QUOTIENTS BY GROUPOIDS

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§1 Introduction and Statement of Results

The main result of this note is the following: (all relevant terms will be defined shortly. The spaces are algebraic spaces of finite type over a locally Noetherian base, unless otherwise mentioned.)

1.1 Theorem. *Let $R \rightrightarrows X$ (i.e. $j : R \to X \times X$) be a flat groupoid such that its stabilizer $j^{-1}(\Delta_X) \to X$ is finite. Then there is an algebraic space which is a uniform geometric quotient, and a uniform categorical quotient. If j is finite, this quotient space is separated.*

Our principal interest in (1.1) is:

1.2 Corollary. *Let G be a flat group scheme acting properly on X with finite stabilizer, then a uniform geometric and uniform categorical quotient for X/G exists as a separated algebraic space.*

A version of (1.2) with stronger assumptions is the main result of [Kollár95], which we refer to for applications, including the existence of many moduli spaces whose existence was previously known only in charateristic zero. Our proof is based on entirely different ideas. We make use of the greater flexibility of groupoids vs. group actions to give an elementary and straightforward argument.

Note that an immediate corollary of (1.1) is:

1.3 Corollary. *(1) A separated algebraic stack* [FaltingsChai80, 4.9] *has a coarse moduli space which is a separated algebraic space.*

(2) An algebraic stack in Artin's sense [Artin74, 5.1, 6.1] *has a GC quotient as an algebraic space if its stabilizer is finite (2.7). If the stack is separated, so is the GC quotient.*

(1.3.1) has a sort of folk status. It appears for example in [FaltingsChai80, 4.10], but without reference or proof.

We note in §9 that for GIT quotients, the assumptions of (1.1) are satisfied, and the quotients obtained are the same.

Several special cases of (1.1) appear in [SGA3,V.4], and our (5.1) can be derived from [SGA3, V.4.1].

1.4 Definitions and Notations. When we say a scheme in this paper, it is not necessarily separated. By a sheaf, we mean a sheaf in the qff (quasi-finite flat) topology of schemes.

We fix a base scheme L which is locally Noetherian, and work only on L-schemes. Thus the product $X \times Y$ means $X \times_L Y$, and if we talk about the properties of X (e.g. separated, of finite type etc.), it is about those of $X \to L$ unless otherwise mentioned.

By a geometric point, we mean the spectrum of an algebraically closed field which is an L-scheme. (This is not of finite type.)

1.5 Definition. By a **relation** we mean any map $j : R \to X \times X$. We say j is a **pre-equivalence relation** if the image of $j(T) : R(T) \to X(T) \times X(T)$ is an equivalence relation (of sets) for all schemes T. If in addition $j(T)$ is always a monomorphism, we call j an **equivalence relation**. For a pre-equivalence relation we write X/R for the quotient sheaf.

Throughout this paper j will indicate a relation, with projections $p_i : R \to X$.

1.6 Definition. A sheaf Q is said to be an **algebraic space** over L if $Q = U/V$ for some schemes U, V over L and an equivalence relation $j : V \to U \times U$ of finite type such that each of the projections $p_i : V \to U$ is étale. We say that Q is **of finite type** (resp. **separated**) if furthermore U, V, j are chosen so that U is of finite type (resp. j is a closed embedding).

We note that fiber products always exist in the category of algebraic spaces [Knudson71, II.1.5]. Therefore the notions of relation, pre-equivalence relation and equivalence relation make sense when X, R are algebraic spaces.

1.7 Remark. If $X/R \to Q$ is a map to an algebraic space and if $Q' \to Q$ is a map of algebraic spaces, then any relation $j : R \to X \times X$ can be pulled back to relation $R' = R \times_Q Q'$ of $X' = X \times_Q Q'$ with $p_{i,R'} = p_{i,R} \times_Q Q' : R' \to X'$. It is easy to check that if R is a pre-equivalence relation (or an equivalence relation) then so is R'. Another form of pullback (which we call restriction) will be discussed in §2.

1.8 Definition. Let $j : R \to X \times X$ be a pre-equivalence relation. And let $q : X/R \to Q$ be a map to an algebraic space Q. Consider the following properties

(G) $X(\xi)/R(\xi) \to Q(\xi)$ is a bijection for any geometric point ξ.

(C) q is universal for maps to (not necessarily separated) algebraic spaces.

(UC) $(X \times_Q Q')/(R \times_Q Q') \to Q'$ satisfies (C) for any flat map $Q' \to Q$.

(US) q is a universal submersion.

(F) The sequence of sheaves in the étale topology

$$0 \to \mathcal{O}_Q \to q_*(\mathcal{O}_X) \overset{p_1^* - p_2^*}{\to} (q \circ p_i)_*(\mathcal{O}_R)$$

is exact (i.e. the regular functions of Q are the R-invariant functions of X).

Note from the definition of quotient sheaf, that (UC) implies (F).

If q satisfies (C) it is called a **categorical quotient**, and if it satisfies (UC) it is called a **uniform categorical quotient**. If it satisfies (G) and (C) it is called a **coarse moduli space**. If it satisfies (G), (US) and (F) it is called a **geometric quotient**. By a **GC quotient** we will mean a quotient satisfying all the above properties.

1.9 Remark. Note (1.8.G) is by definition universal, i.e. is preserved by any pullback $Q' \to Q$. Thus if $U' \subset X' = X \times_Q Q'$ is an R' invariant set ($R' = R \times_Q Q'$), then $U' = {q'}^{-1}(q'(U'))$ (as sets). If the projections p_i are universally open, and q satisfies (1.8.G) and (1.8.US), then q is universally open.

In the category of schemes a geometric quotient is always categorical, and in particular unique (see 0.1 of [MumfordFogarty82]). This however fails for algebraic spaces (see [Kollár95]).

