On the “Smith is Huq” condition in S-protomodular categories

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Joint work with Nelson Martins-Ferreira

Huq commutation

Definition

A pointed finitely complete category is ***unital*** *if, for every pair X, Y of objects of $\mathbb{C}$, the morphisms $\langle 1, 0 \rangle$ and $\langle 0, 1 \rangle$ in the following diagram are jointly extremally epimorphic:*

$$
\begin{tikzcd}
X \arrow[r, shift right, "\langle 1,0 \rangle"'] & X \times Y \arrow[l, shift right, "\pi_X"'] \arrow[r, shift left, "\pi_Y"] & Y \arrow[l, shift left, "\langle 0,1 \rangle"]
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Definition

*Two morphisms f and g with the same codomain **commute** in the sense of Huq if there exists a (unique) morphism φ making the two triangles in the following diagram commute:*

$$\begin{tikzcd} X \arrow[r, "\langle 1,0\rangle"] \arrow[dr, "f"'] & X \times Y \arrow[d, dotted, "\varphi"] & Y \arrow[l, "\langle 0,1\rangle"'] \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

Smith-Pedicchio commutation

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A finitely complete category is ***Mal'tsev*** *if every internal reflexive relation is an equivalence relation.*

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Definition

*Two equivalence relations R and W on the same object X **centralise each other** when there is a (unique) morphism $p\colon R \times_X W \to X$, where $R \times_X W$ is the pullback of r_2 and w_1:*

$$\begin{tikzcd}
R \times_X W \arrow[r, shift left, "p_2"] \arrow[d, shift right, "p_1"'] & W \arrow[l, shift left, "\sigma_R"] \arrow[d, shift right=2, "w_2"'] \arrow[d, shift left=2, "w_1"] \\
R \arrow[u, shift right, "\sigma_W"'] \arrow[r, shift left=2, "r_1"] \arrow[r, shift right=2, "r_2"'] & X \arrow[l, "e_R" description] \arrow[u, "e_W" description]
\end{tikzcd}$$

such that $p\sigma_R = w_2$ and $p\sigma_W = r_1$.

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Examples: groups, rings, Lie algebras, groups with operations.
Not all semi-abelian categories. Counterexamples: digroups, loops.

Consequences: crossed modules and internal structures

Crossed modules of groups: $d\colon H \to G$ + action of G on H such that:

$$\begin{array}{ll} \text{(PXM)} & d(g \cdot h) = gd(h)g^{-1}; \\ \text{(PEIFFER)} & d(h_1) \cdot h_2 = h_1 h_2 h_1^{-1}. \end{array}$$

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$$XM(Gp) \simeq Gpd(Gp)$$

Internal crossed modules

Internal version of this equivalence for semi-abelian categories (Janelidze):

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Theorem (Martins-Ferreira, Van der Linden)

precrossed modules $+$ Peiffer $=$ crossed modules if and only if (SH).

Let $\mathcal{C}$ be pointed and finitely complete, and let S be a class of points stable under pullbacks.

Definition (Bourn, Martins-Ferreira, M., Sobral)

*$\mathcal{C}$ is S-**protomodular** if the points in S are closed under finite limits in $Pt(\mathcal{C})$ and every point in S is a strong point (the kernel and the section are jointly extremally epimorphic).*

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Properties:
Split Short Five Lemma for split extensions in S.
Every S-reflexive relation is transitive.

Examples of S-protomodular categories

A variety is Jónsson-Tarski if its theory contains a unique constant 0 and a binary operation $+$ satisfying

$$0 + x = x + 0 = x$$

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A point $A \underset{f}{\overset{s}{\leftrightarrows}} B$ *in a Jónsson-Tarski variety is a* ***Schreier point*** *when, for any* $a \in A$*, there exists a unique* α *in the kernel of* f *such that* $a = \alpha + sf(a)$.

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Proposition

Every Jónsson-Tarski variety is an S*-protomodular category with the class* S *of Schreier points.*

Definition

*A morphism f is called S-**special** when its kernel equivalence relation $R[f]$ is an S-equivalence relation. An object X is called S-**special** when the terminal morphism $\tau_X : X \to 1$ is S-special (i.e. the undiscrete relation on X is an S-equivalence relation).*

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A reflexive graph

$$\begin{tikzcd} C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e" description] \end{tikzcd}$$

is **multiplicative** if there exists $m \colon C_2 \to C_1$ such that

$$m\langle ed, 1\rangle = m\langle 1, ec\rangle = 1$$

Let $\mathcal{C}$ be S-protomodular, with S closed under composition and such that every point of the form $X \times Y \underset{\pi_Y}{\overset{\langle 0,1 \rangle}{\rightleftarrows}} Y$ is in S.

