cameron.

We first state the result of Ivanov [**3**]:

THEOREM 2. *Suppose Γ is a distance-regular graph of valency k and diameter $d \leqslant \infty$ with parameters a_i, b_i and c_i. (See* [**3**] *for the definition.) Let*

$$t = \sup\{i \mid (a_i, b_i, c_i) = (a_1, b_1, c_1)\}.$$

If $t < \infty$ then d is bounded by a function of t and k.

Now let Γ be an arbitrary undirected graph with vertex set $V(\Gamma)$ and let G be a subgroup of $\operatorname{aut}(\Gamma)$. For each $x \in V(\Gamma)$ we denote by $\Gamma(x)$ the set of vertices adjacent to x and by $G(x)$ the stabilizer of x in G. For $i \in \mathbb{N}$ we set $\Gamma_i(x) = \{u \in V(\Gamma) \mid \partial(u, x) = i\}$ and $G_i(x) = \{a \mid a \in G(u) \text{ for all } u \text{ with } \partial(x, u) \leqslant i\}$. A k-path (for $k \in \mathbb{N}$) is a $(k+1)$-tuple of vertices $(x_0, x_1, \ldots, x_k)$ such that $x_i \in \Gamma(x_{i-1})$ if $1 \leqslant i \leqslant k$ and $x_i \neq x_{i-2}$ if $2 \leqslant i \leqslant k$. Let $G(x_0, \ldots, x_k) = G(x_0) \cap \ldots \cap G(x_k)$ and $G_i(x_0, \ldots, x_k) = G_i(x_0) \cap \ldots \cap G_i(x_k)$ for each k-path $(x_0, \ldots, x_k)$ and each $i \in \mathbb{N}$.

Our main result is the following:

THEOREM 3. *Suppose that*

(∗) *Γ is a connected undirected graph with finite girth g and G is a subgroup of* $\operatorname{aut}(\Gamma)$ *such that for each vertex x, $G(x)$ is finite and acts transitively on the set of s-paths $(x_0, \ldots, x_s)$ with $x_0 = x$.*

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Let M be a postive integer with $M \equiv 3 \pmod 8$ and let $m = (M+9)/4$ and $n = \max(2m+4, 14)$. If $s \geqslant 2(m-1)$ and $s \geqslant (g-M)/2$ then $G_m(x_0, \ldots, x_n) = 1$ for every n-path $(x_0, \ldots, x_n)$.

Suppose that Γ and G are as in (*) with $s \geqslant 1$. Let $\{u, v\}$ be an edge of Γ lying on a circuit and let α and β be the valencies of u and v. For $x \in \{u, v\}$ and each $w \in \Gamma(x)$, we can find an s-path $(x_0, \ldots, x_s)$ with $(x_0, \ldots, x_{s-1})$ on this circuit and $(x_{s-1}, x_s) = (x, w)$. It follows that $G(x)$ acts transitively on $\Gamma(x)$ for $x = u$ and $x = v$. Since Γ is connected, every vertex is in the same $\langle G(u), G(v)\rangle$-orbit as u or v. Hence α and β are the only valencies of Γ. With this observation, we can state:

COROLLARY 1. *Suppose Γ and G are as in (*). Let α and β be the valencies of Γ and suppose at least one of them is $\geqslant 3$. Let M be arbitrary. If $s \geqslant (g-M)/2$ then s is bounded by a function of α, β and M.*