Let us now give a rough idea of the proof, and an overview of the layout of the paper: The basic idea in our proof of 1.1 is to simplify the situation by restriction. The idea comes from the sketch on pg. 218 of [MumfordFogarty82] of a proof of 1.1 in the analytic case, and is roughly as follows: To form the quotient we have to identify elements in the same orbit. If W is a general slice through $x \in X$, then W meets each orbit a finite number of times, and we should be able to form the quotient from W by identifying each $w \in W$ with finitely many "equivalent" points (points in the same orbit in X). This equivalence is no longer described by a group action, but a good deal of structure is preserved. This leads to the notion of a groupoid, defined in §2. The groupoid formalism is very flexible, and allows us to simplify by various restrictions (see §3). We can work étale locally around a point $x \in X$. By taking the slice W we reduce to the case where the projections $p_1, p_2 : R \to X$ are quasi-finite. Then in §4 we reduce to the case where R is *split*, a disjoint union of a finite flat sub-groupoid P and a piece which does not have effects on x (in particular P contains the stabilizer of x). In §5 we treat the case of a finite flat groupoid, very similar to the case where a finite group acts. In §7 we construct the quotient. We first mod out by P, and afterwards we have a free étale action, which thus defines an algebraic space.

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§2 Groupoids

We begin with some trivial categorical remarks, which in the end is all there is to a groupoid.

First, for a small category C, write $R = Hom(C)$ and $X = Obj(C)$. Then we have two natural maps $s, t : R \to X$ giving for each morphism f its source and target objects, and composition gives a map $R \times_{(s,t)} R \xrightarrow{c} R$, and the identity map gives $e : X \to R$, a section for both s and t. There are obvious compatibilities between these maps reflecting the axioms of a category, and a small category can be equivalently defined as a pair of sets R, X with maps s, t, e, c satisfying the various compatibilities.

If in addition every morphism in C is an isomorphism, C is called a **groupoid**. It is denoted by $R \rightrightarrows X$. Then we have a map $i : R \to R$ sending a morphism to its inverse.

Observe that if C is a groupoid, then the image of $j = (t, s) : R \to X \times X$ is an pre-equivalence relation (where two objects are equivalent if they are isomorphic).

Let s, t, e, c, R, X define a category, C. If $A \to X$ is a map (of sets), then fiber product defines a category $C|_A$, with objects A and maps $R|_A = R \times_{X \times X} A \times A$, where a map between two objects a, b in A is just a map between their images in X, with composition taking place on X. If C is a groupoid, so is $C|_A$. We refer to $R|_A$ as the **restriction** of R to A.

Let $p : X \to Z$ be a map (of sets) such that $p \circ s = p \circ t$ and let $Z' \to Z$ be any map. Set $X' = X \times_Z Z'$ and $R' = R \times_Z Z'$. The two maps $s, t : R \to X$ induce $s', t' : R' \to X'$, and it is immediate that these define a groupoid. This is compatible with (1.7). Note there is a natural injection $R' \hookrightarrow R|_{X'}$, and that in this way (R', X') is a subcategory of $(R|_{X'}, X')$. We refer to R' as the **pullback**.

2.1 Definition. A **groupoid space** is a quintuple of maps s, t, c, e, i (all of finite type) of algebraic spaces as above, such that for all T, the quintuple $s(T), t(T), c(T), e(T), i(T)$ define a groupoid $R(T) \rightrightarrows X(T)$, in a functorial way.

For a groupoid (space) $s, t : R \rightrightarrows X$, we use the following convention throughout this paper: $j = (t, s), p_1 = t, p_2 = s$.

From now on we talk about algebraic spaces. For simplicity we will abuse notation and call a groupoid space, a groupoid. When we wish to return to the categorical setup, we will speak of a groupoid of sets.

By a flat (resp étale, etc.) groupoid, we mean one for which the maps s, t are flat (resp étale, etc.).

It is immediate from the categorical remarks that a groupoid is an pre-equivalence relation, and that the restriction or pullback (defined in the obvious way generalizing the definition for sets) of a groupoid is again a groupoid. Note in particular that when X is a geometric point then R is a group scheme, so in particular for any geometric point $x \in X$, $R|_x$ is a group scheme, called the **stabilizer** at x. We call the group scheme $S = j^{-1}(\Delta_X) \to X$ the stabilizer.

By a **map** $g : (R', X') \to (R, X)$ **between groupoids** we mean a pair of maps $g : R' \to R$ and $g : X' \to X$ such that the induced map between categories $(R'(T), X'(T)) \to (R(T), X(T))$ is a functor for all T (this of course can be equivalently expressed by saying various diagrams commute).

We say a geometric point $x \in X$ is fixed by a pre-equivalence relation, if x is the only geometric point of its orbit $t(s^{-1}(x))$.

It will be useful to know when a map of groupoids is a pullback.

2.2 Definition. Let $g : (R', X') \to (R, X)$ be a map of groupoids.

(1) We say that g is a *square* when the commutative diagram

$$\begin{array}{ccc} R' & \longrightarrow & X' \\ {\scriptstyle g}\big\downarrow & & {\scriptstyle g}\big\downarrow \\ R & \longrightarrow & X \end{array}$$

is a fiber square, if we take for the horizontal maps either both source maps, or both target maps.

(2) We say that g is *fixed point reflecting* if for each geometric point $x' \in X'$ the induced map of stabilizers $S_{x'} \to S_{g(x')}$ is a set bijection.

2.3 Remark. It is immediate that if $X \to Z$ is R-invariant, and (R', X') is the pullback along any map $Z' \to Z$, then $(R', X') \to (R, X)$ is square and fixed point reflecting.

Note that in (2.2.1), if the diagram is a fiber square for the source, then the same follows for the target because $t = s \circ i$.

2.4. We will say that a GC quotient $X \to Q$ for a groupoid $R \rightrightarrows X$ satisfies the **descent condition** if whenever $g : (R', X') \to (R, X)$ is an étale square fixed point reflecting map of groupoids such that a GC quotient $q' : X' \to Q'$ for X'/R' exists, then the induced map $Q' \to Q$ is étale and $X' \simeq X \times_Q Q'$.

2.5. One checks easily that a map of groupoids of sets $g : (R', X') \to (R, X)$ is obtained by pullback from the induced map $Q' \to Q$ iff g is square and fixed point reflecting. That is, the descent condition always holds for groupoids of sets.

2.6 Remark. We note for convenience that for any map $f : W \to X$, the restriction $R|_W$ is described by the following diagram, in which every square is a fiber square:

$$\begin{array}{ccccc} R|_W & \longrightarrow & R \times_{(p_2, f)} W & \longrightarrow & W \\ \big\downarrow & & \big\downarrow & & {\scriptstyle f}\big\downarrow \\ W \times_{(f, p_1)} R & \longrightarrow & R & \xrightarrow{p_2} & X \\ \big\downarrow & & {\scriptstyle p_1}\big\downarrow & & \\ W & \xrightarrow{f} & X & & \end{array}$$

The following will alleviate separation anxiety:

2.7 Lemma-Definition. *Let $R \rightrightarrows X$ be a groupoid and let $j : R \to X \times X$ be the induced map and S the stabilizer group scheme $j^{-1}(\Delta_X) \to X$ over X. Then j is separated iff $S \to X$ is separated.*

In this case, we say that $R \rightrightarrows X$ has a separated stabilizer.