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Theorem

Consider the following conditions:

(a) *Every reflexive graph such that both points (d, e) and (c, e) belong to S and $[X, Y] = 0$ is multiplicative.*

(b) *Two effective S-equivalence relations centralise each other as soon as their normalisations commute.*

(c) *Every reflexive graph such that both morphisms d and c are S-special and $[X, Y] = 0$ is multiplicative.*

Then we have the following chain of implications:
$(a) \implies (b) \implies (c)$.

The absolute case

Theorem (Martins-Ferreira, Van der Linden)

In a pointed protomodular category (SH) holds if and only if every reflexive graph such that $[X, Y] = 0$ *is multiplicative.*

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A reflexive graph is $*$-**multiplicative** if
$$\begin{tikzcd} 0 \arrow[r] \arrow[rr, bend right, dotted] & \cdot \arrow[r] & \cdot \end{tikzcd}$$

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A reflexive graph is $*$-**multiplicative** if
$$\begin{tikzcd} 0 \arrow[r, tail] \arrow[rr, dotted, bend right, two heads] & \cdot \arrow[r, two heads] & \cdot \end{tikzcd}$$
It is $*$-**divisible** if
$$\begin{tikzcd} & \cdot \arrow[dd, dotted, bend left] \\ 0 \arrow[ur] \arrow[dr] & \\ & \cdot \end{tikzcd}$$

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$$\begin{tikzcd} & \cdot \arrow[dd, bend left, dotted] \\ 0 \arrow[ur] \arrow[dr] & \\ & \cdot \end{tikzcd}$$

Theorem (Martins-Ferreira, Van der Linden)

In a semi-abelian category, a reflexive graph is $*$-multiplicative if and only if it is $*$-divisible if and only if $[X, Y] = 0$.

Back to the relative case

Let $\mathcal{C}$ be S-protomodular.

Proposition

An S-reflexive graph such that X is an S-special object is $*$-multiplicative if and only if it is $*$-divisible. If, moreover, the square

$$\begin{tikzcd} X \times X \arrow[r, "\pi_1"] \arrow[d, "\omega"'] & X \arrow[d, "ck"] \\ C_1 \arrow[r, "c"'] & C_0 \end{tikzcd}$$

is a pullback (ω is the $*$-division), then $[X, Y] = 0$.

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is a pullback (ω is the $*$-division), then $[X, Y] = 0$.

The converse holds under additional technical hypotheses.

Some stronger conditions

Theorem

Let $\mathcal{C}$ be S-protomodular. If, for every $X \in \mathcal{C}$ the kernel functor $\mathrm{Ker}_X \colon \mathrm{Pt}_X(\mathcal{C}) \to \mathcal{C}$ reflects the commutation of normal subobjects, whenever the domains of the two subobjects are points belonging to S, then (SH) holds.

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In the absolute case, the converse holds, too.

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In the absolute case, the converse holds, too.

For categories of **monoids with operations** and Schreier points, the kernel functors $\mathrm{Ker}_X \colon Pt_X(\mathcal{C}) \to \mathcal{C}$ reflect the commutation of any pair of morphisms, whenever their domains are points belonging to S.

Monoids with operations

Definition (Martins-Ferreira, M., Sobral)

*A category of **monoids with operations** is a variety such that the following conditions hold: if Ω_i is the set of i-ary operations, then:*

(1) $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$;

(2) *There are a binary operation* $+ \in \Omega_2$ *and a constant* $0 \in \Omega_0$ *satisfying the usual monoid axioms;*

(3) $\Omega_0 = \{0\}$;

(4) *Let* $\Omega_2' = \Omega_2 \backslash \{+\}$; *if* $* \in \Omega_2'$, *then* $*^\circ$, *defined by* $x *^\circ y = y * x$, *is also in* Ω_2';

(5) *Any* $* \in \Omega_2'$ *is left distributive w.r.t.* $+$;

(6) *For any* $* \in \Omega_2'$ *we have* $b * 0 = 0$;

(7) *Any* $\omega \in \Omega_1$ *satisfies the following conditions:*

$$\omega(x+y) = \omega(x) + \omega(y); \quad \text{for any } * \in \Omega_2', \omega(a * b) = \omega(a) * b.$$

A consequence of (SH)

Proposition

Let $\mathcal{C}$ be an S-protomodular category in which (SH) holds. Consider an S-reflexive graph such that d is an S-special morphism:

$$\begin{tikzcd} C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e" description] \end{tikzcd}.$$

The following conditions are equivalent:

1. *the graph underlies an internal S-category;*
2. *the graph underlies an internal S-groupoid;*
3. *the kernels of d and c commute.*