Before proving these results, we indicate how they, together with Theorem 2, imply Theorem 1. Let Γ be a finite distance-transitive graph of valency $k \geqslant 3$ and finite girth g with $G = \operatorname{aut}(\Gamma)$. Let t be defined as in Theorem 2. Since $k \geqslant 3$ and $g < \infty$, we can choose s as in (*) to be maximal (see [7, p. 6]). If $g > 3$ then $t = (g-e)/2$ for $e = 2$ or 3 and, because Γ is distance-transitive, $s \geqslant (g-2)/2 \geqslant t$. By Corollary 1 with $M = 2$, t is bounded by a function of k. If $g = 3$ and $t \geqslant 2$ then the subgraph spanned by $\Gamma(x)$ is a disjoint union of complete graphs; let α be the number of components. Let Δ be the bipartite graph with vertex set the disjoint union of $V(\Gamma)$ and the set of maximal cliques of Γ, where an element $x \in V(\Gamma)$ is joined to a maximal clique C whenever $x \in C$. If γ is the girth of Δ then $\gamma = 4t+f$ where $f = 4$ or 6. The valencies α and β of Δ satisfy $\alpha(\beta-1) = k$ and hence cannot both be equal to 2. Thus there is a largest integer σ such that, for each vertex u of Δ, $G(u)$ acts transitively on the set of σ-paths $(u_0, \ldots, u_\sigma)$ in Δ with $u_0 = u$. Because Γ is distance-transitive, $\sigma \geqslant (\gamma-6)/2 \geqslant 2t-1$. By Corollary 1 with $M = 6$, σ is bounded by a function of α and β. Thus t is bounded by a function of k in this case too. By Theorem 2, we conclude that d is bounded by a function of k. Thus Theorem 1 holds.

In the proof of Theorem 3 we require only the following result (compare **[6]** and **[11]**):

THEOREM 4. *Suppose Γ is a connected undirected graph. Suppose G is a subgroup of $\operatorname{aut}(\Gamma)$ such that, for each $x \in V(\Gamma)$, $G(x)$ is finite and acts primitively on $\Gamma(x)$. Then for some prime p, $G_3(x)$ is a p-group (possibly trivial) for all $x \in V(\Gamma)$.*

Theorem 4 is proved in **[8]**. It is assumed there that Γ is finite, but in fact only the hypothesis that $G(x)$ is finite for each $x \in V(\Gamma)$ is actually used. Theorem 3 is related to results in **[2]**, **[5]**, **[9]** and **[10]**.

We begin the proof of Theorem 3. Suppose Γ and G fulfill the hypotheses. Let $(x_0, \ldots, x_{s+m})$ be an arbitrary $(s+m)$-path. If $G_m(x_0) \leqslant G_m(x_1) \leqslant G_m(x_2)$ then $G_m(x_0) = G_m(x_1)$ since $G_m(x_0)$ and $G_m(x_2)$, being conjugate, have the same order. Hence $G_m(x_0) \trianglelefteq \langle G(x_0), G(x_1)\rangle$ and so $G_m(x_0) = G_m(x_1) = 1$. Assuming $G_m(x_0) \neq 1$ or $G_m(x_1) \neq 1$, we may thus choose $k \leqslant s$ to be maximal such that $G_m(x_0, \ldots, x_{k-1}) \nleqslant G_m(x_k)$ or $G_m(x_1, \ldots, x_k) \nleqslant G_m(x_{k+1})$. Since $k \leqslant s$, every k-path is in the same G-orbit as $(x_0, \ldots, x_k)$ or $(x_1, \ldots, x_{k+1})$. If $k \leqslant s-1$ then $G_m(x_0, \ldots, x_k) \leqslant G_m(x_{k+1})$ and $G_m(x_1, \ldots, x_{k+1}) \leqslant G_m(x_{k+2})$. It follows that

(**) if $k \leqslant s-1$ then $G_m(u_0, \ldots, u_k) = 1$ for every k-path $(u_0, \ldots, u_k)$

since Γ is connected. Without loss of generality, we may assume that $G_m(x_0, \ldots, x_{k-1}) \nleqslant G_m(x_k)$. If $(v_0, \ldots, v_h)$ is an h-path with $h \geqslant k$ and v_0 is in the same G-orbit as x_0 then $(v_0, \ldots, v_k)$ is in the same G-orbit as $(x_0, \ldots, x_k)$ and so $|G_m(v_0, \ldots, v_h)| \leqslant |G_m(v_0, \ldots, v_k)| = |G_m(x_0, \ldots, x_k)| < |G_m(x_0, \ldots, x_{k-1})|$. Hence

(***) if $(v_0, \ldots, v_h)$ is an h-path with v_0 in the same G-orbit as x_0 and $|G_m(v_0, \ldots, v_h)| \geqslant |G_m(x_0, \ldots, x_{k-1})|$, then $h \leqslant k-1$.