Proof. The only if part is obvious because $S \to X$ is the base change of j by $\Delta_X \to X \times X$.

Assume that $S \to X$ is separated. Hence any section in particular the identity section $e : X \to S$ is a closed immersion (cf. [Knudson71, II.3.11]). Look at the commutative diagram:

$$\begin{array}{ccc} R \times_{X \times X} R & = & R \times_{(s,s)} S \\ \Delta \uparrow & & \uparrow \\ R & = & R \times_{(s,s)} e(X) \end{array}$$

where the top identity is obtained by the isomorphism $(r_1, r_2) \mapsto (r_1, i(r_1) \circ r_2)$. Since $e(X) \to S$ is a closed embedding, so are the vertical maps. Thus j is separated. $\square$

2.8 Corollary. *If $R \rightrightarrows X$ has a separated stabilizer, so does $R|_W$ for any map $g : W \to X$.*

We will assume throughout that every groupoid has a separated stabilizer.

§3 Localizing the construction of quotients

3.1 Lemma. *Let R be an pre-equivalence relation. Let $g : W \to X$ be a map, and $R|_W$ defined as in (2.6).*

(1) The canonical map $W/(R|_W) \to X/R$ of sheaves is injective, and is an isomorphism if the composition

$$p : W \times_{(g,p_1)} R \to R \xrightarrow{p_2} X$$

is surjective in the qff topology.

*(2) (**Open**) Assume that $p_1, p_2 : R \to X$ are universally open, that p above is universally open, and as a map to its image, is surjective in the qff topology. Suppose $X \to Z$ is a GC quotient. Then the image, V, of the composition $W \to X \to Z$ is open, and the induced map $W \to V$ is a GC quotient.*

*(3) (**Surjection**) Assume that $p_1, p_2 : R \to X$ are universally open and p above is surjective in the qff topology. Suppose $W/(R|_W) \to Z$ is a GC quotient. By (1) there is an induced map $X/R = W/(R|_W) \to Z$. It is a GC quotient.*

Proof. Let q_X and q_W be the quotient maps (with domains X and W).

If $Z' \to Z$ is a map, then for W', X', R' the fiber products of W, X, R with Z' over Z, we have a fiber diagram

$$(*) \qquad \begin{array}{ccccccc} W' \times_{g',p_1'} R' & \longrightarrow & R' & \xrightarrow{p_2'} & X' & \longrightarrow & Z' \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ W \times_{g,p_1} R & \longrightarrow & R & \xrightarrow{p_2} & X & \longrightarrow & Z \end{array}$$

(here for a map f, f' indicates the pullback).

(1) follows from the definitions of sheaves.

For (2): V is open and q_X is universally open, by (1.9). Analogous remarks using $*$ show $q_X \circ g$ is universally open.

By definition the result holds if W is an R-invariant open set, thus we may replace X by U. Then $W/(R|_W) = X/R$ so (1.8.G) and (1.8.C) hold.

By $*$ the assumptions are preserved by any flat base extension $Z' \to Z$. Thus (1.8.UC) holds.

For (3): By $*$ the assumptions are preserved by pullback along any flat $Z' \to Z$, so it is enough to consider properties (1.8.G), (1.8.C) and (1.8.US). (1.8.G) and (1.8.C) depend only on the sheaf $W/(R|_W) = X/R$ and thus hold for q_X iff they hold for q_W. Since q_W factors through q_X, if q_W is a submersion, so is q_X. $\square$

3.2 Lemma. *Assume that $p_1, p_2 : R \to X$ are universally open. Suppose $\{U_i\}$ is a finite étale cover of X Suppose GC quotients $q_i : U_i/(R|_{U_i}) \to Q_i$ exist for all i. Then a GC quotient $q : X \to Q$ exists.*

Proof. By (3.1.3), and induction, we may assume we have a Zariski cover by two R-invariant open sets. By (3.1.2), the GC quotient of $(U_1 \cap U_2)/(R|_{U_1 \cap U_2})$ exists as open set of the GC quotient of $U_i/(R|_{U_i})$ for $i = 1, 2$. Since the categorical quotients are unique, these glue. The rest is easy. $\square$

Corollary 3.2.1. *Suppose $q : X/R \to Q$ is a map. To check it is a GC quotient we may work étale locally on Q.*

Proof. If Q is étale locally the GC quotient, then q satisfies (1.8.C) by decent. Also X has an GC quotient by (3.2). Thus since a categorical quotient is unique, q is the GC quotient. $\square$

Lemma 3.3. *Assume that $s, t : R \to X$ is flat and j is quasi-finite. Let $x \in X$ be a geometric closed point. To prove that there is an étale neighborhood U of $x \in X$ such that a GC quotient exists for $U/(R|_U)$, we may assume R and X are separated schemes, $s, t : R \to X$ are quasi-finite and flat and x is fixed. To show the GC quotient has any additional property for which (3.1.2) and (3.1.3) hold, we can make the same assumptions.*

Proof. By (3.1.2) we can assume X is a separated scheme. Since R is separated and quasi-finite over $X \times X$, R is also a separated scheme by [Knudson71, II.6.16].

Claim 1. *We can assume $F = s^{-1}(x)$ is Cohen Macaulay along $j^{-1}(x, x)$.*

Proof. Since j is quasi-finite, there is a geometric closed point $w \in F$ such that F is Cohen Macaulay along $t^{-1}t(w)$. (Keep cutting down $\overline{t(F)}$ with a general hypersurface of high degree till one gets a 0-dimensional component V. Then take $w \in F \cap t^{-1}(V)$.) We have an isomorphism $F \xrightarrow{\circ i(w)} s^{-1}(t(w))$ sending w to $e(t(w))$. We can replace x by $t(w)$, and thus the claim follows after a Zariski shrinking around x. □

Claim 2. *We can assume s, t are flat and quasi-finite.*

Proof. Let $x \in W \subset X$ be the closed subscheme defined by $\dim_x F$ elements of m_x, whose intersection with $\overline{o_x} = \overline{t(s^{-1}(x))}$ is zero dimensional at x (lift any parameters of the maximal ideal of $\overline{o_x}$ at x). Since F is Cohen Macaulay along $j^{-1}(x, x)$, by [Matsumura80] $p : W \times_{(g,t)} R \xrightarrow{s} X$ is flat over x, and $s : R|_W \to W$ is flat and quasi-finite over x. After a Zariski shrinking, this holds globally. The assertion on t holds by $t = s \circ i$. By 3.1.3 we may restrict to W. □

Once s, t are quasi-finite, we can assume x is fixed after a Zariski shrinking. □

§4 Splitting

4.1 Definition. We say that a flat groupoid $R \rightrightarrows X$ is **split** over a point $x \in X$ if R is a disjoint union of open and closed subschemes $R = P \coprod R_2$, with P a subgroupoid finite and flat, and $j^{-1}(x, x) \subset P$.

Our main goal in this section is to prove:

4.2 Proposition. *Let $s, t : R \rightrightarrows X$ be a quasi-finite flat groupoid of separated schemes. Then every point $x \in X$ has an affine étale neighborhood (W, w) such that $R|_W$ is split over w.*

Lemma 4.3. *Let $p : X \to Y$ be a map, and $F, G \subset X$ two closed subschemes. If F is finite flat over Y then there is unique closed subscheme $i : I \to Y$ such that $f : T \to Y$ factors through i iff F_T is a subscheme of G_T.*

Proof. I is obviously unique, so we can construct it locally, and so can assume $p_*(\mathcal{O}_F)$ is free. Then there is a presentation

$$K \xrightarrow{h} p_*(\mathcal{O}_F) \to p_*(\mathcal{O}_{F\cap G}) \to 0$$

with K free. I is defined by the vanishing of h. □

Now let $s, t : R \rightrightarrows X$ be as in (4.2). Note that $s : R \to X$ is quasi-affine since it is separated and quasi-finite [EGA, IV.18.12.12]. Thus one can embed $s : R \to X$ into a projective scheme of finite type over X. Thus the standard theory of Hilbert schemes applies to our set up.

Let $g : H \to X$ be the relative Hilbert scheme $\mathrm{Hilb}_{R/X}$ parametrizing closed subschemes of R which are proper flat over X via s. Let $W \subset H$ be the closed subscheme of (4.3) of families containing the identity section $e : X \to R$. Let $s : P \to W$ be the universal family.

An S point of P consists of a pair (F, a) of a map $a : S \to R$, and a family $F \subset R \times_{(s,s(a))} S$ flat over S, with a factoring through F. Note that g maps (F, a) to the S point $[F]$ of W representing $F \subset R \times_{(s,s(a))} S$, and s sends $[F]$ to the S point $s(a)$ of X. The composition of component R with $i(a) = a^{-1}$ gives an isomorphism $R \times_{(s,s(a))} S \xrightarrow{\circ a^{-1}} R \times_{(s,t(a))} S$. We write the image of F under this isomorphism as $F \circ a^{-1}$. Since a factors through F, the identity map e factors through $F \circ a^{-1}$, and $F \circ a^{-1} \subset R \times_{(s,t(a))} S$ defines an S point $[F \circ a^{-1}]$ of W. Thus we have a map $t : P \to W$ given by $t(F, a) = [F \circ a^{-1}]$, and $g([F \circ a^{-1}]) = t(a)$.

4.4 Lemma. *The pair $s, t : P \rightrightarrows W$ is a finite flat subgroupoid of $R|_W$.*

Proof. In the big diagram (2.6) defining $R|_W$, $P \subset R \times_{(s,g)} W$. Let $pr_R : P \to R$ be the induced map. Then $g \circ t = t \circ pr_R$, and thus $(F, a) \mapsto ([F \circ a^{-1}], a, [F])$ embeds P in $R|_W$ (2.6).

Let $((F, a), (G, b))$ be an S point of $P \times_{(s,t)} P$. Then we have $F = G \circ b^{-1}$ and

$$([F \circ a^{-1}], a, [F]) \circ ([G \circ b^{-1}], b, [G]) = ([G \circ (a \circ b)^{-1}], a \circ b, [G])$$

by the composition law of $R|_W$. Thus P has the induced composition law

$$(F, a) \circ (G, b) = (G, a \circ b) \quad \text{if} \quad F = G \circ b^{-1},$$

the identity section is given by $e([F]) = (F, e)$, and the inverse by $i(F, a) = (F \circ a^{-1}, a^{-1})$. (Note $i(F, a) \in P$ by our definition of W, this is the reason for restricting from H to W.) □

Let $x \in X$ be an arbitrary point and let $P_x \subset R$ be the open closed set of R_x whose support is $j^{-1}(x, x)$. Indeed if M is the defining ideal of x in X, then P_x is defined in R_x by $(s^*M + t^*M)^n$ for $n \gg 0$.

Let $w = [P_x] \in W$, that is $P_w = P_x$.

4.5 Lemma. *$f : H \to X$ is étale at w, isomorphic on residue fields $\mathbf{k}(w) \simeq \mathbf{k}(x)$ and $H = W$ in a neighborhood of w.*

Proof. The construction of $P_w \subset R \times_X \mathbf{k}(x)$ shows $\mathbf{k}(w) \simeq \mathbf{k}(x)$. It remains to see

$$Hom_{(X,x)}((S, y), (W, w)) = Hom_{(X,x)}((S, y), (H, w)) = (\text{one point set})$$

for any artin schemes (S, y) over (X, x). Since R_S contains an open closed subscheme P' such that $P' \times_S y = P_x \times_x y$, we see $P' \supset e(S)$ and these are obvious. □

4.6 Corollary. *$R|_W = P \cup R_2$, for R_2 a closed subscheme with $R_2 \cap P \cap s^{-1}(w) = \emptyset$.*

Proof. Let $R' = R|_W$, and let y be an arbitrary point of the finite set P_w. Then $P_w = R'_w$ as schemes at y. Let $I \subset \mathcal{O}_{R'}$ be the defining ideal of P in R'. Then $P_w = R'_w$ at y implies $I \otimes_{\mathcal{O}_W} \mathbf{k}(w) = 0$ at y by the flatness of P over W. Thus $I \otimes_{\mathcal{O}_{R'}} \mathbf{k}(y) = 0$. Hence in a neighborhood of y, we have $I = 0$ (that is $P = R'$) by Nakayama's Lemma. The result follows. □

Now to prove (4.2), we will shrink W so that R_2 and P become disjoint. In order to preserve the finiteness of P, we want to shrink using P invariant open sets. For this a general construction will be useful:

A geometric construction of equivalence classes.