Since $s \geqslant 2$, $G(x)$ acts 2-transitively on $\Gamma(x)$ for each $x \in V(\Gamma)$ so, by Theorem 4, there exists a prime p such that $G_3(x)$ is a p-group for each $x \in V(\Gamma)$. Let $P = 0_p(G_1(x_0))$. Then $G_m(x_i) \trianglelefteq G_{m-1}(x_{i-1}) \trianglelefteq \ldots \trianglelefteq G_{m-i}(x_0) \trianglelefteq G_1(x_0)$ so $G_m(x_i) \leqslant P$ for $i \leqslant m-1$. Choose $a \in Z(P)$ and let $t \leqslant s$ be maximal such that $a \in G(x_0, \ldots, x_t)$. If $t \leqslant m-2$ then $G_m(x_{t+1}) = G_m(a(x_{t+1}))$ with $a(x_{t+1}) \neq x_{t+1}$. Since $G(x_t)$ acts 2-transitively on $\Gamma(x_t)$, $G_m(x_{t+1}) = G_{m+1}(x_t)$. Thus $G_m(x_{t+1}) \trianglelefteq \langle G(x_t), G(x_{t+1}) \rangle$ and so $G_m(x_{t+1}) = 1$. Since $G_m(x_0, \ldots, x_{k-1}) \neq 1$ and $G_m(x_{t+1})$ is conjugate to $G_m(x_0)$ or $G_m(x_1)$, we conclude that $k = 1$. By (**), we may assume that $t \geqslant m-1$.

Let $d = k-2$ if k is even and $d = k-1$ if k is odd. Let $e = \frac{1}{2}d+m-1$. Suppose $t < e$. Then $t < d+m-1$ and $a(x_{t+1}) \neq x_{t+1}$ so $(x_{d+m-1}, \ldots, x_{t+1}, x_t, a(x_{t+1}), \ldots, a(x_{d+m-1}))$ is a $2(d+m-1-t)$-path. Since $t \geqslant m-1$ and $G_m(x_{m-1}, \ldots, x_{d+m-1}) \leqslant P$, we have

$$G_m(x_{m-1}, \ldots, x_{d+m-1}) \leqslant G_m(x_{d+m-1}, \ldots, x_{t+1}, x_t, a(x_{t+1}), \ldots, a(x_{d+m-1})).$$

Since m is odd, x_{m-1} is in the same G-orbit as x_0. Thus $|G_m(x_{m-1}, \ldots, x_{d+m-1})| \geqslant |G_m(x_0, \ldots, x_{k-1})|$ since $d \leqslant k-1$. Since $d+m-1$ is even, it follows from (***) above that $2(d+m-1-t) \leqslant k-1$ and hence $2(d+m-1-t) \leqslant d$. This contradicts our assumption that $t < e$. Since we may replace $(x_0, \ldots, x_{m+s})$ in this argument by any other $(s+m)$-path beginning at x_0, we conclude that $Z(P) \leqslant G_e(x_0)$. Hence $G_e(x_0) \neq 1$.

Since $G_e(x_0) \neq 1$, $G_e(x_0) \neq G_e(x_1)$ because otherwise $G_e(x_0) \trianglelefteq \langle G(x_0), G(x_1) \rangle$. Thus $G_e(x_0) \nleqslant G_e(x_1)$ or $G_e(x_1) \nleqslant G_e(x_0)$. Let $z = x_0$ if $G_e(x_0) \nleqslant G_e(x_1)$ and $z = x_1$ otherwise. Let $(z_0, \ldots, z_{e+2})$ be an arbitrary $(e+2)$-path with $z_0 = z$. Then $G_e(z_0) \nleqslant G_e(z_1)$. Since $s \geqslant 2(m-1)$, we have $e \leqslant s-1$. Hence $G(z_0)$ acts transitively on the set of $(e+1)$-paths $(u_0, \ldots, u_{e+1})$ with $u_0 = z_0$. Therefore $G_e(z_0)$ contains an element b which does not fix z_{e+1} and there exists a g-path $(v_0, v_1, \ldots, v_g)$ with $v_0 = v_g$ and $(v_0, \ldots, v_{e+1}) = (z_0, \ldots, z_{e+1})$. Since b fixes $(v_g, v_{g-1}, \ldots, v_{g-e})$, we have $g-e > e+1$ and $(v_{g-e}, v_{g-e-1}, \ldots, v_{e+1}, v_e, b(v_{e+1}), \ldots, b(v_{g-e}))$ is a $2(g-2e)$-path with $v_{g-e} = b(v_{g-e})$. Hence $2(g-2e) \geqslant g$. Since $s \geqslant (g-M)/2$ by hypothesis, we conclude that $k \leqslant s-1$. By (**) above, $G_m(u_0, \ldots, u_k) = 1$ for every k-path $(u_0, \ldots, u_k)$.