4.7 Lemma. *Let $A \to B$ be a local map of local Noetherian rings, with $B/(m_A \cdot B)$ of finite dimension k over A/m_A. Then $m_B^k \subset m_A \cdot B$.*

Proof. We can replace B by $B/(m_A \cdot B)$, and so assume A is a field. Now the result follows by Nakayama's Lemma. □

Let $P \rightrightarrows X$ be a finite flat groupoid, $x \in X$ a point fixed by P, and k the degree of s over x.

We consider the functor on X-schemes, whose $(\alpha : T \to X)$-point consists of k sections $g_i : P \times_{(s,\alpha)} T \to T$ of P whose *scheme theoretic union* in $T \times R$ (the closed subscheme defined by the product of the defining ideals of the k sections g_i) contains the pullback Γ_T, where $\Gamma \subset X \times P$ is the graph of s. By 4.3, this functor is represented by a closed subscheme I of the k-fold fiber product of $s : P \to X$.

A geometric point of I consists of a k-tuple of points in a fiber of s, whose scheme theoretic union contains the fiber. Let $p : I \to X$ be the map induced by s. By 4.7, p is surjective on a neighborhood of x. Let π_i be the map

$$\pi_i : I \subset P^{\times_X k}(= \overbrace{P \times_X \cdots \times_X P}^{k \text{ times}}) \xrightarrow{pr_i} P \xrightarrow{t} X.$$

For a point $\alpha \in I$, note that the set $\bigcup_i^k \pi_i(\alpha) \subset X$ is the full P-equivalence class $[p(\alpha)]$.

4.8 Lemma. *With the above notation and assumptions, let ξ be a geometric point at x and U an open set $U \supset [\xi]$ such that s is of constant degree k over U. Let $V = s(t^{-1}(U^c))^c$. Then V is an P-invariant open neighborhood of x, $V \subset U$, and $V = \{\eta \in X | [\eta] \subset U\}$. If U is affine then V is affine. In particular, x has a base of P-invariant open affine neighborhoods.*

Proof. Since the projections are closed, everything before the final assertion is clear.

Now assume U is affine. Set

$$J = \bigcap_{i=1}^{k} \pi_i^{-1}(U).$$

Note (as a set) $J = \{\alpha | [s(\alpha)] \subset U\}$. Since p is surjective, $J = p^{-1}(V)$, and hence $p : J \to V$ is finite. Since π_i is finite, and I is separated, J is affine. Thus so is V. □

Proof of (4.2). After a Zariski shrinking, we can assume x is fixed by R. We have by (4.6), $R|_W = P \cup R_2$, with w outside the image (under either projection) of R_2. This is preserved by restriction to P-invariant open sets. Since $R_2 \cap P \subset P$ is closed, and does not meet the fiber, we can assume by (4.8) and (4.5), that R_2 and P are disjoint, and $W \to X$ is étale. □

§5 QUOTIENTS FOR FINITE FLAT GROUPOIDS WITH AFFINE BASE

5.1 Proposition (Finite-Over-Affine case). *Let A be a Noetherian ring and B an A-algebra of finite type. Let R be an affine groupoid finite and free over* Spec B. *Then B^R is an A-algebra of finite type and*

$$q : X = \operatorname{Spec} B \to Q = \operatorname{Spec}(B^R)$$

is finite and the GC quotient of X/R.

For some large number n, let $x_1 = 1, x_2, \cdots, x_n \in B$ be a set of generators as an A-algebra. Let $\xi_1, \cdots, \xi_n$ be indeterminates over B and by the flat base change $A \to A[\xi_1, \cdots, \xi_n]$ we may treat ξ's as R-invariant indeterminates. Let $\phi = \sum_i x_i \xi_i$.

5.2 Lemma. *We have $Nm_t(s^*\phi) \in B^R[\xi]$, where Nm_t is the norm for the finite free morphism $t : R \to X$.*

Proof of (5.2). It is enough to prove $t^*Nm_t(s^*\phi) = s^*Nm_t(s^*\phi)$. By the definition of norm, we have

$$s^*Nm_t(s^*\phi) = Nm_{pr_1}({pr_2}^*s^*\phi), \qquad (5.2.1)$$

where pr_1, pr_2 are defined in the following diagram.

$$\begin{array}{ccccc} R \times_{(s,t)} R & \xrightarrow{pr_2} & R & \xrightarrow{s} & X \\ {\scriptstyle pr_1}\downarrow & & {\scriptstyle t}\downarrow & & \\ R & \xrightarrow{s} & X & & \end{array}$$

Let

$$\tau : R \times_{(s,t)} R \simeq R \times_{(t,t)} R$$

be a morphism defined by $(r_1, r_2) \mapsto (r_1, r_1 \circ r_2)$, which is an R-isomorphism via the first projection. Then we have a commutative diagram.

$$\begin{array}{ccccccc} R & \xleftarrow{pr_1} & R \times_{(s,t)} R & \xrightarrow{pr_2} & R & \xrightarrow{s} & X \\ \| & & {\scriptstyle \tau}\downarrow & & & & \| \\ R & \xleftarrow{pr_1'} & R \times_{(t,t)} R & \xrightarrow{pr_2'} & R & \xrightarrow{s} & X \end{array}$$

Similarly to (5.2.1), we see $t^*Nm_t(s^*\phi) = Nm_{pr_1'}({pr_2'}^*s^*\phi)$. Then starting with ϕ on X on the right, we can send it along the diagram either by pull back or by norm. On the left, we get the equality

$$s^*Nm_t(s^*\phi) = Nm_{pr_1}({pr_2}^*s^*\phi) = Nm_{pr_1'}({pr_2'}^*s^*\phi) = t^*Nm_t(s^*\phi),$$

which proves the R-invariance as required. $\square$

5.3 Lemma. *B^R is an A-algebra of finite type and B is a finite B^R-module. In particular q is finite and surjective. The formation of Q commutes with any flat base extension $Q' \to Q$ with Q' affine.*

Proof of (5.3). Let $F(\xi) = Nm_t(s^*\phi) \in B^R[\xi]$ by (5.2). Let C be the A-subalgebra of B^R generated by all the coefficients of F. We note that

$$F(x_i, 0, \cdots, 0, \overset{i\text{-th}}{-1}, 0, \cdots, 0) = 0 \qquad \forall i \geq 2$$

because $t : R \to X$ has a section e such that $s \circ e = t \circ e = id$. Since $F(\xi)$ is monic in ξ_1, each x_i is integral over C. Since x_i generate B as an A-algebra, B is finite over C. Thus B^R is also a finite C-module and hence the lemma is proved. $\square$

5.4 Lemma. *q satisfies the condition (1.8.G).*

Proof of (5.4). Since q is finite dominating, the map in (1.8.G) is surjective. It is enough to prove the injectivity. Let us look at $F(\xi)$ in the proof of (5.3) as $F_t(\xi)$ parametered by $t \in \operatorname{Spec} B^R$. Let $a \in X$ be a geometric point. Then by the definition of norm, we have a polynomial identity

$$F_{q(a)}(\xi) = \prod_{b \in t^{-1}(a)} (\xi_1 + \sum_{i=2}^{n} x_i(s(b))\xi_i),$$

where b is chosen with multiplicity. So if $q(a) = q(a')$, then there exists $b \in t^{-1}(a)$ such that $x_i(s(b)) = x_i(s(a'))$ for all i. Since x_i generate the A-algebra B, we have $a' = b$. $\square$

Proof of (5.1). By 3.2.1 and 5.3 we can work locally in the étale topology on Q.

For the condition (1.8.UC), let $g : X \to Z$ be an R-invariant map. It is enough to prove that for any geometric point $q_0 \in Q$ there is an étale neighborhood Q' of q_0 such that $X \times_Q Q' \to Z$ factors through Q'. Passing to the strict henselization, we may assume (Q, q_0) is strictly henselian. Then X, since it is finite over Q, is a disjoint union of strictly henselian local schemes, and R acts transitively on the components by (5.4). By (3.1.3), we may drop all but one component without affecting Q, or the existence of a factorization, and so assume X is strictly henselian (X, x). Then $X \to Z$ factors through a strict henselization of $(Z, g(x))$ which is R-invariant since R acts trivially on the residue field of X. Thus g factors through Q by construction of Q.

q is a universal submersion since it is a finite surjection. $\square$

Let $g : (R', X') \to (R, X)$ be a map of groupoids. Let $T \to X$ be a map and $T' = T \times_X X'$. There is then an induced map of groupoids $g_T : (R'|_{T'}, T') \to (R|_T, T)$.

6.1 Lemma. *If g is square, so is g_T.*

Proof. Passing to Hom we can assume we are working with groupoids of sets. One simply considers the natural map $R'|_{T'} \xrightarrow{g_T \times s} R|_T \times_T T'$ and checks it is a bijection. $\square$

6.2 Remark. *Let $f : X \to Y$ be a map of algebraic spaces. Then*

(1) $g : X \times_Y X \to X \times X$ is separated.

(2) If X is separated, then f is separated.

Proof. The map g is a monomorphism (of sheaves) by the definition of $X \times_Y X$. Hence g is separated. Assume that X is separated. Since the composition $X \to X \times_Y X \to X \times X$ is a closed immersion, so is $X \to X \times_Y X$ [Knudson71, I.1.21]. $\square$

6.3 Lemma. *Let $R \rightrightarrows X$ be a finite flat groupoid with separated X. Let $q : X \to Q$ be a GC quotient. Then q is finite, it is the GC quotient, and also satisfies the descent condition (2.4).*

Proof. Since X is separated, so is q (6.2). By (3.2.1) we can assume Q is an affine scheme by a base change. Since q is separated and quasi-finite by (1.8.G), X and R are separated schemes [Knudson71, I.I6.16].

After shrinking Q we can assume q is quasi-projective [EGA, IV.18.12.12]: We can complete q to $X \subset X' \xrightarrow{q'} Q$ with $X \subset X'$ open and $X' \xrightarrow{q'} Q$ projective.

Let $F \subset X$ be the fiber over a geometric point $p \in Q$. Then $F \subset X'$ is closed. Let $Z = X' \setminus X$, and let $H \subset X'$ be a general hypersurface such that $Z \subset H$. Then $H \cap F = \emptyset$. Thus $F \subset H^c = U \subset X$ is an open affine set. By 4.8 we can assume U is R-invariant, and so can assume X is affine. Thus q is finite, and so it is a universal submersion. The additional properties follow from 5.1.

Now let $g : (R', X') \to (R, X)$ be as in the definition 2.4. We can check the descent condition locally in the étale topology on Q. Thus we may assume $Q = (Q, z)$ is strictly henselian and $x' \in X'$ any point lying over z. Then X is a disjoint union of strictly henselian schemes. We can obviously prove $X' = X \times_Q Q'$ one connected component of X at a time. Note that if W is a connected component of X, then $R|_W \rightrightarrows W$ is again finite. By (3.2) replacing X by a connected component containing $g(x')$ does not affect $Q' \to Q$, and by (6.1) the assumptions are preserved. So we may assume X is strictly henselian. Then X' is a disjoint union $\coprod X_i \coprod Y_1$ with each X_i isomorphic under g to X, and $z \notin g(Y_1)$. Let i be such that $\{x'\} = g^{-1}(x) \cap X_i$. Since $R' \rightrightarrows X'$ is finite flat, $t'({s'}^{-1}X_i)$ is an open closed set of X' and finite over X. Thus $t'({s'}^{-1}X_i)$ is a union of some X_j's. The fixed point reflecting condition $S_x = S_{x'}$ means that $g^{-1}(x) \cap t'({s'}^{-1}X_i) = \{x'\}$. Thus $t'({s'}^{-1}X_i) = X_i$, and X_i is R'-invariant. Thus $(R'|_{X_i}, X_i) \to (R, X)$ is square, and we can assume $X' = X_i$. Then we have $(R', X') \simeq (R, X)$. $\square$

6.4 Lemma. *Let $s, t : R \to X$ be a groupoid with j proper. Then if $q : X \to Q$ is a GC quotient for X/R, then Q is separated.*