Let $j = 2$ if e is even and $j = 1$ if e is odd. Let $H = [G_e(z_0), G_e(z_{e+j})]$. We have $G_e(z_0) \leqslant G_{e-j}(z_j)$ and $G_e(z_{e+j}) \leqslant G_{e-j}(z_e)$. Since $G_e(z_0)$ normalizes $G_{e-j}(z_e)$ and $G_e(z_{e+j})$ normalizes $G_{e-j}(z_j)$, it follows that $H \leqslant G_{e-j}(z_j) \cap G_{e-j}(z_e)$. Suppose $k \geqslant 2m+5$ so that $e-j \geqslant 2m$. Then $e-j \geqslant m+(e-j)/2$ and so $G_{e-j}(z_j) \cap G_{e-j}(z_e) \leqslant G_m(z_j, \ldots, z_e)$. Letting $f = 3(e-j)-2m$, we have therefore $G_{e-j}(z_j) \cap G_{e-j}(z_e) \leqslant G_m(u_0, \ldots, u_f)$ for any f-path $(u_0, \ldots, u_f)$ with $(u_{e-j-m}, \ldots, u_{2(e-j)-m}) = (z_j, \ldots, z_e)$. Now suppose that $k \geqslant n+1 \geqslant 15$. Then $k \geqslant 21-2m$ so $f \geqslant k$. Hence $H = 1$.

Recall that $G_e(z_0)$ contains an element b which does not fix z_{e+1}. Since $H = 1$,

$G_e(z_{e+j}) = G_e(b(z_{e+j}))$. *Since* $e \geqslant 3$, we can choose an $(e+1)$-path $(u_0, \ldots, u_{e+1})$ with $u_0 = z_{e+j}$ and $u_{2j} = b(z_{e+j})$. Then $G_e(z_{e+j}) = G_e(b(z_{e+j})) \leqslant G(u_{e+j})$. Since $e+j$ is even and $e+1 \leqslant s$, there exists an element of G mapping $(u_0, \ldots, u_{e+1})$ to $(z_0, \ldots, z_{e+1})$. It follows that $G_e(z_0) \leqslant G(z_{e+1})$. From this contradiction we deduce that $k \leqslant n$. This concludes the proof of Theorem 3.

We now prove Corollary 1. Suppose Γ and G fulfill the hypotheses. Let x be a vertex of Γ. Let m be the smallest odd integer $\geqslant 3$ and $\geqslant (M+9)/4$; we may assume that $s \geqslant 2(m-1)$. Thus by Theorem 3, $|G(x)|$ is bounded by a function of α, β and M. If $i \leqslant (g-M)/2$ then $G(x)$ acts transitively on $\Gamma_i(x)$. If $2 \leqslant i \leqslant (g-1)/2$ then $|\Gamma_i(x)| > |\Gamma_{i-2}(x)|$. It follows that g is bounded by a function of α, β and M. But $s \leqslant (g+2)/2$ by [7, p. 62].

COROLLARY 2. *Suppose* Γ *and* G *are as in* $(*)$. *Choose* $r \in \mathbb{N}$ *minimal such that* $r \geqslant (g+13)/8$ *and let* $n = \max(2r+4, 14)$. *If* $s \geqslant (g+12)/4$ *then* $G_r(x_0, \ldots, x_n) = 1$ for every n-path $(x_0, \ldots, x_n)$.

Proof. Let $M \equiv 3 \pmod 8$ be such that $s \geqslant (g-M)/2$ and $M-g+2s \leqslant 7$. Let $m = (M+9)/4$. Since $s \geqslant (g+12)/4$, we have $M \leqslant g-2s+7 \leqslant (g+2)/2$ and so $m \leqslant r$ and $2(m-1) \leqslant (g+12)/4 \leqslant s$. Apply Theorem 3.

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