Proof. We can replace X with its étale affine cover (3.1.2). The proof of (2.9) of [Kollár95] extends to groupoids without change. $\square$

6.5 Fine Quotients Lemma. *Let $R \rightrightarrows X$ be a finite flat groupoid of separated schemes such that $j : R \to X \times X$ is a closed embedding. Then X/R is represented by a finite flat morphism $q : X \to Q$ for the qff topology. The map q is flat, the construction commutes with pullback along any $Q' \to Q$, and q is a GC quotient.*

Proof. Since $R \subset X \times X$ is finite and flat over X, $t : R \to X$ defines an R-invariant map $q : X \to \mathrm{Hilb}_X$. Let $i : Z \subset \mathrm{Hilb}_X \times X$ be the universal family. Then $R = X \times_{\mathrm{Hilb}} Z$. By the universal property of Z, the section e of t induces a map $\gamma : X \to Z$ such that $i \circ \gamma = (q, id)$. thus γ is a closed embedding. In particular, since $\pi : Z \to \mathrm{Hilb}_X$ is finite, $q = \pi \circ \gamma$ is finite. Let $Q \subset \mathrm{Hilb}_X$ be the scheme theoretic image of q. We abuse notation, and write for Z its restriction to $X \times Q$, whence $Z \times_Q X = R \subset X \times X$.

Let $q : R \to Z$ be the projection.

We now show $\gamma : X \hookrightarrow Z$ is an isomorphism. Since γ is a closed embedding, $X \times_Q X \subset X \times X$ factors as

$$X \times_Q X = X \times_Q Z \times_Z \gamma(X) = R \times_Z \gamma(X) \subset R \subset X \times X.$$

Since q is R-invariant, $R \subset X \times_Q X$. Thus $R = R \times_Z \gamma(X)$.

Since q is affine, by the definition of the scheme theoretic image, $\mathcal{O}_Q \to q_*(\mathcal{O}_X)$ is injective. Since π is flat, $\mathcal{O}_Z \to q_*(\mathcal{O}_R)$ is injective. Thus $R = R \times_Z \gamma(X)$ implies γ is an isomorphism.

Thus we have that q is flat and $R = X \times_Q X$. Note these two properties are preserved by pullback along any $Q' \to Q$. They also imply $Q = X/R$ as sheaves in the qff topology. Now the rest follows easily. □

§7 The Boot Strap Theorem

We begin with a number of categorical remarks, which will guide our construction when we move to algebraic spaces. In addition, for GC quotients, these categorical remarks will describe what happens on T-valued points for arbitrary schemes T.

Suppose $s, t : R \to X$ defines a groupoid of sets.

Let $S \subset R$ be the subgroupoid of automorphism. Let $P \subset R$ be a subgroupoid, and $S_P = S \cap P$.

7.1 Definition. P is **normal** if the map

$$S_P \times_{(s,t)} R \to S$$

given by $(s, r) \to i(r) \circ s \circ r$ factors through S_P.

(7.2) P acts on R by either pre-composition, or post-composition, or both. Thus we have groupoids (of sets):

(1) $P \rightrightarrows X$
(2) $P \times_{(s,t)} R \rightrightarrows R$
(3) $R \times_{(s,t)} P \rightrightarrows R$
(4) $P \times_{(s,t)} R \times_{(s,t)} P \rightrightarrows R$

(7.3) We indicate their quotients respectively by X', R^t, R^s, R''. Note there are natural maps $s : R^t \to X$, $t : R^s \to X$, and $s'', t'' : R'' \to X'$. and groupoids

(1) $P \times_{(s,t)} R^s \rightrightarrows R^s$
(2) $R^t \times_{(s,t)} P \rightrightarrows R^t$.

It is elementary to check that R'' is the quotient of either 7.3.1 or 7.3.2.

The following is easy (if a bit tedious) to check:

7.4 Lemma. *s'', t'' define a groupoid such that the map $(R, X) \to (R'', X')$ is a map of groupoids iff P is normal.*

Pre-composition on the first factor, and post composition composed with the inverse on the second factor give a groupoid

$$M = P \times_{(t, s \circ pr_1)} (R^t \times_{(s,t)} R^s) \rightrightarrows R^t \times_{(s,t)} R^s \tag{7.5}$$

(informally, the action is $(p, a, b) \to (a \circ p, i(p) \circ b)$). There are maps of groupoids

$$g : (R^t \times_{(s,t)} P, R^t) \to (P, X)$$
$$h : (M, R^t \times_{(s,t)} R^s) \to (R^t \times_{(s,t)} P, R^t)$$

which are square by construction.

7.6 Lemma. *If P is normal then g and h are fixed point reflecting.*

Proof. First for any P, note that for $\beta \in R$, and $p \in P$, $[\beta \circ p] = [\beta]$ in R^t iff $\beta \circ p \circ i(\beta) \in P$. Thus if P is normal, the stabilizer of $[\beta]$ is the fiber of S_P over $s(\beta)$. Of course similar result apply to R^s. Now the lemma follows easily. □

Let $Z = R^t \times_{(s,t)} R^s/M$. There is a natural M invariant surjection

$$R^t \times_{(s,t)} R^s \to R'' \times_{(s'',t'')} R''$$

and thus a surjection $p : Z \to R'' \times_{(s'',t'')} R''$.

7.7 Lemma. *If P is normal, p is an isomorphism. If $S \subset P$ then $R'' \to X' \times X'$ is injective.*

Proof. This follows easily from the stabilizer remarks in the proof of 7.6. □

Now we work with separated schemes:

7.8 Boot Strap Theorem. *We follow the above notation. Assume $R \rightrightarrows X$, is a quasi-finite flat groupoid of separated schemes, and $P \subset R$ a closed and open subgroupoid which is finite (and necessarily flat) over X. Assume GC quotients exist in (7.2.1), (7.2.4), (7.3.1) and (7.3.2) (Note quotients exist for (7.2.2) and (7.2.3) by (6.5)).*

1. *If $S \subset P$, then $R'' \rightrightarrows X'$ is an étale equivalence relation and the algebraic space X'/R'' is the GC quotient for X/R.*
2. *If P is normal, $R'' \rightrightarrows X'$ is an an étale groupoid such that GC quotient for X/R exists iff one for X'/R'' does, and if it does then they are isomorphic. I.e. X/R and X'/R'' define the same GC quotient problem in a neighborhood of x.*

Proof. Note the groupoids (7.2.2) and (7.2.3) are free, and thus fine quotients exist by (6.5). The map

$$R \times_{(s,t)} R \to R^t \times_{(s,t)} R^s$$

is also a fine quotient, as it is obtained from the fine quotient $R \times R \to R^t \times R^s$ by pullback. In particular these quotients are flat and commute with any base extension.

We show next $s : R^t \to X$ is étale. It is flat since $R \to R^t$ is flat. e gives a section $\overline{e}$ of s, and $\overline{e}(X)$ is a connected component since P is a connected component of R. Thus s is unramified along $\overline{e}$. Translating by the inverse gives an automorphism of the fiber carrying a prescribed point to $\overline{e}$, thus s is unramified, and hence étale.

Now g and h are étale, square, and fixed point reflecting by (7.6) (the only thing left to check is fixed point reflecting, which is a set theoretic question, and thus follows from the categorical case since the quotients are geometric).

Now by (6.3) s'', t'', $\overline{h} : Z \to R''$ are étale. The universal property of Z induces a commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{p} & R'' \times_{(s'',t'')} R'' \\ {\scriptstyle \overline{h}}\downarrow & & {\scriptstyle pr_1}\downarrow \\ R'' & = & R''. \end{array}$$

p is a bijection on geometric points by (7.7). Since both sides are étale over R'', the map is an isomorphism. The universal properties of both maps

$$R \times_{(s,t)} R \to R^t \times_{(s,t)} R^s \to Z$$

together with the composition of R induce a map $c : R'' \times_{(s'',t'')} R'' \to R''$. Such that the diagram

$$\begin{array}{ccc} R \times_{(s,t)} R & \xrightarrow{c} & R \\ \downarrow & & \downarrow \\ R'' \times_{(s'',t'')} R'' & \xrightarrow{c} & R'' \end{array}$$

commutes. Similarly we have induced commutative diagrams

$$\begin{array}{ccc} R & \xrightarrow{i} & R \\ \downarrow & & \downarrow \\ R'' & \xrightarrow{i} & R'' \end{array}$$

and

$$\begin{array}{ccc} X & \xrightarrow{e} & R \\ \downarrow & & \downarrow \\ X' & \xrightarrow{e} & R''. \end{array}$$

To check these define a groupoid, we need to check various diagrams commute. In most cases this follows from the corresponding diagram of (R, X) by the universal property. For example to see

$$\begin{array}{ccc} R'' \times_{(s'',t'')} R'' & \xrightarrow{c} & R'' \\ {\scriptstyle pr_1}\downarrow & & {\scriptstyle t''}\downarrow \\ R'' & \xrightarrow{t''} & X' \end{array}$$

commutes, one compares this with the commutative diagram

$$\begin{array}{ccc} R \times_{(s,t)} R & \xrightarrow{c} & R \\ {\scriptstyle pr_1}\downarrow & & {\scriptstyle t}\downarrow \\ R & \xrightarrow{t} & X \end{array}$$

and uses that

$$R \times_{(s,t)} R \to R'' \times_{(s'',t'')} R''$$
$$R \to R''$$
$$X \to X$$

are all categorical surjections, by the universal properties. Note in particular that since t'' and s'' are étale, this gives that c is étale. The only diagram which must be treated differently is for the associativity of composition. Since c is étale, this amounts to checking equality between étale maps. This is a set theoretic question, and so follows from (7.4). One could also express the three fold fiber products of R'' as geometric quotients and then apply the universal property.

Now suppose $S \subset P$. $R'' \to X' \times X'$ is unramified, since the projections are étale. It is injective on geometric points by (7.4). Thus it is a monomorphism. Now let $Q = X'/R''$. Since $q : X \to Q$ is the composition of two GC quotient maps, the result follows easily. $\square$

§8 Proof of Theorem 1.1

8.1 Lemma. *Let $R \rightrightarrows X$ be a flat quasi-finite groupoid with finite j. Then a GC quotient exists.*

Proof. Note the assumptions are preserved by any flat pullback. By (3.2) its enough to construct the quotient locally in a neighborhood of $x \in X$. Thus we can assume x is fixed by R. By (4.1) we can assume R is split over x, and that X, and hence R (since j is finite) are affine schemes. Then each of the necessary GC quotients in (7.8) exists by (5.1) $\square$

Proof of Theorem 1. Since the stabilizer (2.7) is finite, j is quasi-finite. We can assume we are in the situation of (3.3). We reduce to R split as in the proof of (8.1). By (8.1) each of the necessary quotients in (7.8) exists. $\square$

§9 Relations to GIT

Throughout this section we assume L is the spectrum of a field. We follow the notation of [MumfordFogarty82].

Note first that for any map $p : S \to X$ there is a unique maximal open subset of X over which p is proper. When p is the stabilizer of a groupoid, we call this open set $X(PropStab)$.

9.1 Proposition. *Let G be a reductive group acting on a scheme X. Then $U = X_0^s(Pre) \subset X(PropStab)$, and the quotients of U given by* [MumfordFogarty82,1.9] *is isomorphic to the GC quotient of U/G.*

Proof. Let $q : U \to Y$ be the GIT quotient. To check finiteness of the stabilizer, we can replace Y by any etale cover, and so can assume Y and thus U are affine (note q is affine), and in particular separated. Then by [MumfordFogarty82,0.8] the action of G is proper.

Now let $g : U \to Q$ be the GC quotient of (1.1). Q is universal among algebraic spaces, while Y is universal among schemes. We have an induced map $h : Q \to Y$. We can check h is an isomorphism locally on Y, and so can assume as above that G acts properly. Then Q is separated, and h is set theoretically one to one. Thus by [Knudson71, I.I6.16] Q is a scheme, and so h is an isomorphism by the universality of Y. $\square$

9.2 Remark. A reductive group acting with quasi-finite stabilizer does not in general have finite stabilizer. For example if $G = Aut(P^1)$ acting on the n-th symmetric product X of P^1 for $n \geq 4$ and U is the open set of X where the stabilizers are quasi-finite, then $U \not\subset X(PropStab)$.

9.3 Remark. In (1.1), assume that j is finite, s, t are affine, and L is the spectrum of a field. Then the argument of [MumfordFogarty82,0.7], which extends without change to groupoids, shows that the quotient map is affine. Mumford also remarks on pg. 16 that (0.7) holds for arbitrary base.

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